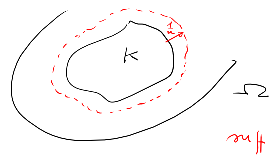
Lezione 12 - 20/03/2026

i) $f \in L^1(\Omega)$ with support in a compact of Ω, say K,
then $\exists \bar{n} : \rho_n * f \in C_0^\infty(\Omega)$ $\forall n \geq \bar{n}$

proof: immediate from the behaviour of convolution w.r.t. supports and regularity



$\operatorname{supp} f * \rho_n \subseteq \overline{\operatorname{supp} f + B(0, \frac{1}{n})}$

ii) if $f \in C_0^0(\Omega)$ then $\rho_n * f \xrightarrow{n} f$ uniformly.

$$\rho_n * f(x) - f(x) = \int_{\mathbb{R}^d} \rho_n(y) f(x-y)\,dy - f(x) \qquad \int_{\mathbb{R}^d} \rho_n(y)\,dy = 1$$

$$= \int_{\mathbb{R}^d} \rho_n(y)\big(f(x-y) - f(x)\big)\,dy$$

now f is uniformly continuous (since f has compact support)

$\forall \varepsilon > 0, \exists \delta > 0 \ \forall y \ |y| < \delta \Rightarrow |f(x-y) - f(x)| < \varepsilon \ \forall x$

$$|(\rho_n * f - f)(x)| \leq \int_{\mathbb{R}^d} \rho_n(y)\,|f(x-y) - f(x)|\,dy \quad (\text{but } |y| < \tfrac{1}{n})$$

$$\leq \varepsilon \int_{\mathbb{R}^d} \rho_n(y)\,dy = \varepsilon \qquad \text{if } \tfrac{1}{n} < \delta$$

iii) $f \in L^p(\Omega)$ with $p \in [1, +\infty[$

define $\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}$ $\quad \bar{f} \in L^p(\mathbb{R}^d)$

then $\| \rho_n * \bar{f}|_\Omega - f \|_{L^p(\Omega)} \xrightarrow{n} 0$ $\quad (\rho_n * \bar{f}|_\Omega \to f \text{ in } L^p(\Omega))$

proof

$$\| \rho_n * \bar{f}|_\Omega - f \|_{L^p(\Omega)} \leq \| \rho_n * \bar{f} - \bar{f} \|_{L^p(\mathbb{R}^d)}$$

$$\| \rho_n * \bar{f} - \bar{f} \|_{L^p} \leq \| \rho_n * \bar{f} - \rho_n * g \|_{L^p} + \| \rho_n * g - g \|_{L^p} + \| g - \bar{f} \|_{L^p}$$

$\forall g \in C_0^0(\mathbb{R}^d)$

$$\leq \underbrace{\| \rho_n * (\bar{f} - g) \|_{L^p}}_{\leq \|\rho_n\|_{L^1} \|\bar{f} - g\|_{L^p}} + \| \rho_n * g - g \|_{L^p} + \| g - \bar{f} \|_{L^p}$$

$\|\rho_n\|_{L^1} = 1$

$\rho_n * g \to g$ uniformly (previous result) then also in L^p

$$\| \rho_n * \bar{f} - \bar{f} \|_{L^p} \leq 2 \| \bar{f} - g \|_{L^p} + \| \rho_n * g - g \|_{L^p}$$

use $C_0^0(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ — small as you want

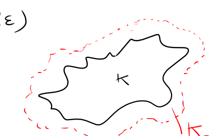
we conclude easily

QED

Remark

suppose to have a compact $K \subseteq \mathbb{R}^d$

consider $K_{2\varepsilon} = \bigcup_{x \in K} B(x, 2\varepsilon)$



consider $\rho_\varepsilon * \chi_{K_{2\varepsilon}}$

Th (partition of unity)

let K compact and let $\Omega_1, \Omega_2, \dots \Omega_N$ open sets

suppose $K \subseteq \bigcup_{i=1}^{N} \Omega_i$

then $\exists\ \varphi_1 \in C_0^\infty(\Omega_1),\ \varphi_2 \in C_0^\infty(\Omega_2), \dots, \varphi_N \in C_0^\infty(\Omega_N)$

s.t. $\forall x \in K, \quad \sum_{i=1}^{N} \varphi_i(x) = 1$

proof. We can write $K = \bigcup_{i=1}^{N} K_i$,

where K_i is a compact set in Ω_i

(one can suppose that $K \cap \Omega_i \neq \emptyset$ $\forall i$)

let $\psi_j \in C_0^\infty(\Omega_j)$ s.t. $\psi_j(x) = 1$ in a nbh of K_j

define $\varphi_1 = \psi_1$

$\varphi_2 = \psi_2(1-\psi_1)$

$\varphi_3 = \psi_3(1-\psi_2)(1-\psi_1)$

$\vdots$

$\varphi_N = \psi_N(1-\psi_{N-1})(1-\psi_{N-2})\dots(1-\psi_1)$

$\varphi_j \in C_0^\infty(\Omega_j)$ ($\psi_j \in C_0^\infty(\Omega_j)$ and the rest is $C^\infty(\mathbb{R}^n)$)

by induction

$\varphi_1 + \varphi_2 + \dots + \varphi_N = 1 - (1-\psi_1)(1-\psi_2)\dots(1-\psi_N)$

