LECTURE 6: THE RIEMANN CURVATURE TENSOR

1. The curvature tensor

Let M be any smooth manifold with linear connection ∇, then we know that

$$R(X,Y)Z := -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z.$$

defines a $(1,3)$-tensor field on M, called the curvature tensor of ∇. Locally if we write

$$R = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_j,$$

then the coefficients can be expressed via the Christoffel symbols of ∇ as

$$R_{ijk}{}^l = -\Gamma_{jk}^s\Gamma_{is}^l + \Gamma_{ik}^s\Gamma_{js}^l - \partial_i\Gamma_{jk}^l + \partial_j\Gamma_{ik}^l,$$

Obviously the curvature tensor for the standard connection on $\mathbb{R}^n$ is identically zero, since its Christoffel's symbols are all zero.

Example. Consider $M = S^n$. Last time we have seen that

$$\nabla_X Y = \overline{\nabla}_X Y - \langle \overline{\nabla}_X Y, \vec{n}\rangle \vec{n}.$$

defines a (Levi-Civita) connection on S^n, where $\overline{\nabla}$ is the standard connection on $\mathbb{R}^{n+1}$:

$$\overline{\nabla}_{X^i\partial_i}(Y^j\partial_j) = X^i\partial_i(Y^j)\partial_j.$$

To calculate its curvature tensor, we need rewrite it into a simpler form. Since $\vec{n} = (x^1, x^2, \cdots, x^{n+1})$, one get

$$\overline{\nabla}_X \vec{n} = X^i\partial_i(x^j)\partial_j = X.$$

It follows

$$\langle \overline{\nabla}_X Y, \vec{n}\rangle\vec{n} = -\langle Y, \overline{\nabla}_X \vec{n}\rangle \vec{n} = -\langle X, Y\rangle\vec{n}.$$

So

$$\nabla_X Y = \overline{\nabla}_X Y + \langle X, Y\rangle \vec{n}.$$

Thus

$$\begin{aligned}
\nabla_Y\nabla_X Z &= \overline{\nabla}_Y \nabla_X Z + \langle Y, \nabla_X Z\rangle\vec{n} \\
&= \overline{\nabla}_Y(\overline{\nabla}_X Z + \langle X, Z\rangle\vec{n}) + X\langle Y, Z\rangle\vec{n} - \langle \nabla_X Y, Z\rangle \vec{n} \\
&= \overline{\nabla}_Y\overline{\nabla}_X Z + Y(\langle X, Z\rangle)\vec{n} + \langle X, Z\rangle Y + X(\langle Y, Z\rangle)\vec{n} - \langle \nabla_X Y, Z\rangle\vec{n}.
\end{aligned}$$

In view of the fact $\overline{R} = 0$, we get

$$\begin{aligned}
R(X,Y)Z =& - X(\langle Y, Z\rangle)\vec{n} - \langle Y, Z\rangle X - Y(\langle X, Z\rangle)\vec{n} + \langle \nabla_Y X, Z\rangle\vec{n} \\
&+ Y(\langle X, Z\rangle)\vec{n} + \langle X, Z\rangle Y + X(\langle Y, Z\rangle)\vec{n} - \langle \nabla_X Y, Z\rangle \vec{n} + \langle [X,Y], Z\rangle\vec{n} \\
=& \langle X, Z\rangle Y - \langle Y, Z\rangle X.
\end{aligned}$$

By definition one immediately gets the following anti-symmetry:

$$R(X,Y)Z = -R(Y,X)Z \tag{1}$$

For the curvature tensor R, one has

Proposition 1.1 (The First Bianchi identity)**.** *If ∇ is a torsion-free, then*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0. \tag{2}$$

Proof. Recall that ∇ is torsion-free means

$$\nabla_X Y - \nabla_Y X - [X,Y] = 0.$$

So we have

$$\begin{aligned}
&R(X,Y)Z + R(Y,Z)X + R(Z,X)Y \\
&= -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z - \nabla_Y\nabla_Z X + \nabla_Z\nabla_Y X + \nabla_{[Y,Z]}X \\
&\qquad - \nabla_Z\nabla_X Y + \nabla_X\nabla_Z Y + \nabla_{[Z,X]}Y \\
&= -\nabla_X[Y,Z] - \nabla_Y[Z,X] - \nabla_Z[X,Y] + \nabla_{[X,Y]}Z + \nabla_{[Y,Z]}X + \nabla_{[Z,X]}Y \\
&= -[X,[Y,Z]] - [Y,[Z,X]] - [Z,[X,Y]] \\
&= 0,
\end{aligned}$$

