Exercise set 1

AI

1 Thermalization

Consider a system with 3 energy levels, $k\epsilon$, $k = 0, 1, 2$.

At $t \leq 0$ the system is at equilibrium at $T_0 = T_c$, then at $t > 0$ it is connected to a bath at temperature $T_h > T_c$. Introduce the jump rates between the states $0 \rightleftarrows 1 \rightleftarrows 2$, assume they satisfy the detailed balance, and express the general solution of the master equations as an expansion in eigenfunctions along the lines of section V.7 in van Kampen's book.

Repeat the calculation with $T_0 = T_h$, and T_c at $t > 0$.

Which processes converges faster to equilibrium, heating or cooling? Can you change the system parameters to change the order of the convergence rates?

2 Fokker-Planck equation

A particle in a viscous medium (temperature T, friction coefficient ζ) moves in a 1D potential $U(x)$. In the following assume the overdamped regime.

a)

1. Show that the differential operator in the Smoluchowski equation is not Hermitian.
2. Can you make the equation self-adjoint?
3. Assume that the Brownian particle diffuses in a harmonic potential $U(x) = kx^2/2$ and find the general solution to the Smoluchowski equation. Hint: use the result from the previous point.

b) Now consider a periodic potential $U_0(x)$, with period L: $U_0(0) = U_0(L)$. By using the Smoluchowski equation:

1. find the steady state probability distribution function (PDF) for the particle position $P(x, t \to \infty)$, with $0 \leq x \leq L$
2. find the steady state PDF when the same particle is also subject to a constant force f, so the total potential reads $U(x) = U_0(x) - fx$.

Hint: consider the general solution discussed during class, and impose the boundary conditions.

3 Quantum Langevin equation for the harmonic oscillator

Verify first the equations for the anti-commutator relation and the correlation for the quantum noise on page 6 of the slides.

Consider the expression for the steady state variance of the oscillator as given in the notes ($k_B = 1$)

$$\left\langle x(t)^2 \right\rangle = \int \frac{d\omega}{2\pi} \frac{\gamma \hbar \omega \coth(\hbar\omega/2T)}{(m\omega^2 - m\omega_s^2)^2 + (\omega\gamma)^2}.$$

Rederive this relation from the Langevin equation on page 8 of the slides.

In the upper half complex plane the integral has

- two simple poles at $z_\pm = \frac{\mathbb{i}}{2m}(\gamma \pm \sqrt{\Delta})$ with $\Delta = \gamma^2 - 4m^2\omega_s^2$.
- an infinite (but isolated) number of poles along the positive complex axis $z_n = 2\pi \mathbb{i} n T/\hbar$, $n = 1, 2 \ldots$

The residues read

$$R_\pm = \pm \frac{\mathbb{i}\hbar}{4\pi\sqrt{\Delta}} \cot\left[\frac{\hbar\left(\gamma \pm \sqrt{\Delta}\right)}{4mT}\right],$$

$$R_n = \frac{2\mathbb{i}\gamma\hbar^3 n T^2}{\left(\hbar^2 m\omega_s^2 + 2\pi n T(2\pi n m T - \gamma\hbar)\right)\left(\hbar^2 m\omega_s^2 + 2\pi n T(\gamma\hbar + 2\pi n m T)\right)}.$$

Calculate the value of $\langle x(t)^2 \rangle$ first taking the weak coupling limit $\gamma \to 0$ and then the classical limit $\hbar \to 0$. Repeat the calculation by inverting the order of the limits.

In which case do you recover the equilibrium result?