Introduction to Lie Groups and Lie Algebras

Fabio Vlacci MIGe Università di Trieste

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Definition of Group

Definition

A *group* is any set G with an operation $\cdot : G \times G \to G$ such that
1) $\forall g, h, l \in G \; g \cdot (h \cdot l) = (g \cdot h) \cdot l$;
2) $\exists \; e_G \in G$ such that $e_G \cdot g = e_G \cdot g = g \; \forall g \in G$;
3) $\forall g \in G$, $\exists \; \widetilde{g} \in G$ such that $e = g \cdot \widetilde{g} = \widetilde{g} \cdot g$.

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In general, the operation $\cdot$ is not commutative but if so the group is said to be *abelian*. Very often $\widetilde{g} = g^{-1}$.

Definition of Action of a Group

Definition

We say that a group $(G, \cdot)$ acts on a set S if there exists a mappting

$$\mu : G \times S \to S$$

$$(g, x) \mapsto gx = \mu(g, x)$$

such that
1) $\mu(e_G, x) = x \quad \forall x \in S$
2) $\mu(h, gx) = \mu(h, \mu(g, x)) = \mu(h \cdot g, x) \quad \forall x \in S$ and $\forall g, h \in G$.

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If S is a topological space, we say that the action of $(G, \cdot)$ on S is *properly discontinuous* if every $x \in S$ has an open neighborhood U in S such that $U \cap g(U) = \varnothing$ for all $g \in G$ $g \neq e_G$.

Definition of Topological Group

Definition

A group $(G, \cdot)$ where the set G is a topological space with

$$\cdot : G \times G \to G, \quad (g, h) \mapsto g \cdot h$$

and

$$\iota : G \to G, \quad g \mapsto g^{-1}$$

continuous maps is called a *topological group*.

Definition of Lie Group

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A group $(G, \cdot)$ where G is a smooth manifold with

$$\cdot : G \times G \to G, \quad (g, h) \mapsto g \cdot h$$

and

$$\iota : G \to G, \quad g \mapsto g^{-1}$$

smooth maps is called a *Lie group*.

Examples of Lie Groups

Examples include

$$(\mathbb{R}^n, +) \quad (\mathbb{C} \setminus \{0\} \simeq \mathbb{R}^2 \setminus \{0\}, \cdot_{\mathbb{C}}) \quad (S^1, \cdot_{\mathbb{C}}) \quad (GL(n, \mathbb{R}), \cdot)$$

Definition of Lie Algebra

Definition

A real vector space $\mathcal{G}$ equipped with a $\mathbb{R}$ bilinear skew symmetric product[a] which satisfies the Jacobi identity is called a *Lie Algebra*

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[a]
$$[\cdot,\cdot] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$$

- Bilinear
- Skew symmetric $[X, Y] = -[Y, X]$
- Jacobi identity $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

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$$M(n, \mathbb{R}) = \{(a_{ij})_{1 \leq i,j \leq n} \quad a_{ij} \in \mathbb{R}\} \quad \text{(as a subspace of } \mathbb{R}^{n^2}\text{)}$$

$$\text{and } (A, B) \mapsto [A, B] = AB - BA$$

Left Translation

For $g \in G$, the function

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Left and right translations are diffeomorphisms and so are inner automorphisms $R_{g^{-1}} \circ L_g : G \to G$

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Remark

Question: since (left translations, $\circ$) is a group acting on G, when is this action properly discontinuous?

Left-Invariant Vector Fields

Definition

A vector field $X \in \Gamma(TG)$ is *left-invariant* if

$$(dL_g)_h X_h = X_{gh} = X_{L_g(h)}$$

for all $g, h \in G$.

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Theorem

The set of left-invariant vector fields on G forms a Lie algebra with $[,]$ Lie bracket as product.

Proof (Step 1)

Let X, Y be left-invariant vector fields.
We have only to show that

$$[X, Y] = XY - YX$$

is also left-invariant.

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Take $f \in \mathcal{D}(G)$ and consider $(dL_g)[X, Y]f$.

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$$dF_p(v)f = \frac{d}{dt}(f \circ F \circ \alpha)|_{t=0}$$

where $\alpha : (-\varepsilon, \varepsilon) \to M$ is a differentiable curve such that $\alpha(0) = p$ and $\alpha'(0) = v$ or

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Hence

$$(dL_g)[X, Y]f = [X, Y](f \circ L_g) =$$

$$= X(Y(f \circ L_g)) - Y(X(f \circ L_g) = X(dL_g Yf) - Y(dL_g Xf)$$

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$$= X(Y(f \circ L_g)) - Y(X(f \circ L_g) = X(dL_g Yf) - Y(dL_g Xf)$$

But since X, Y are left invariant (or $dL_g X = X \ dL_g Y = Y$), the commutator of X and Y is left invariant.