QED

Notations

$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$ multi-index

$|\alpha| = \alpha_1 + \alpha_2 \dots + \alpha_n$ order of the multi-index

$x \in \mathbb{R}^n,\ \alpha \in \mathbb{N}^n \quad x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$

$\alpha, \beta \in \mathbb{N}^n \quad \alpha \leq \beta$ means $\alpha_1 \leq \beta_1, \dots \alpha_n \leq \beta_n$

if $\alpha \leq \beta \quad \binom{\beta}{\alpha} = \frac{\beta_1!}{\alpha_1!(\beta_1-\alpha_1)!} \cdot \frac{\beta_2!}{\alpha_2!(\beta_2-\alpha_2)!} \dots \frac{\beta_n!}{\alpha_n!(\beta_n-\alpha_n)!}$

$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}}$

finally Hörmander's notation for derivatives

$D_x^\alpha = (-i)^{|\alpha|} \cdot \partial_x^\alpha \qquad \left(D_{x_1} = \frac{1}{i} \frac{\partial}{\partial x_1} \right)$

def. let Ω be an open set of $\mathbb{R}^n$

let $T : \underset{C_0^\infty(\Omega)}{\underset{\parallel}{\mathcal{D}(\Omega)}} \longrightarrow \mathbb{C}$ (or $\mathbb{R}$)

space of distributions

T is called distribution ($T \in \mathcal{D}'(\Omega)$)

if 1) T is linear

def. let Ω be an open set of $\mathbb{R}^n$

let $T : \mathcal{D}(\Omega) \longrightarrow \mathbb{C}$ (or $\mathbb{R}$)

$\mathcal{D}(\Omega) = \mathcal{C}_0^\infty(\Omega)$

T is called distribution ($T \in \mathcal{D}'(\Omega)$) ← space of distributions

if

1) T is linear

2) $\forall K$ compact in Ω there exist $C_K > 0$, $m_K \geq 0$ s.t.

$$|T(\varphi)| \leq C_K \cdot \sum_{|\alpha| \leq m_K} \sup_{x \in \Omega} |D^\alpha \varphi(x)|$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$ having support ~~in~~ K

def. if T is a distribution s.t. $\exists\, m$ s.t. $m = m_K$ for all K compact in Ω then the minimum of such m is called the order of the distribution and T is a distribution of finite order

important examples

1) take $f \in L^1_{loc}(\Omega)$

consider $T_f(\varphi) = \int_\Omega f \varphi$

I verify that $T_f \in \mathcal{D}'(\Omega)$ (actually of order 0)

i) T_f is linear in φ ($T_f(a\varphi + b\psi) = \int_\Omega f(a\varphi + b\psi) = a\int_\Omega f\varphi + b\int_\Omega f\psi = a\,T_f(\varphi) + b\,T_f(\psi)$)

ii) $|T_f(\varphi)| = \left|\int_\Omega f\varphi\right| \leq \left|\int_K f\varphi\right| \leq \underbrace{\left(\int_K |f|\right)}_{C_K} \underbrace{\|\varphi\|_{L^\infty}}_{= \sup_{|\alpha|=0,\, x\in\Omega} |D^\alpha\varphi|}$

$\leq C_K \sum_{|\alpha| \leq 0} \sup_{x\in\Omega} |D^\alpha \varphi| \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega)$ with support in K

C_K depends on K; 0 does not depend on K

so $\Phi : L^1_{loc}(\Omega) \longrightarrow \mathcal{D}'(\Omega)$

$f \longmapsto T_f$

Φ is injective.

$T_f = T_{\tilde f}$

$\int_\Omega f\varphi = \int_\Omega \tilde f \varphi \quad \forall \varphi \in \mathcal{D}(\Omega) = \mathcal{C}_0^\infty(\Omega)$

$\int_\Omega (f - \tilde f)\varphi = 0 \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega)$

$\Rightarrow f = \tilde f$ a.e.

so " $L^1_{loc}(\Omega) \subseteq \mathcal{D}'_0(\Omega)$ " ← distributions of order 0

Ex. let $x_0 \in \Omega$

consider $\delta_{x_0} : \mathcal{C}^\infty_0(\Omega) \longrightarrow \mathbb{C}$

$$\varphi \longmapsto \varphi(x_0) = \delta_{x_0}(\varphi)$$

(Dirac's delta)

i) δ_{x_0} is linear

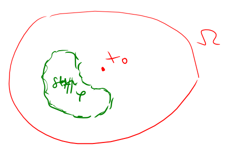
ii) $|\delta_{x_0}(\varphi)| = |\varphi(x_0)| \leq \|\varphi\|_{L^\infty}$ so choosing $C_K = 1$, $m_K = 0$

