Lesson 14 - 27/3/2026

Please correct the following

on $\mathcal{C}_0^\infty(\Omega) = \mathcal{D}(\Omega)$ we the maximal topology

s.t. $\forall j$: $\mathcal{C}_0^\infty(\bar{\Omega}_j) \hookrightarrow \mathcal{C}_0^\infty(\Omega)$, $\varphi \longmapsto \varphi$

are continuous

Remark

def. Consider Ω open set in $\mathbb{R}^n$

if $\mu : \mathcal{C}_0(\Omega) \longrightarrow \mathbb{C}$

- μ linear
- $\forall K$ compact in Ω, $\exists C>0$ s.t. $|\mu(f)| \le C \sup_\Omega |f|$ for all $f \in \mathcal{C}_0(\Omega)$ with supp in K

μ is called Radon measure

It is possible to prove

i) if μ is a Radon measure then $\mu|_{\mathcal{D}(\Omega)}$ is of order 0 (trivial)

ii) given $T \in \mathcal{D}'(\Omega)$, T of order 0 then $\exists!\, \mu$ Radon measure s.t. $\mu|_{\mathcal{D}} = T$

def. let ν be a signed or complex measure on $\mathcal{B}(\Omega)$ (Borelian sets in Ω)

ν is called regular Borel measure if

i) $|\nu|$ is finite on compact sets of Ω

ii) $\forall B \in \mathcal{B}(\Omega)$

$\sup\{|\nu|(K),\ K \text{ compact}, K \subseteq B\} = |\nu|(B) = \inf\{|\nu|(A),\ A \text{ open and } B \subseteq A\}$

Theorem (Riesz) let Ω be an open set in $\mathbb{R}^n$

μ is a Radon measure

if and only if

there exists ν, regular Borel measure, s.t.

$\forall f \in \mathcal{C}_0(\Omega)$, $\mu(f) = \int_\Omega f\, d\nu$

Rudin

Ex. Let T be a linear functional defined on $\mathcal{D}(\Omega)$

Suppose that $\forall \varphi \in \mathcal{D}(\Omega)$

if $\varphi(x) \geq 0\ \forall x \in \Omega$ then $T(\varphi) \geq 0$

Prove that T is a distribution of order 0.

solution. I have to show that

$\forall K$ compact in Ω, $\exists C > 0$

s.t. $|T(\psi)| \le C \cdot \sup_\Omega |\psi|$ for all $\psi \in \mathcal{D}(\Omega)$ with support in K

consider K compact set in Ω.

consider $\chi \in \mathcal{C}_0^\infty(\Omega)$ s.t. $\begin{cases} \chi(x) \geq 0 & \forall x \in \Omega \\ \chi(x) = 1 & \text{in nbh of } K \end{cases}$

now take $\psi \in \mathcal{D}(\Omega)$ s.t. supp $\psi \subseteq K$

consider $\varphi(x) = \chi(x) \cdot \sup_K |\psi(x)| - \psi(x)$

$\varphi \in \mathcal{D}(\Omega)$ and $\varphi(x) \geq 0$ $\forall x \in \Omega$

$T(\varphi) \geq 0$

$T(\chi \cdot \sup |\psi|) \geq T(\psi)$

similarly considering $\varphi(x) = \chi(x) \sup_K |\psi(x)| + \psi(x)$

$T(\psi) \geq -T(\chi \sup|\psi|)$

$T(\psi) \geq 0$

$T(\chi \cdot \sup|\psi|) \geq T(\psi)$

similarly considering $\varphi(x) = \chi(x) \sup_K |\psi(x)| + \psi(x)$

$T(\psi) \geq -T(\chi \sup|\psi|)$

then $-T(\chi) \sup_\Omega|\psi| \leq T(\psi) \leq T(\chi) \cdot \sup_\Omega |\psi|$

finally $|T(\psi)| \leq \underbrace{(T(\chi))}_{=C} \cdot \sup_\Omega |\psi|$ $\forall \psi$ having support in K.