where in the last step we used the Jacobi identity for vector fields. □

Obviously one can then write (1) and (2) in local coordinates as

$$R_{ijk}{}^l = -R_{jik}{}^l,$$
$$R_{ijk}{}^l + R_{jki}{}^l + R_{kij}{}^l = 0.$$

Recall that one can always extend a linear connection on the tangent bundle to a linear connection on tensor bundles. In particular, for the tensor field R of type $(1,3)$, $\nabla_X R$ is also a tensor field of type $(1,3)$, given by

$$(\nabla_X R)(Y,Z,W) := \nabla_X(R(Y,Z)W) - R(\nabla_X Y, Z)W - R(Y,\nabla_X Z)W - R(Y,Z)\nabla_X W.$$

Proposition 1.2 (The Second Bianchi Identity)**.** *Suppose ∇ is torsion free, then*

$$(\nabla_X R)(Y,Z,W) + (\nabla_Y R)(Z,X,W) + (\nabla_Z R)(X,Y,W) = 0.$$

Proof. By definition,

$$\begin{aligned}
&(\nabla_X R)(Y,Z,W) + (\nabla_Y R)(Z,X,W) + (\nabla_Z R)(X,Y,W) \\
=&\nabla_X(R(Y,Z)W) - R(\nabla_X Y,Z)W - R(Y,\nabla_X Z)W - R(Y,Z)\nabla_X W + \\
&\nabla_Y(R(Z,X)W) - R(\nabla_Y Z,X)W - R(Z,\nabla_Y X)W - R(Z,X)\nabla_Y W + \\
&\nabla_Z(R(X,Y)W) - R(\nabla_Z X,Y)W - R(X,\nabla_Z Y)W - R(X,Y)\nabla_Z W.
\end{aligned}$$

Using the torsion-freeness and (1), one can simplify the middle two columns to

$$R([X,Z],Y)W + R([Y,X],Z)W + R([Y,X],Z)W.$$

Now expand each R using its definition, the whole expression becomes a summation of 27 terms, the first 9 terms being

$$\begin{aligned} &- \nabla_X\nabla_Y\nabla_Z W + \nabla_X\nabla_Z\nabla_Y W + \nabla_X\nabla_{[Y,Z]}W \\ &- \nabla_{[X,Z]}\nabla_Y W + \nabla_Y\nabla_{[X,Z]}W + \nabla_{[[X,Z],Y]}W \\ &+ \nabla_Y\nabla_Z\nabla_X W - \nabla_Z\nabla_Y\nabla_X W - \nabla_{[Y,Z]}\nabla_X W, \end{aligned}$$

the second and third 9 terms are similar to the first 9 terms above: one just replace X, Y, Z by Y, Z, X and Z, X, Y respectively. It is not hard to check that all those expressions containing three ∇'s (12 terms in total) cancel out trivially, all those expressions containing two ∇'s (also 12 terms in total) cancel out by using the fact $[X, Y] = -[Y, X]$, and the remaining three terms

$$\nabla_{[[X,Z],Y]}W + \nabla_{[[Y,X],Z]}W + \nabla_{[[Z,Y],X]}W = 0$$

in view of the Jacobi identity. □

In local coordinates we can write

$$\nabla_{\partial_n} R = R_{ijk}{}^{l}{}_{;n} dx^i \otimes dx^j \otimes dx^k \otimes \partial_j,$$

Then the second Bianchi identity can be written as

$$R_{ijk}{}^{l}{}_{;n} + R_{jnk}{}^{l}{}_{;i} + R_{nik}{}^{l}{}_{;j} = 0.$$

2. The Riemann curvature tensor

Now suppose (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection. As last time, by using the Riemannian metric g one can convert the $(1, 3)$-tensor R to a $(0, 4)$-tensor $Rm \in \Gamma(\otimes^4 TM)$:

$$Rm(X, Y, Z, W) = g(R(X, Y)Z, W).$$

We shall call Rm the *Riemann curvature tensor* of (M, g). Locally if we write

$$Rm = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l,$$

then

$$R_{ijkl} = Rm(\partial_i, \partial_j, \partial_k, \partial_l) = g(R_{ijk}{}^{m}\partial_m, \partial_l) = g_{ml}R_{ijk}{}^{m}.$$

In other words, the Riemannian metric "lower one of the the index".

