Elicoide circolare

$$\begin{cases} x(t) = (a\cos t, a\sin t, bt) & , t\in\mathbb{R} \\ \text{i.e.} \quad \begin{cases} 2\cos t - \frac{t}{2}\sin t \\ 2\sin t + \frac{t}{2}\cos t \\ 2\cos t - \frac{t}{2}\sin t - \frac{t^2}{4}\cos t \end{cases} \end{cases}$$

per la sua lunga durata

$$\begin{cases} 2\cos t - \left(\frac{t}{2}\right) = 0 \Rightarrow \frac{t}{2} = \cos t \\ \quad \Rightarrow \frac{t}{2} = \frac{1}{\cos t} \end{cases}$$

$$\sin t + \frac{t}{2}\cos t = \frac{t}{2}$$

$x(t) = (a\cos t, a\sin t, bt)$

$x(t) = (\cos t, \sin t, t)$

$x(t) = (1, 0, t)$

L è lunga $x(t) = 2\pi b$

una linea.

Applichiamo il procedimento di spiegazione

$x(t)$ per lunghezza d'arco.

$$\begin{cases} x(t) = (a\cos t, a\sin t, bt) & , t\in\mathbb{R} \\ s(t) = \int_0^t |x'(u)|\,du = \int_0^t \sqrt{a^2+b^2}\,du = \sqrt{a^2+b^2}\,(t-0) \end{cases}$$

$t = s(u)/\sqrt{a^2+b^2}$

$x(t) = (a\cos t, a\sin t, t)$

posso scegliere $t_0 = 0$, $s(t) = \sqrt{a^2+b^2}\,t$

$t = s(t)/\sqrt{a^2+b^2}$

$t = s(t) = \frac{s}{\sqrt{a^2+b^2}}$, $c = \sqrt{a^2+b^2}$

$\Rightarrow \gamma(s) = x(t(s)) = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, b\frac{s}{c}\right)$

Semp. per semplicità $a > 0$.

$T(s) = \gamma'(s) = \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right)$

$\gamma''(s) = \left(-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0\right)$

$\|\gamma''(s)\| = k(s) = \sqrt{\frac{a^2}{c^4}} = \frac{a}{c^2} > 0$

$$n(s) = \frac{\gamma''(s)}{\|\gamma''(s)\|} = \left(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0\right)$$

$$b(s) = T(s) \wedge n(s) \qquad \begin{pmatrix} -\frac{a}{c}\sin\frac{s}{c} & -\cos\frac{s}{c} \\ \frac{a}{c}\cos\frac{s}{c} & -\sin\frac{s}{c} \\ \frac{b}{c} & 0 \end{pmatrix}$$

$$b(s) = \begin{pmatrix} \frac{b}{c}\sin\frac{s}{c} \\ -\frac{b}{c}\cos\frac{s}{c} \\ \frac{a}{c} \end{pmatrix}$$

$$b'(s) = \begin{pmatrix} \frac{b}{c^2}\cos\frac{s}{c} \\ \frac{b}{c^2}\sin\frac{s}{c} \\ 0 \end{pmatrix} \qquad n(s) = \begin{pmatrix} -\cos\frac{s}{c} \\ -\sin\frac{s}{c} \\ 0 \end{pmatrix}$$

$b'(s) = \tau(s)\, n(s) = -\frac{b}{c^2}\, n(s)$

$\tau(s) = -\frac{b}{c^2} = \frac{-b}{a^2+b^2}$ (non dipende da s o non varia)

Pb.: Posso usarli tutti le curve con k e τ costanti?