Derivatives in the sense of distributions.

Rem. suppose $f \in C^1(\Omega)$ ($\Rightarrow f \in L^1_{loc}(\Omega)$)

consider T_f

define "$\partial_{x_j} T_f$" in such a way that

$$\partial_{x_j} T_f = T_{\partial_{x_j} f}$$

so that $(\partial_{x_j} T_f)(\varphi) = T_{\partial_{x_j} f}(\varphi) = \int_\Omega \partial_{x_j} f \cdot \varphi$

$\partial_{x_j} f \in C^0(\Omega) \subseteq L^1_{loc}$

integrate by parts $\int_\Omega \partial_{x_j} f \varphi = \underbrace{\int_\Omega \partial_{x_j}(f\varphi)}_{=0} - \int_\Omega f \partial_{x_j} \varphi$

since $f\varphi$ has compact support in Ω

$$(\partial_{x_j} T_f)(\varphi) = -\int_\Omega f \partial_{x_j} \varphi = -T_f(\partial_{x_j}\varphi)$$

$$(\partial_{x_j} T_f)(\varphi) = -T_f(\partial_{x_j}\varphi)$$

Hint define $\boxed{(\partial_{x_j} T)(\varphi) = -T(\partial_{x_j}\varphi)}$

def. let $T \in \mathcal{D}'(\Omega)$

define $\partial_{x_j} T$ as $\boxed{\partial_{x_j} T(\varphi) = -T(\partial_{x_j}\varphi)}$

new operator on T

this is the usual derivative of a C^∞_c function

Th. $\partial_{x_j} T$ is a distribution (of order $\leq m+1$ if T was of order m)

proof. the operator $\partial_{x_j} T$ is linear

$$\partial_{x_j} T(\alpha\varphi + \beta\psi) = -T(\partial_{x_j}(\alpha\varphi+\beta\psi))$$
$$\stackrel{!}{=} -T(\alpha\partial_{x_j}\varphi + \beta\partial_{x_j}\psi)$$
$$\stackrel{!}{=} -T(\alpha\partial_{x_j}\varphi) - T(\beta\partial_{x_j}\psi)$$
$$\stackrel{!}{=} -\alpha T(\partial_{x_j}\varphi) - \beta T(\partial_{x_j}\psi)$$
$$\stackrel{!}{=} +\alpha(\partial_{x_j}T)(\varphi) + \beta(\partial_{x_j}T)(\psi)$$

take K compact $\subset$

we know that $\exists C_K, \exists m_K$ $|T(\varphi)| \leq C_K \sum_{|\alpha|\leq m_K} \sup_\Omega |D^\alpha \varphi|$ $\forall \varphi$ with supp. in K

consequently $|\partial_{x_j} T(\varphi)| = |T(\partial_{x_j}\varphi)| \leq C_K \sum_{|\alpha|\leq m_K} \sup_\Omega |D^\alpha \partial_{x_j}\varphi|$

take K compact $]$

we know that $\exists C_K, \exists m_K$ $|T(\varphi)| \le C_K \sum_{|\alpha|\le m_K} \sup_\Omega |D^\alpha \varphi|$ $\forall \varphi$ with supp. in K

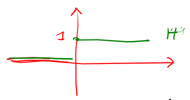
consequently $|\partial_j T(\varphi)| = |T(\partial_j \varphi)| \le C_K \sum_{|\alpha|\le m_K} \sup_\Omega |D^\alpha \partial_j \varphi|$

$$|\partial_j T(\varphi)| \le C_K \sum_{|\beta|\le m_K+1} \sup_\Omega |D^\beta \varphi| \quad \forall \varphi \text{ with supp in } K$$

QED

Ex. let $H(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$

H is the Heaviside function.



