Lesson 13 - 24/03/2026

Distributions $\quad \Omega$ is open set in $\mathbb{R}^n$

def let $T : \mathcal{D}(\Omega) \to \mathbb{C}$ (or $\mathbb{R}$)

T is a distribution, $T \in \mathcal{D}'(\Omega)$ if

i) T is linear

ii) $\forall K$ compact $\subseteq \Omega$, $\exists C_K > 0$, $m_K \in \mathbb{N}$ s.t.

$$|T(\varphi)| \le C_K \cdot \sum_{|\alpha| \le m_K} \sup_{\Omega} |D^\alpha \varphi|$$

$\forall \varphi \in \mathcal{D}(\Omega)$ s.t. $\operatorname{supp} \varphi \subseteq K$.

def if m_K does not depend on K ($\exists m \in \mathbb{N}$ s.t. $\forall K$, $m = m_K$) then T is said to be of finite order (m is the minimal when)

important examples

1) if $f \in L^1_{loc}(\Omega)$ then $T_f(\varphi) = \int_\Omega f \varphi$

$T_f \in \mathcal{D}'(\Omega)$

and $\Phi : \underset{\overset{\cap}{L^1_{loc}}}{f} \longmapsto \underset{\overset{\cap}{\mathcal{D}'}}{T_f}$ is injective $\Longrightarrow$ $\underline{L^1_{loc}(\Omega) \subseteq \mathcal{D}'(\Omega)}$

$T_f \in \mathcal{D}_0'(\Omega)$ ← distributions of order 0

2)

Dirac's delta $\quad x_0 \in \Omega$, $\delta_{x_0}(\varphi) = \varphi(x_0)$

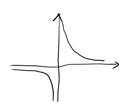
$\delta_{x_0} \in \mathcal{D}_0'(\Omega) \setminus L^1_{loc}(\Omega)$

3) $\exists T \notin \mathcal{D}_F'(\Omega)$ ← distributions of finite order

$\Omega =]0, 2[$, $\quad T(\varphi) = \sum_{n=0}^{+\infty} \varphi^{(n)}\left(\frac{1}{n+1}\right)$

4) $PV_{\frac{1}{x}}(\varphi) = \lim_{\varepsilon \to 0^+} \int_{|x| > \varepsilon} \frac{\varphi(y)}{y} dy \quad$ "Principal value of $\frac{1}{x}$"

remark the function $x \longmapsto \frac{1}{x}$, $\mathbb{R} \setminus \{0\} \to \mathbb{R}$ is not in $L^1_{loc}(\mathbb{R})$



" $\int_{\mathbb{R}} \varphi(x) \frac{1}{x} dx$ " naively

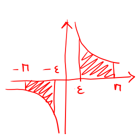
I have to prove that

$\forall \varphi \in \mathcal{D}(\mathbb{R})$, the $\lim_{\varepsilon \to 0^+} \left(\int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x} dx + \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x} dx \right)$ exists

consider $\varphi \in \mathcal{D}(\mathbb{R})$ then $\exists M > 0$ $\operatorname{supp} \varphi \subseteq [-M, M]$

$$\int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x} dx + \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x} dx = \int_{-M}^{-\varepsilon} \frac{\varphi(x)}{x} dx + \int_{\varepsilon}^{M} \frac{\varphi(x)}{x} dx$$

remark that $\int_{-M}^{-\varepsilon} \frac{1}{x} dx + \int_{\varepsilon}^{M} \frac{1}{x} dx = 0$



$$= \int_{-M}^{-\varepsilon} \frac{\varphi(x) - \varphi(0)}{x} dx + \int_{\varepsilon}^{M} \frac{\varphi(x) - \varphi(0)}{x} dx$$

consider $\psi(x) = \begin{cases} \frac{\varphi(x) - \varphi(0)}{x} & \text{if } x \neq 0 \\ \varphi'(0) & \text{if } x = 0 \end{cases} \qquad \psi(x) \in \mathcal{C}(\mathbb{R})$

