Introduction to Geodesic Flows

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Proposition (1)

If X is a vector field on the open set $V \subset M$ differentiable manifold, take $p \in V$; then there exist an open set $V_0 \subset V$, $p \in V_0$, a positive real number ε and a mapping $\varphi : (-\varepsilon, \varepsilon) \times V_0 \to V$ such that, for every $q \in V_0$, the curve $(-\varepsilon, \varepsilon) \ni t \mapsto \varphi(t, q)$ is the unique trajectory of X (i.e. $\frac{d}{dt}\varphi(t, q)|_{t_0} = X(\varphi(t_0, q))$ for every $t_0 \in (-\varepsilon, \varepsilon)$) such that $\varphi(0, q) = q$.

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The map $\varphi_t : V_0 \to V$ given by $\varphi_t(q) = \varphi(t, q)$ is called the *flow* of X on V.

Proposition (2)

Given $p \in M$ there exist an open set $V \subset M$ differentiable manifold, with $p \in V$, positive real number ε and r and a differentiable mapping $\gamma : (-\varepsilon, \varepsilon) \times U \to M$, where $U = \{(q, v) \ : \ q \in V \ v \in T_q M \ |v| < r\}$ such that the curve $t \mapsto \gamma(t, q, v)$ is the unique geodesic of M which passes through p with velocity v for each $(q, v) \in U$.

Homogeneneity of a geodesic

Proposition

If the geodesic $\gamma(t, q, v)$ *is defined on* $(-\varepsilon, \varepsilon)$*, then the geodesic* $\gamma(t, q, \delta v)$ $(\delta > 0)$ *is defined in the interval* $(-\varepsilon/\delta, \varepsilon/\delta)$ *and* $\gamma(t, q, \delta v) = \gamma(\delta t, q, v)$*.*

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Proof.

Let $h(t) = \gamma(\delta t,q,v)$; then $h : (-\varepsilon/\delta,\varepsilon/\delta) \to M$ is such that $h(0) = q$ and $\dfrac{d}{dt}h(0) = \delta v$. In addition, $h'(t) = \delta\gamma'(\delta t,q,v)$ and so

$$\frac{D}{dt}\frac{dh}{dt} = \nabla_{h'(t)}h'(t) = \delta^2\nabla_{\gamma'(\delta t,q,v)}\gamma'(\delta t,q,v) = 0.$$

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Hence $h(t) = \gamma(\delta t, q, v)$ is a geodesic such that $h(0) = q$ and $\dfrac{d}{dt}h(0) = \delta v$. Therefore, from the uniqueness of geodesic as (local) solution of Cauchy problem, it follows that $\gamma(\delta t, q, v) = \gamma(t, q, \delta v)$.

☐

Proposition (3)

Given $p \in M$ *there exist a neighborhood* V *of* p *in* M*, differentiable manifold, with* $p \in V$*, a positive real number* R *and a differentiable mapping* $\gamma : (-2,2) \times U \to M$*, where* $U = \{(q,v)\ :\ q \in V\ v \in T_qM\ |v| < R\}$ *such that the curve* $t \mapsto \gamma(t,q,v)$ $t \in (-2,2)$ *is the unique geodesic of* M *which passes through* p *with velocity* v *for each* $(q,v) \in U$*.*

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The geodesic $\gamma(t, q, v)$ was defined for $|t| < \varepsilon$ and $|v| < r$ (thanks to Proposition 2);

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Proof.

The geodesic $\gamma(t, q, v)$ was defined for $|t| < \varepsilon$ and $|v| < r$ (thanks to Proposition 2); from the previous Lemma, $\gamma(t, q, v\varepsilon/2)$ is defined for $|t| < 2$. Take $R < \varepsilon r/2$, then the geodesic $\gamma(t, q, w)$ is defined for $t \in (-2, 2)$ and $|w| < R$. □

Exponential map

Definition

Let $p \in M$ differentiable manifold and let $U \subset TM$ as in Proposition (3). Then it is well defined the map

$$(q, v) \in U \underset{\exp}{\longmapsto} \gamma(1, q, v)$$

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Notice that if $(q, v) \in U$

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Furthermore, $\exp(q, v)$ will be often denoted by $\exp_q(v)$.