H is in $L^1_{loc}(\mathbb{R})$

I consider T_H $\quad T_H(\varphi) = \int_{-\infty}^{+\infty} H(x)\varphi(x)\,dx = \int_0^{+\infty} \varphi(x)\,dx$

derivative in the sense of $\mathcal{D}'$

compute $T_H'(\varphi)$

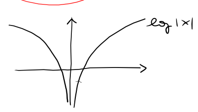
$$T_H'(\varphi) = -T_H(\varphi') = -\int_0^{+\infty} \varphi'(t)\,dt = -\varphi(t)\Big|_0^{+\infty} = \varphi(0) = \delta_0(\varphi)$$

$$\boxed{T_H' = \delta_0}$$

take $2H(x) = \begin{cases} 2 & x \ge 0 \\ 0 & x < 0 \end{cases}$ $\quad 2H' = 2\delta_0$

Ex. consider $x \mapsto \log|x|$

this function is $L^1_{loc}(\mathbb{R})$



compute $T'_{\log|x|}$

Hint: $T'_{\log|x|} = PV\frac{1}{x}$

(remark that if one proves that $T'_{\log|x|}(\varphi) = \lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} \frac{\varphi(y)}{y}\,dy$ then the limit is a distribution)

Th ("Théorème de structure")

Let $T \in \mathcal{D}'(\Omega)$

consider $\underbrace{\omega \subseteq \Omega, \ \omega \text{ open and } \bar\omega \text{ compact inside } \Omega}_{\omega \subset\subset \Omega \ (\text{relatively compact in } \Omega)}$

then there exists $f \in L^\infty(\omega)$ and there exists $m \in \mathbb{N}$ s.t.

$$\forall \varphi \in \mathcal{C}_0^\infty(\omega), \quad T(\varphi) = \int_\omega f\, \partial_{x_1}^m \partial_{x_2}^m \cdots \partial_{x_n}^m \varphi \, dx = (-1)^{nm} \partial_{x_1}^m \cdots \partial_{x_n}^m T_f(\varphi) \qquad (*)$$

proof. I claim that the condition $(*)$ is equivalent to

$\exists C > 0, \exists m \in \mathbb{N}$ s.t.

$$|T(\varphi)| \le C \, \|\partial_{x_1}^m \cdots \partial_{x_n}^m \varphi\|_{L^1(\omega)} \quad \text{for all } \varphi \in \mathcal{D}(\omega) \qquad (**)$$

$(*) \Rightarrow (**)$ in fact

$$|T(\varphi)| = \left|\int_\omega f\, \partial_{x_1}^m \cdots \partial_{x_n}^m \varphi\right| \le \|f\|_{L^\infty(\omega)} \|\partial_{x_1}^m \cdots \partial_{x_n}^m \varphi\|_{L^1(\omega)}$$

calling $C = \|f\|_{L^\infty(\omega)}$ you obtain $(**)$

2. I claim that the condition (*) is equivalent to

$\exists C>0,\ \exists m \in \mathbb{N}$ s.t.

$|T(\varphi)| \le C \, \| \partial_{x_1}^m \cdots \partial_{x_n}^m \varphi \|_{L^1(\omega)}$ for all $\varphi \in \mathcal{D}(\omega)$ (**)

(*) $\Rightarrow$ (**) in fact

$$|T(\varphi)| = \left| \int_\omega f \, \partial_{x_1}^m \cdots \partial_{x_n}^m \varphi \right| \le \|f\|_{L^\infty(\omega)} \, \|\partial_{x_1}^m \cdots \partial_{x_n}^m \varphi\|_{L^1(\omega)}$$

calling $C = \|f\|_{L^\infty(\omega)}$ you obtain (**)

conversely

suppose (**) $\quad |T(\varphi)| \le C \, \|\partial_{x_1}^m \cdots \partial_{x_n}^m \varphi\|_{L^1(\omega)}$

consider $V \subseteq L^1(\omega)$

$$V = \{ \partial_{x_1}^m \cdots \partial_{x_n}^m \varphi \ : \ \varphi \in \mathcal{C}_0^\infty(\omega) \}$$

V is subspace of $L^1(\omega)$

consider $\Phi : V \longrightarrow \mathbb{C}$

$$\partial_{x_1}^m \cdots \partial_{x_n}^m \varphi \longmapsto T(\varphi)$$

condition (**) says that Φ is (linear and) bounded on V w.r.t. L^1_ω-norm.