$$= \int_{-M}^{-\varepsilon} \frac{\varphi(x)-\varphi(0)}{x}\,dx + \int_{\varepsilon}^{M} \frac{\varphi(x)-\varphi(0)}{x}\,dx$$

consider $\psi(x) = \begin{cases} \frac{\varphi(x)-\varphi(0)}{x} & \text{if } x \neq 0 \\ \varphi'(0) & \text{if } x = 0 \end{cases}$ $\qquad \psi(x) \in \mathcal{C}(\mathbb{R})$

then
$$\lim_{\varepsilon\to 0} \int_{-M}^{-\varepsilon} \frac{\varphi(x)-\varphi(0)}{x}\,dx + \int_{\varepsilon}^{M} \frac{\varphi(x)-\varphi(0)}{x}\,dx$$
$$= \lim_{\varepsilon\to 0} \int_{-M}^{-\varepsilon} \psi(x)\,dx + \int_{\varepsilon}^{M} \psi(x)\,dx = \int_{-M}^{M} \psi(x)\,dx$$

from the continuity $\Rightarrow$ integrable in $[-M,M]$

$\Rightarrow$ the limit exists. $\quad PV_{\frac{1}{x}}(\varphi)$ is linear (easy)

the last condition

$$|PV_{\frac{1}{x}}(\varphi)| = \left|\int_{-M}^{M} \psi(x)\,dx\right| \le 2M \cdot \sup_{x\in\mathbb{R}} |\varphi'| \quad *$$

for $\varphi \in \mathcal{D}(\mathbb{R})$ s.t. $\operatorname{supp}\varphi \subseteq [-M,M]$

$|\psi(x)| \le \sup_{[-M,M]} |\varphi'(x)|$

$\Rightarrow$ $PV_{\frac{1}{x}}$ is a distribution of order ≤ 1

from $\circledast$ we have that
$\forall K$ compact in $\mathbb{R}$ $\exists C > 0$,
s.t. $|PV_{\frac{1}{x}}(\varphi)| \le C \cdot \sum_{|\alpha|\le 1} \sup_{x\in\mathbb{R}} |D^\alpha \varphi|$
$C = 2M$ if $K \subseteq [-M,M]$

Ex. show that $PV_{\frac{1}{x}}$ has (exactly) order 1

Th. (characterization of $\mathcal{D}'$)

Let Ω be an open set in $\mathbb{R}^n$

Let $T : \mathcal{D}(\Omega) \to \mathbb{C}$, T linear operator.

T is a distribution

$\Updownarrow$

given $(\varphi_n)_n$ sequence of $\mathcal{D}(\Omega)$ such that

a) $\exists K$ compact in Ω s.t. $\forall n$, $\operatorname{supp}\varphi_n \subseteq K$

b) $\forall \alpha$ $\|D^\alpha \varphi_n\|_{L^\infty} \xrightarrow{n} 0$

$(\varphi_n)_n$ goes to 0 uniformly and the same for all the derivatives of φ_n

then $T(\varphi_n) \xrightarrow{n} 0$ $\qquad \circledast$

we will say that $(\varphi_n)_n$ goes to 0 in the sense of $\mathcal{D}(\Omega)$

proof. if $T \in \mathcal{D}'(\Omega)$ then $\circledast$ is valid.
(this is easy)

in fact take $(\varphi_n)_n$ satisfying a) and b)

in particular $\operatorname{supp}\varphi_n \subseteq K$ $\forall n$

we know that

$$|T(\psi)| \le C \sum_{|\alpha|\le m} \sup_{\Omega} |D^\alpha \psi| \quad \forall \psi \in \mathcal{D}(\Omega) \text{ s.t. } \operatorname{supp}\psi \subseteq K$$

consequently $|T(\varphi_n)| \le C \sum_{|\alpha|\le m} \sup_{\Omega} |D^\alpha \varphi_n|$

this quantity goes to 0 from b)