OSS.: $\forall$ coppia $(k,\tau)\in\mathbb{R}^2 \setminus R$

$\exists?$ (a,b) t.c. $(k,\tau) = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$?

con $a+ib \neq 0$, $(a+ib)^{-1} = \frac{\bar{a}}{|a+ib|^2} = \frac{b}{|a+ib|^2}$;

$\Rightarrow 0$

Pongo $(a+ib) = (k+i\tau)^{-1}$, $k > 0$

$\Rightarrow (k+i\tau) = (a+ib)^{-1}$

Lemma: $\forall (k,\tau)\in\mathbb{R}^2 \setminus \mathbb{R}$, $\exists$ unica circolare con k curvatura e τ torsione.

Conc. del Teo. Fond. per curve dell'$\wedge\mathbb{R}^2$

COR.: Le uniche curve regolari e biregolari di $\mathbb{R}^3$ con curvat. e torsione costanti sono le eliche circolari.

In part., l'unica curva piana con k costante è la circonferenza (o un arco di circonferenza)

CURVATURA & TORSIONE
IN PARAM. ARBITRARIA

Prop.: Sia $\sigma : I \to \mathbb{R}$ curva param. di classe C^k, $k\geq 3$, regolare

Allora: $k(t) = \frac{\|\sigma'(t)\wedge\sigma''(t)\|}{\|\sigma'(t)\|^3}$, dove $k(t)$ è la curvatura (prima ripar. per lunghezza d'arco di σ nel punto $s(t)$).

$$\tau(t) = \frac{-(\sigma'(t)\wedge\sigma''(t))\cdot\sigma'''(t)}{\|\sigma'(t)\wedge\sigma''(t)\|^2}$$

Dim.: σ regolare $\Rightarrow$ posso riparametrizzare per lunghezza d'arco

$\sigma : I \to \mathbb{R}^3$, $t_0 \in I$

$s(t) : I \to \mathbb{R}$, $s(t) = \int_{t_0}^{t} \|\sigma'(u)\|\,du$

$s(I) = J$, $\boxed{s'(t) = \|\sigma'(t)\|}$

$\exists$ $f : J \to I$, $t = f(s)$

$\alpha(s) := \sigma(f(s))$ è PLA

$\alpha(s(t)) = \sigma(f(s(t))) = \sigma(t)$

$\sigma'(t) = \alpha'(s(t))\cdot s'(t) = \alpha'(s(t))\cdot\|\sigma'(t)\|$

$= T_\alpha(s(t))\cdot\|\sigma'(t)\|$

$\sigma''(t) = \left(\alpha''(s(t))\cdot s'(t)\right)\cdot s'(t) + \alpha'(s(t))\cdot s''(t)$

$= k_\alpha(s(t))\cdot n_\alpha(s(t))\cdot\frac{(s'(t))^2}{\|\sigma'(t)\|^2} + T_\alpha(s(t))\cdot s''(t)$

$\Rightarrow \sigma''(t) \in W_{s(t)}$ PIANO VETT. osculatore

OSS. $k_\alpha(s(t)) = 0 \Leftrightarrow \sigma''(t) \parallel T_\alpha(s(t))$

Supp. $k_\alpha(s(t)) \neq 0 \Rightarrow \sigma'(t)$ & $\sigma''(t)$ sono linearmente indipendenti (e quindi formano base di $W_{s(t)}$)

Conc. $\sigma'(t)\wedge\sigma''(t) \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\sigma'(t)\wedge\sigma''(t) = (s'(t)\cdot T_\alpha(s(t)))\wedge\left(k_\alpha(s(t))(s'(t))^2 n_\alpha(s(t)) + s''(t)T_\alpha(s(t))\right)$

| Lin. & antisim.

