Introduction to Curvature and other Tensors on Riemannian manifolds

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$$= f(\nabla_Y \nabla_X - \nabla_X \nabla_Y)Z + ([Y, X]f)Z)$$

Consider, for $X, Y, Z \in \mathcal{X}(M)$, the map

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$R(X,Y)$ is bilinear since

$$R(X,Y)(fZ) = f(\nabla_Y \nabla_X - \nabla_X \nabla_Y)Z + ([Y,X]f)Z) + f\nabla_{[X,Y]}Z + ([X,Y]f)Z =$$

$$= fR(X,Y)Z$$

Furthermore the following properties hold

1. *Linearity in the first entry*
   $R(fX_1 + gX_2, Y) = fR(X_1, Y) + gR(X_2, Y)$ for any $X_1, X_2, Y \in \mathcal{X}(M)$ and $f, g \in \mathcal{D}(M)$
2. *Linearity in the second entry*
   $R(X, fY_1 + gY_2) = fR(X, Y_1) + gR(X, Y_2)$ for any $X, Y_1, Y_2 \in \mathcal{X}(M)$ and $f, g \in \mathcal{D}(M)$
3. *First Bianchi identity*
   $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$

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5. $\mathcal{R}(X,Y,Z,W) = \mathcal{R}(W,Z,X,Y)$ $\forall X, Y, Z, W \in \mathcal{X}(M)$

Definition (Sectional curvature)

Let $S \subset T_pM$ be a two dimensional subspace of T_pM (M Riemmanian manifold). If $x, y \in S$ are two linearly independent vectors, one can calculate

$$K_p(x,y) := \frac{\mathcal{R}(x,y,x,y)}{|x \wedge y|^2}$$

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Proposition

The sectional curvature is independent of the choice of linearly independent vectors in S, so one can consider $K_p(S)$.

Algebraic Lemma

Lemma

Let $\mathcal{V}$ be a vector space of dimension at least 2, equipped with an inner product $\langle\cdot,\cdot\rangle$. Let $r, r' : \mathcal{V}\times\mathcal{V}\times\mathcal{V}\to\mathcal{V}$ be trilinear mappings which satisfy the symmetry conditions proved for R (and $\mathcal{R}$) and define

$\mathfrak{r}(x,y,z,w) := \langle r(x,y,z),w\rangle$ $k(\mathcal{S}) := \dfrac{\mathfrak{r}(x,y,x,y)}{|x\wedge y|^2}$

$\mathfrak{r}'(x,y,z,w) := \langle r'(x,y,z),w\rangle$ $k'(\mathcal{S}) := \dfrac{\mathfrak{r}'(x,y,x,y)}{|x\wedge y|^2}$ *where x, y are linearly independent vectors of $\mathcal{V}$ which span the two dimensional subspace $\mathcal{S}$ of $\mathcal{V}$.*

If $k(\mathcal{S}) = k'(\mathcal{S})$ for any two dimensional subspace $\mathcal{S}$ of $\mathcal{V}$, then $r = r'$ (and, equivalently, $\mathfrak{r} = \mathfrak{r}'$).

Costant sectional curvature

Proposition

Let M be a Riemannian manifold, $p \in M$ and $r : T_pM \times T_pM \times T_pM \to T_pM$ be the tri-linear mapping defined by

$$\langle r(x, y, z), w\rangle = \langle x, z\rangle \cdot \langle y, w\rangle - \langle y, z\rangle \cdot \langle x, w\rangle$$

with $x, y, z, w \in T_pM$. Then M has constant sectional curvature K_0 if and only if $R = K_0 r$ where R is the curvature of M.

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Proof.

If M has constant sectional curvature K_0 , then for x, y linearly independent vectors in T_pM

$$\mathcal{R}(x, y, x, y) = \langle R(x, y)x, y\rangle = K_0(|x \wedge y|^2) = K_0(|x|^2|y|^2 - \langle x, y\rangle^2).$$

□

Proof.

On the other hand,

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Remark

In coordinates, with respect to an orthonormal basis $(e_1,\dots,e_n)$ of T_pM , it turns out that a Riemannian manifold M has constant sectional curvature if and only if

$$R_{ijkl} = \langle R(e_i,e_j)e_k,e_l\rangle = K_0(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

Remark

Since the Kronecker delta is such that $\delta_{ij} = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$ *, it turns out that a Riemannian manifold* M *has constant sectional curvature if and only if* $R_{ijij} = -R_{ijji} = K_0$ *for* $i \neq j$ *and* $R_{ijkl} = 0$ *in all the other cases.*

Ricci and scalar curvature

Definition

Let M be a Riemannian manifold, $p \in M$. Consider in T_pM a unit vector x and an orthogonal basis $(e_1, \dots, e_{n-1})$ of the hyperplane in T_pM orhogonal to x and put $x = e_n$.

$$Ric_p(x) := \frac{1}{n-1}\sum_{i=1}^{n-1}\langle R(x, e_i)x, e_i\rangle =$$

$$= \frac{1}{n-1}\sum_{i=1}^{n-1}\mathcal{R}(x, e_i, x, e_i) = \frac{1}{n-1}\sum_{i=1}^{n-1}\mathcal{R}(e_n, e_i, e_n, e_i) = Ric_p(e_n)$$

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$$K(p) := \frac{1}{n(n-1)}\sum_{i,j=1}^{n}\langle R(e_j, e_i)e_j, e_i\rangle =$$

$$= \frac{1}{n(n-1)}\sum_{i,j=1}^{n}\mathcal{R}(e_j, e_i, e_j, e_i) = \frac{1}{n}\sum_{j=1}^{n} Ric_p(e_j)$$

Ricci and scalar curvature

For $x, y \in TpM$ consider the linear mapping $\Phi : \mathcal{X}(M) \to \mathcal{X}(M)$

$$\Phi(z) = R(x, z)y$$

and define

$$Q(x, y) = \mathrm{tr}(\Phi)$$

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- Q is symmetric.

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- Q is symmetric. If x is a unit vector in T_pM, consider the orthonormal basis $(e_1, \ldots, e_{n-1}, x = e_n)$ of T_pM.

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- Q is bilinear
- Q is symmetric. If x is a unit vector in T_pM, consider the orthonormal basis $(e_1, \ldots, e_{n-1}, x = e_n)$ of T_pM. Then, for any $y \in T_pM$,

$$Q(x, y) = \sum_{j=1}^{n} \langle R(x, e_j)y, e_j \rangle =$$

$$= \sum_{j=1}^{n} \mathcal{R}(x, e_j, y, e_j) = \sum_{j=1}^{n} \mathcal{R}(y, e_j, x, e_j) = Q(y, x)$$

Ricci and scalar curvature

In particular, we have

$$Q(x,x) = \sum_{j=1}^{n} \mathcal{R}(x, e_j, x, e_j) = (n-1)Ric_p(x)$$

or

$$\frac{1}{n-1} Q(x,x) = Ric_p(x).$$

Ricci and scalar curvature

Furthermore, there exists a self-adjoint linear operator $\mu : T_pM \to T_pM$ such that

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Taking an orthonormal basis $(e_1, \dots, e_n)$ of T_pM, we have

$$\mathrm{tr}(\mu) = \sum_{j=1}^{n} \langle \mu(e_j), e_j \rangle = \sum_{j=1}^{n} Q(e_j, e_j) = \sum_{j=1}^{n} (n-1) Ric_p(e_j) = n(n-1)K(p).$$