consequently $\lim_n T(\varphi_n) = 0$ then $*$ is valid

Conversely

suppose T is not a distribution

$\exists K$ s.t. $\forall C>0, \forall m \in \mathbb{N}$, $\exists \phi \in \mathcal{D}(\Omega)$

s.t. $\operatorname{supp} \phi \subseteq K$

and $|T(\phi)| > C \sum_{|\alpha| \le m} \sup_\Omega |D^\alpha \phi|$

in particular take $C = m = j \in \mathbb{N}$

then $\exists \psi_j \in C_0^\infty(\Omega)$ s.t. $\operatorname{supp} \psi_j \subseteq K$

and $|T(\psi_j)| > j \cdot \sum_{|\alpha| \le j} \sup_\Omega |D^\alpha \psi_j|$. ***

I define $\varphi_j(x) = \dfrac{\psi_j(x)}{j \cdot \sum_{|\alpha|\le j} \sup_\Omega |D^\alpha \psi_j|}$

$i)$ $\operatorname{supp} \varphi_j$? $\operatorname{supp} \varphi_j \subseteq K$ $\forall j$

$ii)$ $\lim_{j \to +\infty} \sup_\Omega |D^\beta \varphi_j|$

fix β

$\sup_\Omega \left| D^\beta \left(\dfrac{\psi_j(x)}{j \sum_{|\alpha|\le j} \sup_\Omega |D^\alpha \psi_j|} \right) \right|$

$= \dfrac{\sup_\Omega |D^\beta \psi_j|}{j \sum_{|\alpha|\le j} \sup_\Omega |D^\alpha \psi_j|}$ this quantity $\le \dfrac{1}{j}$

as soon as $j \ge |\beta|$

condition $\lim_j \sup_\Omega |D^\beta \varphi_j| = 0$

for $(\varphi_n)_n$ a) b) } are valid

but $|T(\varphi_j)| = \dfrac{|T(\psi_j)|}{j \cdot \sum_{|\alpha|\le j} \sup_\Omega D^\alpha \psi_j|} \ge 1$ $\forall j$

***

so $\lim_j T(\varphi_j) \ne 0$

QED

def let Ω open set in $\mathbb{R}^n$

let $T \in \mathcal{D}'(\Omega)$ let $x_0 \in \Omega$

$x_0 \notin \operatorname{supp} T$ if $\exists U$ open nbh of x_0, $U \subseteq \Omega$

s.t. $\forall \psi \in C_0^\infty(U)$, $T(\psi) = 0$

$\operatorname{supp} T$ is the minimal relatively closed set in Ω outside of which T is locally equal to 0.

remark take $f \in C(\Omega)$

I can consider $\operatorname{supp} f$ in the sense of continuous functions

$\operatorname{supp} f$ " " " of L^1_{loc} functions

$\operatorname{supp} T_f$ " " " " distributions

they are the same

Th (local character of a distribution)

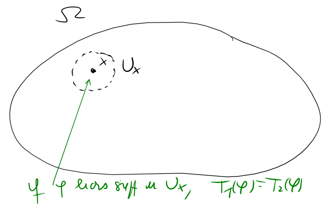
Th (local character of a distribution)

let $T_1, T_2 \in \mathcal{D}'(\Omega)$

suppose that $\forall x \in \Omega$, $\exists U_x$ nbh of x in Ω

s.t. $\forall \varphi \in \mathcal{D}(\Omega)$ if $\operatorname{supp} \varphi \subseteq U_x$ then

$T_1(\varphi) = T_2(\varphi)$



Then $T_1 = T_2$

proof. let $\psi \in \mathcal{D}(\Omega)$

then take $x \in \operatorname{supp}\psi$ and consider U_x as in the statement

consider $\{U_x,\ x \in \operatorname{supp}\psi\}$ this is an open covering of the compact set $\operatorname{supp}\psi$