Proposition

Given $q \in M$ *there exists* $\varrho > 0$ *such that* $\exp_q : B_\varrho(0) = \{v \in T_qM \; |v| < \varrho\} \to M$ *is a diffeomorphism.*

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Proof.

Indeed

$$d(\exp_q)|_0 = \frac{d}{dt}(\exp_q(tv))|_0 = \frac{d}{dt}\gamma(t, q, v)|_0 = v$$

of $d(\exp_q)|_0 = Id$ in $T_q M$; then it follows from the Inverse Function Theorem that $\exp_q$ is a local diffeomorphism in a neighborhood of 0.

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Hence

$$\langle (d\exp_p)_v v, (d\exp_p)_v v\rangle = \langle \gamma'(1), \gamma'(1)\rangle = \langle \gamma'(0), \gamma'(0)\rangle = \langle v, v\rangle$$

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This means that the function $\exp_p$ *is a* radial *isometry.*

Parametrized surface in a manifold

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Let $p \in M$ and $v \in T_pM$ be such that $\exp_p v$ is defined. Then there exists $\varepsilon > 0$ such that $\exp_p u$ is defined for $u = u(t,s) := tv(s)$, where $t \in [0,1]$, $s \in (-\varepsilon, \varepsilon)$ and $v(s)$ is a curve in T_pM with $v(0) = v$ and $|v(s)| = |v|$ constant for $s \in (-\varepsilon, \varepsilon)$.

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$$\exp_p(tv(s)) \quad \text{as} \quad (t,s) \in A$$

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This is an example of *parametrized surface* in M according to the following

Definition

Let A be a connected set in $\mathbb{R}^2$ such that the boundary of A is a piecewise differentiable curve (with vertex angles different from π). A parametrized surface in M is a differentiable mapping[a] $\sigma : U \to M$ restricted to $A \subset U$.

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[a] Very often an immersion

Parametrized surface in a manifold

Let (t, s) be the cartesian coordinates in $\mathbb{R}^2$; then for t_0 fixed, the mapping

$$s \mapsto \sigma(s, t_0)$$

where s belongs to a connected component of $A \cap \{t = t_0\}$ is a curve in M and $d\sigma\left(\frac{d}{dt}\right) = \frac{\partial \sigma}{\partial t}$ is a vector field along this curve.

Similarly, one can define $\frac{\partial \sigma}{\partial s}$.

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Notice

$$t \mapsto \exp_p(t, v_0)$$

(which corresponds to $\sigma(t, v_0)$ with $\exp_p = \sigma$) is a geodesic in M for any $v_0 = v(s)$ $s \in (-\varepsilon, \varepsilon)$.

Parametrized surface in a manifold

Given a vector field V on a parametrized surface $\sigma(A)$ in M equipped with an affine connection ∇, we can define the covariant derivatives $\dfrac{DV}{dt}$ (and $\dfrac{DV}{ds}$) as the covariant derivative of V along the curve $t \mapsto \sigma(t, s_0)$ of the restriction of V on this curve (along the curve $s \mapsto \sigma(t_0, s)$ of the restriction of V on this curve).

Lemma of symmetry

Lemma

If M is a differentiable manifold with a symmetric connection ∇ and $\sigma(A)$ is a parametrized surface in M, then

$$\frac{D}{dt}\frac{\partial\sigma}{\partial s} = \frac{D}{ds}\frac{\partial\sigma}{\partial t}$$

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Proof.

Let $\mathbf{x} : U \subset \mathbb{R}^n$ be a local chart of M in a neighborhood of a point of $\sigma(A)$.

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Proof.