$(*) = (s'(t))^3\, k_\alpha(s(t))\, \overbrace{T_\alpha(s(t))\wedge n_\alpha(s(t))}^{b_\alpha(s(t))}$ $\Rightarrow k_\alpha(s(t)) = \frac{\sigma'(t)\wedge\sigma''(t)}{\|\sigma'(t)\wedge\sigma''(t)\|}$

$\Rightarrow \|\sigma'(t)\wedge\sigma''(t)\| = \|\sigma'(t)\|^3\, k_\alpha(s(t))$

Vediamo ora di calcolare $\tau(s)$

$\sigma'''(t) = \left(k_\alpha(s(t))\cdot(s'(t))^2 n_\alpha(s(t)) + s''(t)\cdot T_\alpha(s(t))\right)'$

$= \left(k_\alpha(s(t))\cdot(s'(t))^2\right)'\cdot n_\alpha(s(t)) +$

$+ \left(k_\alpha(s(t))\cdot(s'(t))^2\right)\cdot n_\alpha'(s(t))\cdot s'(t) +$

$+ s''(t)\cdot T_\alpha(s(t)) +$

$+ s''(t)\cdot T_\alpha'(s(t))\cdot s'(t)$

$(\sigma'(t)\wedge\sigma''(t))\cdot\sigma'''(t) = \left(\|\sigma'(t)\|^3\cdot k_\alpha(s(t))\, b_\alpha(s(t))\right)\cdot\left(k_\alpha(s(t))(s'(t))^3\, n_\alpha'(s(t))\right)$

$(*)$ $\{T_\alpha, n_\alpha, b_\alpha\}$ sono base ortonormale

FS $= -k_\alpha(s(t))\cdot T_\alpha(s(t)) - \tau(s(t))\, b_\alpha(s(t))$

$$\Rightarrow (\sigma'(t)\wedge\sigma''(t))\cdot\sigma'''(t) = -\|\sigma'(t)\|^6\, k_\alpha(s(t))^2\cdot\tau(s(t))$$

$$\frac{(\sigma'(t)\wedge\sigma''(t))\cdot\sigma'''(t)}{\|\sigma'(t)\wedge\sigma''(t)\|^2} = -k_\alpha(s(t))^2\cdot\tau(s(t))$$

$$\Rightarrow \tau(s(t)) = \frac{-(\sigma'(t)\wedge\sigma''(t))\cdot\sigma'''(t)}{\|\sigma'(t)\wedge\sigma''(t)\|^2}$$

$\{T_\alpha, n_\alpha, b_\alpha\}$, $\{k_\alpha, \tau_\alpha\}$ sono ordinati.

Idea: $n_\alpha(s(t)) = b_\alpha(s(t))\wedge T_\alpha(s(t))$

$$\left.\begin{aligned} T_\alpha'(s(t)) &= \frac{\alpha''(t)}{\|\sigma'(t)\|} \\ b_\alpha(s(t)) &= \frac{\sigma'(t)\wedge\sigma''(t)}{\|\sigma'(t)\wedge\sigma''(t)\|} \end{aligned}\right\}$$

Esempio: $\sigma(t) = (t, t^2)$ è una parab.

$\sigma'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix} \neq (0)$

$s(t) = \int_0^t \|\sigma'(u)\|\,du = \int_0^t \sqrt{1+4u^2}\,du$

$k(t) = \frac{\|\sigma'(t)\wedge\sigma''(t)\|}{\|\sigma'(t)\|^3}$

Iniziamo il piano in $\mathbb{R}^3$

$\tilde{\sigma}(t) : \mathbb{R} \to \mathbb{R}^2$

$\tilde{\sigma}(t) = (t, t^2, 0)$

$\tilde{\sigma}'(t) = \begin{pmatrix} 1 \\ 2t \\ 0 \end{pmatrix}$, $\tilde{\sigma}''(t) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

$k(t) = \frac{\|\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}\|}{(1+4t^2)^{3/2}} = \frac{2\cdot 1}{(1+4t^2)^{3/2}} = \frac{2}{(1+4t^2)^{3/2}}$

$\Leftrightarrow$ fine

$\Rightarrow \tilde{\sigma}(t) = (t, t, 0)$ è piana

Es.: $f : I \to \mathbb{R}$

$\alpha(t) = (t, f(t))$ in $\mathbb{R}^2 \longleftrightarrow \mathbb{R}^2$

posso fare