Extract a finite subcovering

$U_1, U_2, U_3, \dots U_N$

$\underset{\text{compact}}{\operatorname{supp}\psi} \subseteq \bigcup_{i=1}^{N} U_i$

Apply the theorem on partition of unity

then $\exists \varphi_1 \in C_0^\infty(U_1)$, $\varphi_2 \in C_0^\infty(U_2)$ ..., $\varphi_N \in C_0^\infty(U_N)$

s.t. $\forall x \in \operatorname{supp}\psi$, $\sum_{i=1}^{N} \varphi_i(x) = 1$

now
$$T_1(\psi) = T_1\Big(\sum_{i=1}^{N} \varphi_i(x)\psi(x)\Big) = \sum_{i=1}^{N} T_1(\varphi_i \psi) = \sum_{i=1}^{N} T_2(\varphi_i \psi) = T_2\Big(\sum_{i=1}^{N} \varphi_i \psi\Big) = T_2(\psi)$$

but $\operatorname{supp}(\varphi_i \psi) \subseteq U_i$

$\varphi_i \psi \in C_0^\infty(U_i)$

QED

Remark
let $(T_n)_n$ be a sequence in $\mathcal{D}'(\Omega)$

let $T \in \mathcal{D}'(\Omega)$

I say that T_n converges to T in sense of distributions

if $\forall \varphi \in \mathcal{D}(\Omega)$ $\lim_n T_n(\varphi) = T(\varphi)$

weak convergence

It is possible to prove a version of Banach-Steinhaus theorem for distributions.

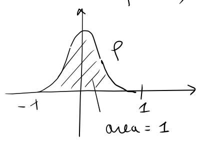
i.e. let $(T_n)_n$ be a sequence of distributions and suppose that, $\forall \varphi \in \mathcal{D}(\Omega)$,

the $\lim_n T_n(\varphi)$ exists in $\mathbb{C}$

then $\exists T \in \mathcal{D}'(\Omega)$ s.t. $\forall \varphi \in \mathcal{D}(\Omega)$, $\lim_n T_n(\varphi) = T(\varphi)$

remark · consider $\rho \in C_0^\infty(\mathbb{R}^n)$

with $\rho \geq 0$, supp $\rho \subseteq B(0,1)$ and $\int_{\mathbb{R}^n} \rho(x)\,dx = 1$



$\rho_k(x) = k^n \rho(kx)$

$(\rho_k)_k$ mollifier

consider $\left(T_{\rho_k}\right)_k$

find $\lim_k T_{\rho_k}$

$T_{\rho_k}(\varphi) = \int_{\mathbb{R}^n} \rho_k(x)\varphi(x)\,dx = \int_{\mathbb{R}^n} k^n \rho(kx)\varphi(x)\,dx$

$kx = y$

$x = \frac{y}{k}$ $\quad dx = \frac{dy}{k^n}$

$= \int \cancel{k^n} \rho(y)\varphi\left(\frac{y}{k}\right)\frac{dy}{\cancel{k^n}}$

$= \int_{\mathbb{R}^n} \rho(y)\varphi\left(\frac{y}{k}\right)dy$

$\rho(y)\varphi\left(\frac{y}{k}\right) \xrightarrow{k} \rho(y)\varphi(0)$ pointwise

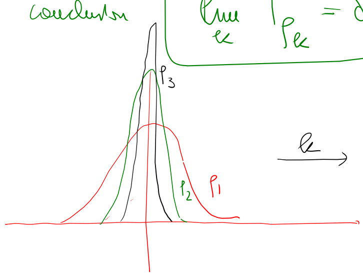
$\left|\rho(y)\varphi\left(\frac{y}{k}\right)\right| \leq \rho(y)\cdot\|\varphi\|_{L^\infty}$

apply dom. conv.

