Lesson 16 - 31/03/2026

derivative in the sense of distributions

def. $T \in \mathcal{D}'(\Omega)$

$$\partial_{x_j} T(\varphi) = - T(\partial_{x_j} \varphi)$$

Th. Let $T \in \mathcal{D}'(\Omega)$. Let $\omega \subset\subset \Omega$

$\exists m \in \mathbb{N}$ $\exists f \in L^\infty(\omega)$ s.t.

$$T(\varphi) = \int_\Omega f \, \partial_{x_1}^m \cdots \partial_{x_n}^m \varphi \qquad \forall \varphi \in \mathcal{D}(\Omega) \text{ st. supp } \varphi \subseteq \omega$$

$$\Big(T = (-1)^{nm} \partial_{x_1}^m \cdots \partial_{x_n}^m T_f \Big) \text{ on } \omega$$

rem. if $T_1, T_2 \in \mathcal{D}(\Omega)$, in general $T_1 \cdot T_2$ makes no sense

multiplication

multiplication of a distribution with a $\mathcal{C}^\infty$ function

def Let $T \in \mathcal{D}'(\Omega)$ Let $a \in \mathcal{C}^\infty(\Omega)$

define aT

$$aT(\varphi) = T(a\varphi)$$

Th $aT \in \mathcal{D}'(\Omega)$

proof. $aT(\varphi) = T(a\varphi)$ it makes sense as $a\varphi \in \mathcal{C}_0^\infty(\Omega)$

$$\mathcal{D}(\Omega) \longrightarrow \mathbb{C}$$
$$\varphi \longmapsto aT(\varphi) = T(a\varphi) \quad \text{is linear}$$

$$\Big(\text{e.g. } aT(\varphi_1 + \varphi_2) = T(a(\varphi_1+\varphi_2)) = T(a\varphi_1 + a\varphi_2) \overset{!}{=} T(a\varphi_1) + T(a\varphi_2) \overset{!}{=} aT(\varphi_1) + aT(\varphi_2) \Big)$$

secondly

let K be a compact
let $\varphi \in \mathcal{C}_0^\infty(\Omega)$ with support in K

$$|aT(\varphi)| = |T(a\varphi)| \underset{T \in \mathcal{D}'(\Omega)}{\le} C_K \sum_{|\alpha| \le m_K} \sup_\Omega |D^\alpha(a\varphi)| \qquad \forall \varphi \in \mathcal{C}_0^\infty(\Omega) \text{ with supp } \varphi \subseteq K$$

$$\sup_\Omega |D^\alpha (a\varphi)| \le \underset{\text{depending on } a}{\tilde{C}} \sum_{|\beta| \le m_K} \sup |D^\beta \varphi|$$

$$D^\alpha(a\varphi) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \varphi \, D^{\alpha-\beta} a$$

so that $aT \in \mathcal{D}'(\Omega)$ and if T has order m then aT has order $\le m$.

Ex. consider $T \in \mathcal{D}'(\Omega)$, $a \in \mathcal{C}^\infty(\Omega)$

evaluate $\partial_{x_j}(aT)$

in the sense of $\mathcal{D}'$

solution
$$\begin{aligned}
\partial_{x_j}(aT)(\varphi) &= aT(-\partial_{x_j}\varphi) \\
&\overset{!}{=} T(a(-\partial_{x_j}\varphi)) \\
&\overset{!}{=} T(-a\,\partial_{x_j}\varphi) \\
&\overset{!}{=} T(-\partial_{x_j}(a\varphi) + \partial_{x_j}a \cdot \varphi) \\
&\overset{!}{=} T(-\partial_{x_j}(a\varphi)) + T(\partial_{x_j}a\, \varphi) \\
&\overset{!}{=} (\partial_{x_j}T)(a\varphi) + (\partial_{x_j}a\, T)(\varphi) \\
&\overset{!}{=} (a\,\partial_{x_j}T)(\varphi) + (\partial_{x_j}a\, T)(\varphi) \\
&\overset{!}{=} (a\,\partial_{x_j}T + \partial_{x_j}a\, T)(\varphi)
\end{aligned}$$