Obviously one can rewrite the identities (1) and (2) using Rm as

$$\begin{aligned} Rm(X, Y, Z, W) + Rm(Y, X, Z, W) &= 0, \\ Rm(X, Y, Z, W) + Rm(Y, Z, X, W) + Rm(Z, X, Y, W) &= 0, \end{aligned}$$

or in local coordinates as

$$\begin{aligned} R_{ijkl} &= -R_{jikl}, \\ R_{ijkl} + R_{jkil} + R_{kijl} &= 0. \end{aligned}$$

Similarly the second Bianchi identity can be written in terms of Rm as

$$(\nabla_X Rm)(Y, Z, W, V) + (\nabla_Y Rm)(Z, X, W, V) + (\nabla_Z Rm)(X, Y, W, V) = 0,$$

and if we denote $R_{ijkl;n} = (\nabla_{\partial_n} R)(\partial_i, \partial_j, \partial_k, \partial_l)$, then

$$R_{ijkl;n} + R_{jnkl;i} + R_{nikl;j} = 0.$$

Example. The Riemann curvature tensor for S^n (equiped with the standard round metric) is

$$Rm(X,Y,Z,W) = \langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle.$$

Note that this can be written as

$$Rm = \frac{1}{2} g \owedge g$$

where $\owedge$ is the Kulkarni-Nomizu product which takes 2 symmetric $(0,2)$-tensor into one $(0,4)$-tensor that has many nice symmetry properties:

$$\begin{aligned}(T_1 \owedge T_2)(X,Y,Z,W) =& T_1(X,Z)T_2(Y,W) - T_1(Y,Z)T_2(X,W) \\ &- T_1(X,W)T_2(Y,Z) + T_1(Y,W)T_2(X,Z).\end{aligned}$$

By staring at the above example, one see that the Riemann curvature tensor Rm on the standard S^n has even more (anti-)symmetries than the ones we have seen, e.g. one can exchange Z with W to get a negative sign, or even exchange X,Y with Z,W. In fact thest two (anti-)symmetries are consequneces of metric compatibilities, and thus hold in general:

Proposition 2.1. *The Riemann curvature tensor Rm satisfies*

$$\begin{aligned} Rm(X,Y,Z,W) &= -Rm(X,Y,W,Z), \\ Rm(X,Y,Z,W) &= Rm(Z,W,X,Y). \end{aligned} \tag{3}$$

Proof. We first notice that if we denote $f = \langle Z,Z\rangle$, then

$$\langle \nabla_X Z, Z\rangle = Xf - \langle Z, \nabla_X Z\rangle,$$

in other words,

$$\langle \nabla_X Z, Z\rangle = \frac{1}{2} Xf.$$

It follows

$$\langle \nabla_X\nabla_Y Z, Z\rangle = X\langle \nabla_Y Z, Z\rangle - \langle \nabla_Y Z, \nabla_X Z\rangle = \frac{1}{2}X(Yf) - \langle \nabla_Y Z, \nabla_X Z\rangle.$$

So

$$\begin{aligned} Rm(X,Y,Z,Z) &= \langle R(X,Y)Z, Z\rangle \\ &= \langle -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z, Z\rangle \\ &= -\frac{1}{2}X(Yf) + \frac{1}{2}Y(Xf) + \frac{1}{2}[X,Y]f = 0. \end{aligned}$$

As a consequence, we get

$$\begin{aligned} & Rm(X,Y,Z,W) + Rm(X,Y,W,Z) \\ =& Rm(X,Y,Z+W,Z+W) - Rm(X,Y,Z,Z) - Rm(X,Y,W,W) \\ =& 0. \end{aligned}$$

The second one is a consequence of the first one together with (1) and (2). In fact, by the first Bianchi identity (2) one has

$$\begin{aligned} Rm(X,Y,Z,W) + Rm(Y,Z,X,W) + Rm(Z,X,Y,W) &= 0, \\ Rm(Y,Z,W,X) + Rm(Z,W,Y,X) + Rm(W,Y,Z,X) &= 0, \\ Rm(Z,W,X,Y) + Rm(W,X,Z,Y) + Rm(X,Z,W,Y) &= 0, \\ Rm(W,X,Y,Z) + Rm(X,Y,W,Z) + Rm(Y,W,X,Z) &= 0, \end{aligned}$$

Summing the equations and using (1) as well as the first one in (3) that we just proved, the first two columns cancel out and we get

$$Rm(Z,X,Y,W) + Rm(W,Y,Z,X) = 0,$$

which is equivalent to the second one in (3). □

Of course if one use local coordinates, then the two identities in (3) can be rewritten as

$$\begin{aligned} R_{ijkl} &= -R_{ijlk}, \\ R_{ijkl} &= R_{klij}. \end{aligned}$$