STATISTICAL METHODS WITH APPLICATION TO FINANCE

a.a. 2025-2026

Exercises Set 1- Solutions

1

$X_1, X_2, \ldots$ iid $N(0.06, 0.47)$

- Find the mean and standard deviation of $X_1+X_2+X_3$:

$$E(X_1+X_2+X_3) = \sum_i E(X_i) = 3 \times 0.06 = 0.18$$

$$SD(X_1+X_2+X_3) = \sqrt{V(\textstyle\sum_i X_i)} \underset{\text{ind.}}{=} \sqrt{V(X_1)+V(X_2)+V(X_3)} = \sqrt{3 \times 0.47} = 1.187$$

- Distribution of $e^{X_1+X_2}$

Let $Y = e^{X_1+X_2}$; since $X_i \sim N(0.06, 0.47)$ $(i=1,2)$ and X_i are independent, $X_1+X_2 \sim N(0.12, 0.94)$

Recall that if $X \sim N(\mu, \sigma^2)$, then $e^X \sim \text{LogN}(\mu, \sigma^2)$.

Hence $Y = e^{X_1+X_2} \sim \text{logN}(0.12, 0.94)$

- Find $P(X_1 < 1.5)$, with $X_1 \sim N(0.06, 0.47)$

$$P(X_1 < 1.5) = P\left(\frac{X_1-\mu_{X_1}}{\sqrt{\sigma_{X_1}^2}} < \frac{1.5-0.06}{\sqrt{0.47}}\right) = \Phi(2.1) = 0.982$$

- Find $Cov(X_1, X_1+X_2)$. Using the covariance properties, we get

$$Cov(X_1, X_1+X_2) = Cov(X_1, X_2) + Cov(X_1, X_1) = 0 + V(X_1) = 0.47.$$

2

log returns are normally distributed with mean 0.1 and standard deviation 0.13: $r_t \sim N(0.1, 0.13^2)$

- The distribution of one-period gross return is

$$1+R_t = e^{r_t} \sim \log N(0.1, 0.13^2)$$

- $P(1+R_t \geq 1.2) = 1 - P(1+R_t < 1.2)$

$$= 1 - P(\log(1+R_t) < \log(1.2))$$
$$= 1 - P(r_t < 0.18) = 1 - \Phi\left(\frac{0.18-0.1}{0.13}\right)$$
$$= 1 - \Phi(0.62) \approx 0.27$$

- What is the 0.85-quantile of $r_t(2)$?

$r_t(2) = r_t + r_{t-1} \implies r_t(2) \sim N(0.2, 0.034)$

since the $r_t \sim iid\ N(0.1, 0.017)$

We want to find $q_{0.85}$: $\Phi\left(\frac{q_{0.85} - 0.2}{\sqrt{0.034}}\right) = 0.85$

$$\implies q_{0.85} = \Phi^{-1}(0.85) \cdot \sqrt{0.034} + 0.2 = 1.032\sqrt{0.034} + 0.2$$
$$= 0.39$$

3

Daily log returns $r_t \overset{iid}{\sim} N(0.001, 0.015^2)$

- $P(r_t < 0.02) = P\left(\frac{r_t - \mu}{\sigma} < \frac{0.02 - 0.001}{0.015}\right) = \Phi(1.27) = 0.898$
- $SD(r_1+r_2+r_3) \overset{iid}{=} \sqrt{3V(r_t)} = \sqrt{3} \cdot 0.015 = 0.026$
- five-day log return: $r_t(5) = \sum_{i=0}^{4} r_{t-i}$

Given that $r_t \overset{iid}{\sim} N(0.001, 0.015^2)$ we get

$$r_t(5) \sim N(0.005, 0.001125)$$

- Let $X_0 = 1000$. What is $P(X_0(1+R_t(5)) < 990)$?

$$P(X_0(1+R_t(5)) < 990) = P(1+R_t(5) < 0.99)$$
$$= P(\log(1+R_t(5)) < \log(0.99))$$
$$= P(r_t(5) < -0.01) = \Phi(-0.45)$$
$$= 1 - \Phi(0.45) = 0.3264$$

4

Prices are assumed to follow a lognormal geometric random walk $(\mu = 0.15, \sigma^2 = 0.04)$

- 5-year expected log return : $E(r_5(5))$

  $r_5(5) = r_1 + r_2 + r_3 + r_4 + r_5$ and $r_i \overset{iid}{\sim} N(0.15, 0.04)$