Use Hahn-Banach

$\exists \ \tilde{\Phi} : L^1(\omega) \to \mathbb{C}$ s.t. $\tilde{\Phi}_{|V} = \Phi$

Use Riesz $\Rightarrow \exists f \in L^\infty(\omega)$ s.t.

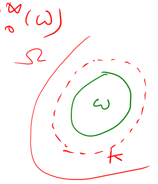
$$\tilde{\Phi}(g) = \int_\omega f g$$

then $\Phi(\partial_{x_1}^m \cdots \partial_{x_n}^m \varphi) = \int_\omega f \, \partial_{x_1}^m \cdots \partial_{x_n}^m \varphi = T(\varphi)$

So now I prove that if $\boxed{T \in \mathcal{D}'(\Omega)}$ then ** is valid

i.e. $\exists C>0, \exists m$ s.t.

$|T(\varphi)| \le C \, \|\partial_{x_1}^m \cdots \partial_{x_n}^m \varphi\|_{L^1(\omega)} \quad \forall \varphi \in \mathcal{C}_0^\infty(\omega)$



$\overline{\omega}$ is compact in Ω

I choose K compact s.t. $\mathring{K} \supseteq \overline{\omega}$

I apply to K the condition of $T \in \mathcal{D}'(\Omega)$

$\exists C_K > 0, \exists m_K \in \mathbb{N}$ s.t. $\quad |T(\psi)| \le C_K \sum_{|\alpha| \le m_K} \sup_\Omega |D^\alpha \psi|$

$\forall \psi$ with supp in K

take $\varphi \in \mathcal{C}_0^\infty(\omega)$ ($\Rightarrow$ supp $\varphi \subseteq K$)

$\sup_\Omega |\varphi| = \sup_K |\varphi| \le$

$\varphi \in \mathcal{C}_0^\infty(\omega)$, $\quad \underset{(x_1, \hat{x})}{\varphi(x)} = \int_{-\infty}^{x_1} \partial_{x_1} \varphi(t, \hat{x}) \, dt \qquad \sup_\omega |\varphi| \le \underbrace{\int_{-\infty}^{+\infty} |\partial_{x_1} \varphi(t, \hat{x})| \, dt}_{\le C \ \sup |\partial_x \varphi|}$

$\varphi \in C_0^\infty(\omega)$ / $\underset{\substack{\| \\ (x_1, \hat{x}_1)}}{\varphi(x)} = \int_{-\infty}^{x_1} \partial_{x_1} \varphi(t, \hat{x}_1)\, dt$ $\quad \sup_\omega |\varphi| \le \underbrace{\int_{-\infty}^{+\infty} |\partial_{x_1} \varphi(t, \hat{x}_1)|\, dt}_{\le C \sup |\partial_x \varphi|}$

depends on ω

at the end $\quad \sup_{|\alpha| \le m} |D^\alpha \varphi| \le \tilde{C} \underbrace{\sup |\partial_{x_1}^{\tilde{m}} \cdots \partial_{x_n}^{\tilde{m}} \varphi|}_{\le \|\partial_{x_1}^{\tilde{m}+1} \cdots \partial_{x_n}^{\tilde{m}+1} \varphi\|_{L^1(\omega)}}$

dep on ω

& obtain

$$|T(\varphi)| \le \tilde{\tilde{C}} \, \|\partial_{x_1}^{\tilde{m}+1} \cdots \partial_{x_n}^{\tilde{m}+1} \varphi\|_{L^1(\omega)}$$

dep. on ω

QED