conclusion

$$\underbrace{\partial_{x_j}}_{\text{in the sense of } \mathcal{D}'}(aT) = \underbrace{\partial_{x_j} a}_{\text{classical derivative}}\, T + a\, \underbrace{\partial_{x_j} T}_{\text{in the sense of } \mathcal{D}'}$$

$$\boxed{\partial_{x_j}(T_{\chi f})(\varphi) = T_{(\chi g + \partial_{x_j}\chi f)}(\varphi)}$$

suppose that $\partial_{x_j} f = g$

then f is differentiable in the direction x_j in classical sense and $\partial_{x_j} f = g$

proof step 1. suppose $f, g \in \mathcal{C}_0(\Omega)$

consider $(\rho_n)_n$ mollifier

we know that $\rho_n * f \xrightarrow{n} f$ uniformly

$\rho_n * g \xrightarrow{n} g$ uniformly

if we have that $\underbrace{\partial_{x_j}}_{\text{classical}}(\rho_n * f) = \rho_n * g$

then the conclusion follows

let's compute $(\rho_n * f)(x) = \int_{\mathbb{R}^d} f(y)\, \rho_n(x-y)\, dy$

$= T_f(\varphi_n) \qquad \varphi_n : y \longmapsto \rho_n(x-y)$

variable

parameter

$$\partial_{x_j}(\rho_n * f)(x) = (\partial_{x_j}(\rho_n)) * f$$

$$= \int_{\mathbb{R}^d} f(y)\, \underbrace{\partial_{x_j}(\rho_n(x-y))}_{= -\partial_{y_j}(\rho_n(x-y))}\, dy$$

$$= T_f(-\partial_{y_j}\varphi_n) = \partial_{y_j} T(\varphi_n) = T_g(\varphi_n) = (\rho_n * g)(x)$$

use the fact that $\partial_{y_j} T_f = T_g$

conclusion $\partial_{x_j}(\rho_n * f)(x) = (\rho_n * g)(x)$

and we conclude.

Step 2 suppose $f, g \in \mathcal{C}(\Omega)$

consider $x_0 \in \Omega$

consider $\chi \in \mathcal{C}_0^\infty(\Omega)$ s.t. $\chi = 1$ in a nbh of x_0

let's compute $\partial_{x_j}(T_{\chi f})$

$$\partial_{x_j}(T_{\chi f})(\varphi) = T_{\chi f}(-\partial_{x_j}\varphi)$$

$$= \chi T_f(-\partial_{x_j}\varphi)$$

$$= T_f(-\chi \partial_{x_j}\varphi)$$

$$= T_f(-\partial_{x_j}(\chi\varphi) + \partial_{x_j}\chi\, \varphi)$$

$$= T_f(-\partial_{x_j}(\chi\varphi)) + T_f(\partial_{x_j}\chi\, \varphi)$$

$$= \underbrace{(\partial_{x_j} T_f)}_{= T_g}(\chi\varphi) + \partial_{x_j}\chi\, T_f(\varphi)$$

$$= \chi T_g(\varphi) + \partial_{x_j}\chi\, T_f(\varphi)$$

$$\boxed{\partial_{x_j}(T_{\chi f})(\varphi) = T_{(\chi g + \partial_{x_j}\chi f)}(\varphi)}$$

$$\partial_{x_j}(T_{\chi f})(\varphi) = T_{(\chi g + \partial_j \chi f)}(\varphi)$$