Let $\mathbf{x} : U \subset \mathbb{R}^n$ be a local chart of M in a neighborhood of a point of $\sigma(A)$. Then we can write

$$\mathbf{x}^{-1} \circ \sigma(t,s) = (x_1(t,s), \ldots, x_n(t,s))$$

and hence

$$\frac{D}{ds}\frac{\partial\sigma}{\partial t} = \frac{D}{ds}\left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial t}\frac{\partial}{\partial x_j}\right)$$

Proof.

$$\frac{D}{ds}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial t}\frac{\partial}{\partial x_j}\right)=\nabla_{\sum_{i=1}^{n}\frac{\partial x_i}{\partial s}\frac{\partial}{\partial x_i}}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial t}\frac{\partial}{\partial x_j}\right)=$$

$$=\sum_{j=1}^{n}\left(\nabla_{\sum_{i=1}^{n}\frac{\partial x_i}{\partial s}\frac{\partial}{\partial x_i}}\frac{\partial x_j}{\partial t}\frac{\partial}{\partial x_j}\right)$$

$$=\sum_{i=1}^{n}\frac{\partial^2 x_i}{\partial s\partial t}\frac{\partial}{\partial x_i}+\sum_{i,j=1}^{n}\frac{\partial x_i}{\partial s}\frac{\partial x_j}{\partial t}\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}$$

Since ∇ is symmetric, it follows that $\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}=\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i}$, and so

$$\frac{D}{ds}\frac{\partial\sigma}{\partial t}=\frac{D}{dt}\frac{\partial\sigma}{\partial s}.$$

□

Gauss' Lemma

Lemma

Let $p \in M$ *and let* $v \in T_pM$ *be such that* $\exp_p v$ *is defined. Let* $w \in T_v(T_pM) \simeq T_pM$*, then*

$$\langle (d\exp_p)_v v, (d\exp_p)_v w \rangle = \langle v, w \rangle$$

Proposition

Let $p \in M$, let U be a normal neighborhood of p and let $B \subset U$ be a normal ball of center p. Let $\gamma : [0,1] \to B$ be a geodesic with $\gamma(0) = p$. If $\alpha : [0,1] \to M$ is any piecewise differentiable curve joining $\gamma(0) = p$ and $\gamma(1)$, then $\ell(\gamma) \leq \ell(\alpha)$ where ℓ represents the length of the curve. Furthermore, if equality $\ell(\gamma) = \ell(\alpha)$ holds, then $\gamma([0,1]) = \alpha([0,1])$.

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Corollary

If a piecewise differentiable curve joining p *to* q *has length less or equal to the length of any other piecewise differentiable curve joining* p *to* q *and is parametrized with a parameter proportional to arc length, then the curve is a geodesic.*

$$\ell(\alpha) = \int_0^1 \left|\frac{d\alpha}{dt}\right| dt \quad E(\alpha) = \frac{1}{2}\int_0^1 \left|\frac{d\alpha}{dt}\right|^2 dt$$

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From Schwarz inequality

$$\left(\int_0^a (fg)dt\right)^2 \leq \int_0^a f^2 dt \cdot \int_0^a g^2 dt$$

(where equaliy holds if and only if g is constant), it follows that

$$\ell(\alpha)^2 \leq 2E(\alpha)$$

and equality occurs if and only if $\left|\frac{d\alpha}{dt}\right|$ is constant or t is proportional to the arc length parameter.

Lemma

Let $p, q \in M$ and let $\gamma : [0, 1] \to M$ be a minimizing geodesic joining p to q. Then for any curve $\alpha : [0, 1] \to M$ joining p to q it turns out that

$$E(\gamma) \leq E(\alpha)$$

with equality if and only if α is a minimizing geodesic.

Lemma

Let $\sigma : A \to M$ *be a parametrized surface and let* V *be a vector field along* $\sigma(A)$*. If* (t,s) *are the usual coordinates in* $A \subseteq \mathbb{R}^2$*, then, in local coordinates,* $V = \sum_{j=1}^{n} v_j(t,s)X_j$ *with* $X_j = \frac{\partial}{\partial x_j}$ *and we have*

$$\frac{D}{\partial t}\frac{D}{\partial s}V - \frac{D}{\partial s}\frac{D}{\partial t}V = \sum_{j=1}^{n} v_j \left(\frac{D}{\partial t}\frac{D}{\partial s} - \frac{D}{\partial s}\frac{D}{\partial t} \right) X_j$$

Proof.

$$\frac{D}{\partial s}V = \frac{D}{\partial s}\left(\sum_{j=1}^{n} v_j(t,s)X_j \right) = \sum_{j=1}^{n} v_j \frac{D}{\partial s}X_j + \sum_{j=1}^{n} \frac{\partial v_j}{\partial s}X_j$$

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Proof.