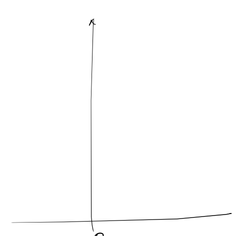
$\lim_k T_{\rho_k}(\varphi) = \lim_k \int_{\mathbb{R}^n} \rho(y)\varphi\left(\frac{y}{k}\right)dy = \left(\int_{\mathbb{R}^n} \rho(y)\,dy\right)\varphi(0) = \varphi(0)$

$= \delta_0(\varphi)$

conclusion $\boxed{\lim_k T_{\rho_k} = \delta_0}$



$\xrightarrow{k}$ δ_0



Remark (the topology of $\mathcal{D}(\Omega)$)

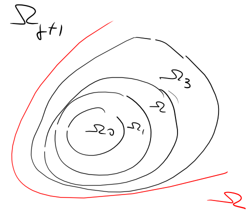
Let Ω be an open set in $\mathbb{R}^n$

Let $m \in \mathbb{N}$

consider $C^m(\Omega) = \{ f$ continuous, differentiable up to the order m, with continuous derivatives $\}$

I consider $\Omega = \bigcup_{j=0}^{+\infty} \Omega_j$

Ω_j open, $\overline{\Omega_j}$ compact, $\overline{\Omega_j} \subseteq \Omega_{j+1}$

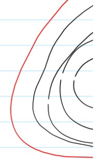


$f \in C^m(\Omega)$ define $p_j(f) = \sum_{|\alpha| \leq m} \sup_{\overline{\Omega_j}} |D^\alpha f|$

$(p_j)_j$ countable family of seminorms

U is a nbd of 0 if $\exists j \in \mathbb{N}, \exists \varepsilon > 0$ s.t.

$f \in \mathcal{C}^m(\Omega)$ define $p_j(f) = \sum_{|\alpha| \le m} \sup_{\Omega_j} |D^\alpha f|$



$(p_j)_j$ countable family of seminorms

- U is a nbd of 0 if $\exists j \in \mathbb{N}, \exists \varepsilon > 0$ s.t.

$$\underbrace{\{g \in \mathcal{C}^m(\Omega) : p_j(g) < \varepsilon\}}_{\text{ball w.r.t. } p_j \text{ seminorm, of radius } \varepsilon} \subseteq U$$

with this topology $\mathcal{C}^m(\Omega)$ is a Fréchet space (metrizable and complete)

- Consider now $\mathcal{C}^\infty(\Omega) = \bigcap_m \mathcal{C}^m(\Omega) = \mathcal{E}(\Omega)$

I consider now

$f \in \mathcal{C}^\infty(\Omega)$, $\tilde{p}_j(f) = \sum_{|\alpha| \le j} \sup_{\Omega_j} |D^\alpha f|$

- U is a nbh of 0 if

$\exists j, \exists \varepsilon$ s.t. $U \supseteq \{g \in \mathcal{C}^\infty(\Omega) : \tilde{p}_j(g) < \varepsilon\}$

$\mathcal{C}^\infty(\Omega) = \mathcal{E}(\Omega)$ is a Fréchet space.

- consider now $\mathcal{C}_0^\infty(\overline{\Omega_j}) = \{\varphi \in \mathcal{C}_0^\infty(\Omega)$ s.t. $\operatorname{supp} \varphi \subseteq \overline{\Omega_j}\}$

on this set I consider the norm

$$q_k(\varphi) = \sum_{|\alpha| \le k} \sup_{\Omega_j} |D^\alpha \varphi|$$

Again $\mathcal{C}_0^\infty(\overline{\Omega_j})$ is a Fréchet space

Consider $\mathcal{C}_0^\infty(\Omega) = \bigcup_{j=0}^{+\infty} \mathcal{C}_0^\infty(\overline{\Omega_j})$

on $\mathcal{C}_0^\infty(\Omega) = \mathcal{D}(\Omega)$ test functions

consider the minimal topology for which all the injection

$$\mathcal{C}_0^\infty(\overline{\Omega_j}) \hookrightarrow \mathcal{C}_0^\infty(\Omega)$$

are continuous (so called inductive limit topology)

It's not a Fréchet space (but it is complete!)