Lesson 11 - 19/03/2026

Density - $\mathcal{C}_0^\infty(\Omega)$ is dense in $L^1(\Omega)$.

- if $f \in L^1_{loc}(\Omega)$ and

$$\int_\Omega f\varphi = 0 \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega)$$

then $f = 0$ (in L^1_{loc}) | lemma

- <u>Theorem</u> let Ω be open set in $\mathbb{R}^d$

let $p \in [1, +\infty[$

then $\mathcal{C}_0^\infty(\Omega)$ is dense in $L^p(\Omega)$

proof. the case $p = 1$ is the first result above

suppose $1 < p < +\infty$

I use a corollary of Hahn-Banach theorem

W subspace of V (V normed space)

suppose the following holds

| if $\phi \in V'$ and ($\phi_W = 0 \Rightarrow \phi = 0$)

| then W is dense in V

So take $\phi \in (L^p(\Omega))'$

From Riesz theorem $\exists\, g \in L^{p'}(\Omega)$ (with $\frac{1}{p} + \frac{1}{p'} = 1$)

s.t. $\phi(f) = \int_\Omega g f$

suppose ϕ is 0 on $\mathcal{C}_0^\infty(\Omega)$ (we think to $\mathcal{C}_0^\infty(\Omega)$ as a subspace of $L^p(\Omega)$)

this means that $\int_\Omega g\varphi = 0 \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega)$

the lemma implies that $g = 0$ a.e.

then ϕ is 0 $\Rightarrow$ $\mathcal{C}_0^\infty(\Omega)$ is dense in $L^p(\Omega)$

<u>QED</u>

2) <u>convolution</u>.

We suppose known the following results.

1) Let $f \in L^p(\mathbb{R}^d)$, $g \in L^1(\mathbb{R}^d)$ $\quad p \in [1, +\infty]$

then for a.e. $x \in \mathbb{R}$, $y \longmapsto f(x-y)g(y)$ is in $L^1(\mathbb{R}^d)$

$\mathbb{R}^d \longrightarrow \mathbb{R}^d$

and $f*g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy$, $\quad f*g \in L^p(\mathbb{R}^d)$

and $\|f*g\|_{L^p(\mathbb{R}^d)} \le \|f\|_{L^p}\|g\|_{L^1}$

2) more generally if $p, q, r \in [1, +\infty]$ (YOUNG INEQ.)

and $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} - 1$

and $f \in L^q(\mathbb{R}^d)$, $g \in L^r(\mathbb{R}^d)$

then $f*g \in L^p(\mathbb{R}^d)$ and $\|f*g\|_{L^p} \le \|f\|_{L^q}\|g\|_{L^r}$

<u>def</u>. let $f \in L^1_{loc}(\Omega)$, let $x_0 \in \Omega$.

I say that $x_0 \notin \operatorname{supp} f$ if $\exists r > 0$ s.t.

$f(x) = 0$ a.e. in $B(x_0, r)$

$\operatorname{supp} f$ is smaller relatively closed set in Ω outside of which $f = 0$ a.e.

3) let $f \in L^p(\mathbb{R}^d)$, $g \in L^1(\mathbb{R}^d)$

$\operatorname{supp}(f*g) \subseteq \overline{\operatorname{supp} f + \operatorname{supp} g}$ ← closure in $\mathbb{R}^d$

4) If $f \in \mathcal{C}^0_0(\mathbb{R}^d)$ and $g \in L^1_{loc}(\mathbb{R}^d)$

we can still define $f * g$

($f*g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy$

fixed x, $y \mapsto f(x-y)$ has compact support

$f(x-y)g(y)$ is integrated

on a compact

and it makes sense)

let $f \in \mathcal{C}^m_0(\mathbb{R}^d) = \mathcal{C}^m(\mathbb{R}^d) \cap \mathcal{C}^0_0(\Omega)$

let $g \in L^1_{loc}(\mathbb{R}^d)$

then $f*g \in \mathcal{C}^m(\mathbb{R}^d)$ and

$$\frac{\partial}{\partial x_j}(f*g) = \frac{\partial f}{\partial x_j} * g$$

3) <u>test functions and mollifiers.</u>

$\mathcal{C}^0_0(\Omega) = \{ f$ continuous in Ω, with compact support in $\Omega \}$

$\mathcal{C}^m_0(\Omega) = \mathcal{C}^m(\Omega) \cap \mathcal{C}^0_0(\Omega)$

$\mathcal{C}^\infty_0(\Omega) = \bigcap_{m \in \mathbb{N}} \mathcal{C}^m_0(\Omega) = \mathcal{D}(\Omega)$ ← set of <u>test functions</u>

<u>Ex</u> consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$

prove that f is $\mathcal{C}^\infty$

Hint use the fact that if

f is continuous in $\mathbb{R}$ and diff in $\mathbb{R}\setminus\{0\}$

Hint: use the fact that if

f is continuous in $\mathbb{R}$ and diff in $\mathbb{R}\setminus\{0\}$

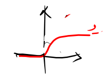
and $\lim_{x\to 0} f'(x)$ exist in $\mathbb{R}$

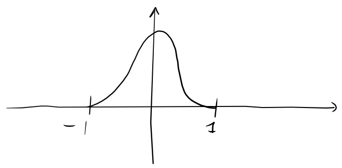
then f is diff in $\mathbb{R}$ and $f'(0) = \lim_{x\to 0} f'(x)$.

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x>0 \\ 0 & x\le 0 \end{cases}$$

considering $g(x) = f(1-|x|^2)$ in $\mathbb{R}^d$

$g \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, supp $g = B(0,1)$, $g \geq 0$





taking $\rho \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $\rho \geq 0$, supp $\rho \subseteq B(0,1)$

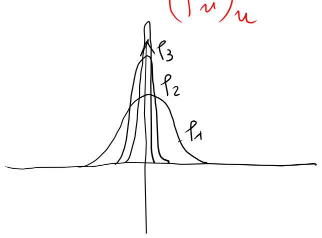
and $\int_{\mathbb{R}^d} \rho(y)\,dy = 1$

I call mollifier (or family of mollifiers)

the set of function $\rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho\left(\frac{x}{\varepsilon}\right)$ for $\varepsilon > 0$

(I denote $(\rho_\varepsilon)_{\varepsilon>0}$)

or $(\rho_n)_n$ with $\rho_n(x) = n^d \rho(nx)$



Th. let $(\rho_n)_n$ be a mollifier

i) if $f \in L^1(\Omega)$ and f has support in a compact set of Ω

then $\exists\, \bar{n}$ s.t. $\forall n > \bar{n}$

$\rho_n * f \in \mathcal{C}_0^\infty(\Omega)$

ii) let $f \in \mathcal{C}_0^0(\Omega)$

then $\rho_n * f \longrightarrow f$ uniformly

iii) let $f \in L^p(\Omega)$ with $1 \le p < +\infty$

let $\bar{f}(x) = \begin{cases} f(x) & \text{for } x \in \Omega \\ 0 & \text{for } x \notin \Omega \end{cases}$

then $\rho_n * \bar{f}\big|_\Omega \longrightarrow f$ in $L^p(\Omega)$