Lie Algebra of a Lie Group

Given a Lie group $(G,\cdot)$, consider $T_{e_G}G$ and the Lie algebra $\mathcal{G}$ of left invariants vector fields on G.

Proposition

$$X \in \mathcal{G} \to X_{e_G}$$

is a linear isomorphism which preserves the Lie brackets (or a Lie algebra homomorphism).

Proof.

If $X_{e_G} = Y_{e_G}$, for any $g \in G$ $dL_g X_{e_G} = L_g Y_{e_G}$ and hence $X_g = Y_g$ for any $g \in G$ or $X = Y$.

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If $X \in T_{e_G}G$, put $X_g = dL_g X$; then
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If $X \in T_{e_G}G$, put $X_g = dL_g X$; then
$dL_h X_g = dL_h(dL_g X_{e_G}) = X_{gh} = dL_{gh} X_{e_G}$.
Finally, if $X, Y \in \mathcal{G}$, then $[X, Y] \in \mathcal{G}$ and $[X, Y]_{e_G} = [X_{e_G}, Y_{e_G}]$.

□

Example

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$$(\mathbb{R},+) \qquad \left(\mathcal{R}=\left\{\lambda\frac{d}{dt}\right\},[\ ,\ ]=0\right)$$

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If $\varphi : G \to H$ is a Lie group homomorphism, then $\varphi(e_G) = e_H$; furthermore

Proposition

$$d\varphi_{e_G} : T_{e_G}G \to T_{e_H}H$$

is a linear homomorphism such that

$$d\varphi([X, Y]_{e_G}) = [d\varphi(X), d\varphi(Y)]_{e_H}$$

i.e. φ *is a Lie algebra homomorphism.*

Exponential map

Let G be a Lie group and consider $X \in T_{e_G}G \simeq \mathcal{G}$. Then there exists a unique $\alpha : (-\varepsilon, \varepsilon) \to G$ such that $\alpha(0) = e_G$ and $\alpha'(0) = X$.

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If $\exp(tX) = \exp_X(t)$ we have

$$\exp_X(t_1 + t_2) = \exp_X(t_1) \cdot \exp_X(t_2)$$

and, if $\varphi : G \to H$ is a Lie group homomorphism, then

$$\varphi \circ \exp^G = \exp^H \circ d\varphi_{e_G}$$

with $\exp^G$ the exponential map of G and $\exp^H$ the exponential map of H.

Theorem (Local Diffeomorphism)

Theorem

The exponential map is a local diffeomorphism near the identity.

Proof

Proof.

Let

$$\gamma(t) = tX$$

Then

$$\frac{d}{dt}\exp(tX)|_{t=0} = X$$

Thus

$$d\exp_0 = Id$$

By the inverse function theorem, exp is a local diffeomorphism. ☐

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$$\mathrm{Exp}(A) = I + A + A^2/2 + \ldots + A^n/n! + \ldots$$

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$$\exp^{GL(n,\mathbb{R})} = \mathrm{Exp}!!$$

Moreover, if $B \in GL(n, \mathbb{R})$

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$$\det(\mathrm{Exp}(A)) = \mathrm{Exp}(\mathrm{trace}(A))$$

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Finally, if $A \cdot A' = A' \cdot A$, then

$$\mathrm{Exp}(A + A') = \mathrm{Exp}A \cdot \mathrm{Exp}A'$$

Invariant metrics on G

Definition

If a Lie group G has a Riemannian metric, we'll say that this Riemannian metric is *left invariant* if, for any $X, Y \in T_pG$ and for any $g \in G$

$$\langle X, Y \rangle_p = \langle (dL_g)_p X, (dL_g)_p Y \rangle_{L_g(p)}$$

or L_g is an isometry for any $g \in G$.

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Analogously, one can define *right invariant* metrics; if a Riemannian metric is both right and left invariant it will be called *bi-invariant*.

Invariant metrics on G

Take any generic inner product $\langle \cdot, \cdot \rangle_{e_G}$ in $\mathcal{G} \simeq T_{e_G} G$ and extend it as follows at $g \in G$

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The corresponding Riemannian metric in G is, by defintion, left invariant.