$$\frac{D}{\partial s}V = \sum_{j=1}^{n} v_j \left(\frac{D}{\partial s}X_j + \frac{\partial v_j}{\partial s}X_j \right)$$

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$$= \sum_{j=1}^{n} \left(v_j \frac{D}{\partial t}\frac{D}{\partial s}X_j + \frac{\partial v_j}{\partial t}\frac{D}{\partial s}X_j + \frac{\partial v_j}{\partial s}\frac{D}{\partial t}X_j + \frac{\partial^2 v_j}{\partial t \partial s}X_j \right) =$$

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Hence, interchanging the role of t and s,

$$\frac{D}{\partial t}\frac{D}{\partial s}V - \frac{D}{\partial s}\frac{D}{\partial t}V = \sum_{j=1}^{n} v_j \left(\frac{D}{\partial t}\frac{D}{\partial s} - \frac{D}{\partial s}\frac{D}{\partial t} \right) X_j$$

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Some additional comments

Since

$$\frac{D}{\partial s}X_j = \nabla_{\sum_{i=1}^{n}\frac{\partial x_i}{\partial s}X_i}X_j = \sum_{i=1}^{n}\frac{\partial x_i}{\partial s}\nabla_{X_i}X_j$$

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$$= \sum_{i=1}^{n}\frac{\partial x_i}{\partial s}\nabla_{\sum\limits_{k=1}^{n}\frac{\partial x_k}{\partial t}X_i}(\nabla_{X_i}X_j) + \sum_{i=1}^{n}\frac{\partial^2 x_i}{\partial t\partial s}\nabla_{X_i}X_j$$

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$$= \sum_{i,k=1}^n \frac{\partial x_i}{\partial s} \frac{\partial x_k}{\partial t} \nabla_{X_k} \nabla_{X_i} X_j + \sum_{i=1}^n \frac{\partial^2 x_i}{\partial t \partial s} \nabla_{X_i} X_j$$

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$$\left(\frac{D}{\partial t}\frac{D}{\partial s} - \frac{D}{\partial s}\frac{D}{\partial t}\right)X_j = \sum_{i,k=1}^{n}\frac{\partial x_i}{\partial s}\frac{\partial x_k}{\partial t}\left(\nabla_{X_k}\nabla_{X_i} - \nabla_{X_i}\nabla_{X_k}\right)X_j$$

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$$(Z_1 + Z_2) \mapsto (\nabla_Y \nabla_X - \nabla_X \nabla_Y)(Z_1 + Z_2) =$$

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$$= \nabla_Y \nabla_X Z_1 + \nabla_Y \nabla_X Z_2 - \nabla_X \nabla_Y Z_1 - \nabla_X \nabla_Y Z_2 =$$

Consider, for $X, Y, Z \in \mathcal{X}(M)$, the map

$$Z \mapsto (\nabla_Y \nabla_X - \nabla_X \nabla_Y)Z$$

If $Z_1, Z_2 \in \mathcal{X}(M)$

$$(Z_1 + Z_2) \mapsto (\nabla_Y \nabla_X - \nabla_X \nabla_Y)(Z_1 + Z_2) =$$

$$= \nabla_Y \nabla_X (Z_1 + Z_2) - \nabla_X \nabla_Y (Z_1 + Z_2) =$$

$$= \nabla_Y \nabla_X Z_1 + \nabla_Y \nabla_X Z_2 - \nabla_X \nabla_Y Z_1 - \nabla_X \nabla_Y Z_2 =$$

$$= (\nabla_Y \nabla_X - \nabla_X \nabla_Y) Z_1 + (\nabla_Y \nabla_X - \nabla_X \nabla_Y) Z_2$$