remark that χf and $\chi g + \partial_j \chi f$ are $\mathcal{C}_0(\Omega)$

We can apply step 1

so that χf is diff. in classical sense in x_j

and $\partial_{x_j}(\chi f) = \chi g + \partial_j \chi \cdot f$ classically

but $\chi = 1$ in nbh of x_0

in this nbh: $\partial_{x_j}(\chi f) = \partial_j f$

$\chi g + \partial_j \chi \cdot f = g$

QED

Ex. Let $T \in \mathcal{D}'(\mathbb{R})$

suppose $T' = 0$

Prove that $\exists c \in \mathbb{C}$ s.t. $T = T_c$

solution

$$T' = 0 \text{ in } \mathcal{D}'(\mathbb{R})$$
$$\Updownarrow$$
$$T'(\varphi) = 0 \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R})$$
$$\Updownarrow$$
$$T(\varphi') = 0 \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R})$$

we have to consider $T(\psi)$ with ψ a general $\mathcal{C}_0^\infty(\mathbb{R})$ function (not only on φ')

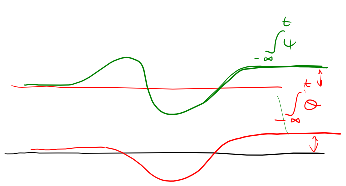
Consider $\theta \in \mathcal{C}_0^\infty(\mathbb{R})$ s.t. $\int_{-\infty}^{+\infty} \theta(t)\,dt = 1$

given $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$, consider $\varphi(t) = \int_{-\infty}^{t} \psi(s)\,ds - \left(\int_{-\infty}^{t} \theta(s)\,ds\right) \cdot \int_{-\infty}^{+\infty} \psi(s)\,ds$

I claim that $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$

for $t \le \underline{t}$ $\quad \varphi(t) = 0$

for $t \ge \overline{t}$



$$\varphi'(t) = \psi(t) - \theta(t) \cdot \int_{-\infty}^{+\infty} \psi(s)\,ds$$

$$0 = T(\varphi') \Rightarrow T\left(\psi - \theta \cdot \int_{-\infty}^{+\infty} \psi(s)\,ds\right) = 0$$

$$T(\psi) = \underbrace{T(\theta)}_{c} \cdot \underbrace{\int_{-\infty}^{+\infty} \psi(s)\,ds}_{T_1(\psi)}$$

at the end $T = T_c$

remark: c does not depend on θ, but only on T

in fact if $\int_{-\infty}^{+\infty} \theta_1(s)\,ds = \int_{-\infty}^{+\infty} \theta_2(s)\,ds = 1$

$$\int_{-\infty}^{+\infty} (\theta_1 - \theta_2) = 0 \Rightarrow \int_{-\infty}^{t} (\theta_1 - \theta_2)\,ds \in \mathcal{C}_0^\infty$$

so that $T(\theta_1 - \theta_2) = 0 \Rightarrow T(\theta_1) = T(\theta_2)$

$$\overset{\shortparallel}{c}$$

Ex. let $T \in \mathcal{D}'(\mathbb{R})$
let $g \in L^1_{loc}(\mathbb{R})$
suppose $T' = T_g$
then, ... $\mathbb{C}$ s.t

U is a nbhd of 0 in $\mathcal{E}^\infty(\Omega)$ if
$\exists j, \exists r>0$ s.t. $\{f \in \mathcal{E}^\infty(\Omega):$

Distributions with compact support

def. Let $T \in \mathcal{D}'(\Omega)$
$x_0 \notin \operatorname{supp} T$ if $\exists r>0$ s.t. $\forall \varphi \in \mathcal{E}^\infty_0(\Omega)$
with $\operatorname{supp} \varphi \subseteq B(x_0, r)$, $T(\varphi) = 0$

supp T is the minimal relatively closed set in Ω outside of which T is identically 0.

Ex. Suppose $T \in \mathcal{D}'(\Omega)$ Suppose $\varphi \in \mathcal{E}^\infty_0(\Omega)$
suppose $\operatorname{supp} T \cap \operatorname{supp} \varphi = \emptyset$
Then $T(\varphi) = 0$

solution. use the same strategy of the proof on local character of distributions.

