8. (*Schur's Theorem*). Let M^n be a connected Riemannian manifold with $n \geq 3$. Suppose that M is *isotropic*, that is, for each $p \in M$, the sectional curvature $K(p,\sigma)$ does not depend on $\sigma \subset T_pM$. Prove that M has constant sectional curvature, that is, $K(p,\sigma)$ also does not depend on p.
Hint: Define a tensor R' of order 4 by

$$R'(W,Z,X,Y) = \langle W,X\rangle\langle Z,Y\rangle - \langle Z,X\rangle\langle W,Y\rangle.$$

If $K(p,\sigma) = K$ does not depend on σ, by Lemma 3.4, $R = KR'$. Therefore, for all $U \in \mathcal{X}(M)$, $\nabla_U R = (UK)R'$. Using the 2nd Bianchi identity (see Exercise 7):

$$\nabla R(W,Z,X,Y,U) + \nabla R(W,Z,Y,U,X) + \nabla R(W,Z,U,X,Y) = 0,$$

we obtain, for all $X,Y,W,Z,U \in \mathcal{X}(M)$,

$$\begin{aligned} 0 = (UK)&(\langle W,X\rangle\langle Z,Y\rangle - \langle Z,X\rangle\langle W,Y\rangle) \\ &+ (XK)(\langle W,Y\rangle\langle Z,U\rangle - \langle Z,Y\rangle\langle W,U\rangle) \\ &+ (YK)(\langle W,U\rangle\langle Z,X\rangle - \langle Z,U\rangle\langle W,X\rangle). \end{aligned}$$

Fix $p \in M$. Because $n \geq 3$, it is possible, fixing X at p, to choose Y and Z at p such that $\langle X,Y\rangle = \langle Y,Z\rangle = \langle Z,X\rangle = 0$, $\langle Z,Z\rangle = 1$. Put $U = Z$ at p. The relation above yields, for all W,

$$\langle (XK)Y - (YK)X, W\rangle = 0.$$

Since X and Y are linearly independent at p, we conclude that $XK = 0$ for all $X \in T_pM$. Thus $K = \text{const}$.