If $f \in \mathcal{D}(M)$

$$fZ \mapsto (\nabla_Y \nabla_X - \nabla_X \nabla_Y)(fZ) =$$

If $f \in \mathcal{D}(M)$

$$fZ \mapsto (\nabla_Y \nabla_X - \nabla_X \nabla_Y)(fZ) =$$

$$= \nabla_Y \nabla_X (fZ) - \nabla_X \nabla_Y (fZ) =$$

If $f \in \mathcal{D}(M)$

$$fZ \mapsto (\nabla_Y \nabla_X - \nabla_X \nabla_Y)(fZ) =$$

$$= \nabla_Y \nabla_X (fZ) - \nabla_X \nabla_Y (fZ) =$$

$$= \nabla_Y (f \nabla_X Z + X(f) Z) - \nabla_X (f \nabla_Y Z + Y(f) Z) =$$
$$= f \nabla_Y \nabla_X Z + Y(f) \nabla_X Z + X(f) \nabla_Y Z + Y(X(f)) Z +$$
$$- f \nabla_X \nabla_Y Z - X(f) \nabla_Y Z - Y(f) \nabla_X Z - X(Y(f)) Z =$$

If $f \in \mathcal{D}(M)$

$$fZ \mapsto (\nabla_Y \nabla_X - \nabla_X \nabla_Y)(fZ) =$$

$$= \nabla_Y \nabla_X (fZ) - \nabla_X \nabla_Y (fZ) =$$

$$= \nabla_Y (f \nabla_X Z + X(f) Z) - \nabla_X (f \nabla_Y Z + Y(f) Z) =$$
$$= f \nabla_Y \nabla_X Z + Y(f) \nabla_X Z + X(f) \nabla_Y Z + Y(X(f)) Z +$$
$$- f \nabla_X \nabla_Y Z - X(f) \nabla_Y Z - Y(f) \nabla_X Z - X(Y(f)) Z =$$

$$= f \nabla_Y \nabla_X Z - f \nabla_X \nabla_Y Z + Y(X(f)) Z - X(Y(f)) Z$$

If $f \in \mathcal{D}(M)$

$$fZ \mapsto (\nabla_Y \nabla_X - \nabla_X \nabla_Y)(fZ) =$$

$$= \nabla_Y \nabla_X (fZ) - \nabla_X \nabla_Y (fZ) =$$

$$= \nabla_Y (f \nabla_X Z + X(f)Z) - \nabla_X (f \nabla_Y Z + Y(f)Z) =$$
$$= f \nabla_Y \nabla_X Z + Y(f) \nabla_X Z + X(f) \nabla_Y Z + Y(X(f))Z +$$
$$- f \nabla_X \nabla_Y Z - X(f) \nabla_Y Z - Y(f) \nabla_X Z - X(Y(f))Z =$$

$$= f \nabla_Y \nabla_X Z - f \nabla_X \nabla_Y Z + Y(X(f))Z - X(Y(f))Z$$

$$= f(\nabla_Y \nabla_X - \nabla_X \nabla_Y)Z + ([Y, X]f)Z)$$

Consider, for $X, Y, Z \in \mathcal{X}(M)$, the map

$$Z \xmapsto[R(X,Y)]{} R(X,Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

Consider, for $X, Y, Z \in \mathcal{X}(M)$, the map

$$Z \underset{R(X,Y)}{\longmapsto} R(X,Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

$R(X,Y)$ is bilinear since

$$R(X,Y)(fZ) = f(\nabla_Y \nabla_X - \nabla_X \nabla_Y)Z + ([Y,X]f)Z) + f\nabla_{[X,Y]}Z + ([X,Y]f)Z$$

$$= fR(X,Y)Z$$

Furthermore the following properties hold

1. *Linearity in the first entry*
   $R(fX_1 + gX_2, Y) = fR(X_1, Y) + gR(X_2, Y)$ for any $X_1, X_2, Y \in \mathcal{X}(M)$ and $f, g \in \mathcal{D}(M)$
2. *Linearity in the second entry*
   $R(X, fY_1 + gY_2) = fR(X, Y_1) + gR(X, Y_2)$ for any $X, Y_1, Y_2 \in \mathcal{X}(M)$ and $f, g \in \mathcal{D}(M)$
3. *First Bianchi identity*
   $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$

Lemma

Let $\sigma : A \to M$ *be a parametrized surface and let* V *be a vector field along* $\sigma(A)$*. If* (t, s) *are the usual coordinates in* $A \subseteq \mathbb{R}^2$*, then*

$$\left(\frac{D}{\partial t}\frac{D}{\partial s} - \frac{D}{\partial s}\frac{D}{\partial t}\right) V = R\left(\frac{\partial\sigma}{\partial s}, \frac{\partial\sigma}{\partial t}\right) V$$