Ex. Let Ω be an open set in $\mathbb{R}^d$
let $x_0 \in \Omega$. let $T \in \mathcal{D}'(\Omega)$
suppose $\operatorname{supp} T = \{x_0\}$
moreover suppose that T is of order 0 ($\forall K$ compact, $\exists C_K$ s.t $|T(\varphi)| \le C \sup_\Omega |\varphi|$ $\forall \varphi$ with support in K)
$\varphi \in \mathcal{D}(\Omega)$
Prove that $\exists c \in \mathbb{C}$ s.t.
$T = c\,\delta_{x_0}$

(more generally if $T \in \mathcal{D}'(\Omega)$ and $\operatorname{supp} T = \{x_0\}$
then $\exists m \in \mathbb{N}$, $\exists c_\alpha \in \mathbb{C}$ for $|\alpha| \le m$

$$T = \sum_{|\alpha| \le m} c_\alpha \partial^\alpha \delta_{x_0}$$)

Remark. Ω is an open set in $\mathbb{R}^d$
Consider $\mathcal{E}^\infty(\Omega)$
we consider this countable set of seminorms:

$\Omega = \bigcup_j \Omega_j$ — Ω_j open, $\overline{\Omega_j}$ compact, $\overline{\Omega_j} \subseteq \Omega_{j+1}$

for $f \in \mathcal{E}^\infty(\Omega)$ we define $\tilde{p}_j(f) = \sum_{|\alpha| \le j} \sup_{\Omega_j} |D^\alpha f|$

seminorm ($\forall f$, $\tilde{p}_j(f) \le \tilde{p}_{j+1}(f)$)

U is a nbhd of 0 in $\mathcal{E}^\infty(\Omega)$ if
$\exists j, \exists r>0$ s.t. $\{f \in \mathcal{E}^\infty(\Omega) : \tilde{p}_j(f) < r\} \subseteq U$

minimum ($\forall f,\ \tilde{p}_j(f) \le \tilde{p}_{j+1}(f)$)

U is a nbh of 0 in $\mathcal{E}^\infty(\Omega)$ if
$\exists j, \exists r>0$ s.t. $\{f \in \mathcal{E}^\infty(\Omega) : \tilde{p}_j(f) < r\} \subseteq U$

$\mathcal{E}^\infty(\Omega)$ with this topology (a Fréchet space) is denoted by $\mathcal{E}(\Omega)$ (mind $\mathcal{D}(\Omega) \subseteq \mathcal{E}(\Omega)$)

problem: how to characterize the elements of $\mathcal{E}'(\Omega)$ (the dual space)

$S : \mathcal{E}(\Omega) \to \mathbb{C}$ linear is in $\mathcal{E}'(\Omega)$ if and only if $\exists j, \exists C>0$ s.t.
$\forall f \in \mathcal{E}(\Omega),\quad |S(f)| \le C\, \tilde{p}_j(f)$

remark
$$\boxed{\begin{array}{l} S \in \mathcal{E}'(\Omega) \text{ iff and only if} \\ \exists K \text{ compact}, \exists m \in \mathbb{N}, \exists C>0 \text{ s.t.} \\ \forall f \in \mathcal{E}^\infty(\Omega),\quad |S(f)| \le C \sum_{|\alpha| \le m} \sup_K |D^\alpha f| \end{array}}$$

Problem: which are the relation between $\mathcal{E}'(\Omega)$ and $\mathcal{D}'(\Omega)$.