so $\delta_{x_0} \in \mathcal{D}'_0(\Omega)$ — we have the def. of distribution

Let's see that $\delta_{x_0} \notin L^1_{loc}(\Omega)$

let by contradiction $f \in L^1_{loc}(\Omega)$ s.t. $\delta_{x_0} = T_f$

take $\varphi \in \mathcal{C}^\infty_0(\Omega \setminus \{x_0\})$



$$0 = \varphi(x_0) = \delta_{x_0}(\varphi) = \int_{\Omega \setminus \{x_0\}} f\varphi \implies$$

$$\underbrace{\int_{\Omega \setminus \{x_0\}} f\varphi = 0 \quad \forall \varphi \in \mathcal{C}^\infty_0(\Omega \setminus \{x_0\})}$$

$$\Downarrow$$

$f = 0$ a.e. in $\Omega \setminus \{x_0\}$

$$\Downarrow$$

$T_f \equiv 0$ also in $\mathcal{D}'(\Omega)$

Ex. let $\Omega =]-1, 1[\subseteq \mathbb{R}$

consider $\mathrm{dip}_0(\varphi) = \varphi'(0)$

$\mathrm{dip}_0 : \mathcal{D}(\Omega) \to \mathbb{C}$

- dip_0 is linear
- $|\mathrm{dip}_0(\varphi)| \leq \sup_{x \in \Omega} |\varphi'(x)|$

$\mathrm{dip}_0 \in \mathcal{D}'(\Omega)$ (of order 1)

Ex. let $\Omega =]0, 2[$

consider $T(\varphi) = \sum_{j=0}^{+\infty} \varphi^{(j)}\left(\frac{1}{j+1}\right) = \varphi(1) + \varphi'\left(\tfrac{1}{2}\right) + \varphi''\left(\tfrac{1}{3}\right) + \dots + \varphi^{(j)}\left(\tfrac{1}{j+1}\right) + \dots$

$\forall \varphi \in \mathcal{C}^\infty_0(]0, 2[)$, this is not a series! is a finite sum.

supp $\varphi \subseteq [a, b] \subseteq]0, 2[$
and if $\frac{1}{n} < a$ then $\varphi^{(n)}\left(\frac{1}{n+1}\right) = 0$

Ex. for $\varphi \in \mathcal{D}(\mathbb{R})$ consider

$$PV_{\frac{1}{x}}(\varphi) = \lim_{\varepsilon \to 0} \left(\int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x} dx + \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x} dx \right)$$

$$= \int_{|x| > \varepsilon} \frac{\varphi(y)}{y} dy$$

Prove that $PV_{\frac{1}{x}}$ is a distribution (of order $= 1$)

Remark

suppose to have a compact $K \subseteq \mathbb{R}^d$

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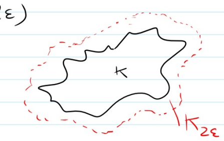

consider $\rho_\varepsilon * \chi_{K_{2\varepsilon}}$

supp $\rho_\varepsilon * \chi_{K_{2\varepsilon}} \subseteq K_{3\varepsilon} = \bigcup_{x\in K} B(x,3\varepsilon)$

$\rho_\varepsilon * \chi_{K_{2\varepsilon}} = 1$ on $K_\varepsilon = \bigcup_{x\in K} B(x,\varepsilon)$

$\cap$

$C_0^\infty(\mathbb{R}^d)$

Corollary 1. Let $f \in L^1_{loc}(\Omega)$

Suppose $\int_\Omega f\varphi = 0 \quad \forall \varphi \in \mathcal{D}(\Omega)$

Then $f = 0$ a.e.

proof. let $g \in C_0^0(\Omega)$

then $\rho_n * g = g_n \in C_0^\infty(\Omega)$ (at least for $n \geq \bar{n}$)

and $g_n \to g$ uniformly (so pointwise)

and $g_n(x) = \int \rho_n(x-y)\,g(y)\,dy$

$|g_n(x)| \leq \max_{x\in\Omega} |g| \cdot \underbrace{\int \rho_n(x-y)\,dy}_{=1}$

$|g_n(x)| \leq C = \max|g|$

then take $g \in C_0^0(\Omega)$

consider $\int_\Omega fg$ (remember, $f \in L^1_{loc}$)

we have $fg_n \longrightarrow fg$ pointwise

$|fg_n| \leq |f| \cdot \chi_K \cdot \max|g|$

↑ all the support of g_n are inside a fixed compact

passing to the limit

$\int_\Omega fg = \lim_n \int_\Omega fg_n$ but $\int_\Omega fg_n = 0$

so $\int_\Omega fg = 0 \quad \forall g \in C_0^0(\Omega) \Rightarrow f = 0$ a.e.

Corollary 2 $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $p \in [1,+\infty[$

4) Partition of unity

Ex. let Ω_1, Ω_2 two open sets in $\mathbb{R}^d$
Let K be a compact set in $\mathbb{R}^d$
suppose $K \subseteq \Omega_1 \cup \Omega_2$

then $\exists K_1, K_2$ compact sets s.t.
$K_1 \subseteq \Omega_1$, $K_2 \subseteq \Omega_2$ and $K_1 \cup K_2 = K$