Th. Let Ω be an open set in $\mathbb{R}^d$
Let $S \in \mathcal{E}'(\Omega)$
then $S_{|\mathcal{D}(\Omega)}$ is in $\mathcal{D}'(\Omega)$ and $\text{supp}(S_{|\mathcal{D}(\Omega)})$ is a compact

conversely
let $T \in \mathcal{D}'(\Omega)$ with compact support
then there exists a unique $S \in \mathcal{E}'(\Omega)$ s.t. $S_{|\mathcal{D}} = T$.

conclusion: $\mathcal{E}'(\Omega)$ is the set of distribution with compact support.

proof. Let $S \in \mathcal{E}'(\Omega)$
we know that $\exists K$ compact, $\exists m \in \mathbb{N}$, $\exists C$
s.t. $\forall f \in \mathcal{E}^\infty(\Omega)$ $\quad |S(f)| \le C \sum_{|\alpha| \le m} \sup_{x \in K} |D^\alpha f(x)|$

now take K_1 compact
take $\varphi \in \mathcal{E}_0^\infty(\Omega)$ with $\text{supp}\,\varphi \le K_1$

$$|S(\varphi)| \le C \sum_{|\alpha| \le m} \underbrace{\sup_{x \in K} |D^\alpha \varphi(x)|}_{\wedge|} \le C \cdot \sum_{|\alpha| \le m} \sup_{x \in \Omega} |D^\alpha \varphi(x)|$$

$\forall \varphi \in \mathcal{E}_0^\infty(\Omega)$ with $\text{supp}\,\varphi \subseteq K_1$

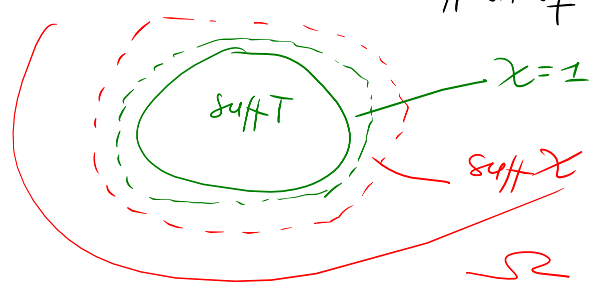
so that taking $C_{K_1} = C$, $m_{K_1} = m$ we have that $S_{|\mathcal{D}} \in \mathcal{D}'$

moreover if $\text{supp}\,\varphi \cap K_1 = \emptyset$
then $S(\varphi) = 0$
$\Rightarrow \text{supp}(S_{|\mathcal{D}}) \subseteq K$

to see the converse

we have $T \in \mathcal{D}'(\Omega)$ and supp T is a compact

consider $\chi \in \mathcal{C}^\infty_0(\Omega)$ s.t. $\chi = 1$ in nbh of the support of T



define $S(f) = T(\chi f)$

S is linear (easy)

now consider $\tilde{K} = \operatorname{supp} \chi$

Apply to T the estimate from the fact that $T \in \mathcal{D}'$, with $\tilde{K}$

$\exists\, C_{\tilde{K}}, m_{\tilde{K}}$ s.t.

$$|T(\varphi)| \le C_{\tilde{K}} \sum_{|\alpha| \le m_{\tilde{K}}} \sup_{\Omega} |D^\alpha \varphi| \quad \forall \varphi \in \mathcal{C}^\infty_0(\Omega) \text{ having support in } \tilde{K}$$

$$|S(f)| = |T(\chi f)| \le C_{\tilde{K}} \sum_{|\alpha| \le m_{\tilde{K}}} \sup_{\Omega} |D^\alpha(\chi f)|$$

$$\sup_{\Omega} |D^\alpha(\chi f)| \le \tilde{C} \sum_{\beta \le m_{\tilde{K}}} \sup_{\tilde{K}} |D^\beta f|$$

$\tilde{C}$ depends on χ but not on f

conclusion $|S(f)| \le \tilde{\tilde{C}} \cdot \sum_{|\alpha| \le m} \sup_{\tilde{K}} |D^\alpha f| \quad \forall f \in \mathcal{C}^\infty$

$\Rightarrow S \in \mathcal{E}'$

Ex. try to prove that S is unique

QED