  $\Rightarrow E(r_5(5)) = 5 \cdot 0.15 = \boxed{0.75}$

- 10-year median log return $me(r_{10}(10))$

  $r_{10}(10) \sim N(1.5, 0.4) \Rightarrow \boxed{me(r_{10}(10)) = 1.5}$

- 10-year median gross return $me(1+R_t(10))$

  $k=10$ gross return $1+R_t(10) \sim \log N(1.5, 0.4)$

  Let $q_{0.5}$ be the 50th percentile (median) of $Y = 1+R_t(10)$

  $$0.5 = P(Y \le q_{0.5}) = P(\log(Y) \le \log(q_{0.5}))$$
  $$= P(X \le \log(q_{0.5})) \quad , X \sim N(1.5, 0.4)$$
  $$= \Phi\left(\frac{\log(q_{0.5}) - \mu_X}{\sigma_X} \right)$$

  $$\Rightarrow 0 = \Phi^{-1}(0.5) = \frac{\log(q_{0.5}) - 1.5}{\sqrt{0.4}} \Rightarrow \boxed{q_{0.5} = e^{1.5} = 4.48}$$

  In general, $me(1+R_k(k)) = e^{k\mu}$, where μ is the log-mean of the gross return $1+R_t = e^{r_t}$.

- Let P_0 be the initial price at time $t=0$, $r_t \overset{iid}{\sim} N(\mu, \sigma^2)$.

  Given that $P_k = P_0 e^{r_1 + \dots + r_k} = P_0(1+R_k(k))$

  - median price after k years $= P_0 e^{k\mu}$
  - mean price after k years $= P_0 E(1+R_k(k))$

  Since $1+R_k(k) \sim \log N(k\mu, k\sigma^2)$ we have

  $$E(1+R_k(k)) = \exp\left(k\mu + \frac{k\sigma^2}{2}\right)$$

  $\Rightarrow$ mean price of P_k is $P_0 E(1+R_k(k)) = P_0 e^{k\mu + k\sigma^2/2}$

  $= 100\, e^{10 \cdot 0.15 + 10 \cdot 0.04}$

The model for log prices is

$$p_t = p_0 + r_t + r_{t-1} + \dots + r_1 \quad , \text{ where } p_t := \log(P_t)$$

and $r_i \overset{iid}{\sim} N(\mu, \sigma^2)$ are log-returns.

5

(a) $P(X_2 > 1.3\,X_0) = P(X_0 e^{r_1+r_2} > 1.3\,X_0)$

$= P(r_1+r_2 > \log(1.3))$

$Y = r_1 + r_2 \sim N(2\mu, 2\sigma^2)$

$\mu = 0.1,\ \sigma^2 = 0.15^2$

$$P(Y > \log(1.3)) = 1 - \Phi\left(\frac{\log(1.3) - 2\times 0.1}{\sqrt{2}\,(0.15)}\right)$$

$= 1 - \Phi(0.29)$

$= 0.386$

(b) Find $q_{0.25}$ such that

$P(X_k \leq q_{0.25}) = P(X_0 e^{r_1+\dots+r_k} \leq q_{0.25})$

$= P\left(e^{r_1+\dots+r_k} \leq \frac{q_{0.25}}{X_0}\right)$

$= P(r_1+\dots+r_k \leq \log(q_{0.25}/X_0))$

$r_1 + \dots + r_k \sim N(k\mu, k\sigma^2)$

$$\Rightarrow 0.25 = P(X_k \leq q_{0.25}) = \Phi\left(\frac{\log\left(\frac{q_{0.25}}{X_0}\right) - k\mu}{\sqrt{k\sigma^2}}\right)$$

$$\Longleftrightarrow \frac{\log\left(\frac{q_{0.25}}{X_0}\right) - k\mu}{\sqrt{k\sigma^2}} = \Phi^{-1}(0.25) = z_{0.25}$$

$= -0.67$

$$\Rightarrow \quad \log\left(\frac{q_{0.25}}{x_0}\right) - k\mu = -0.67\sqrt{k}\,\sigma$$

$$\log\left(\frac{q_{0.25}}{x_0}\right) = k\mu - 0.67\sqrt{k}\,\sigma$$

$$q_{0.25} = X_0 \exp\left(k\mu - 0.67\sqrt{k}\,\sigma\right)$$

(C) $$E(X_k) = X_0\, E\left(e^{r_1 + \ldots + r_k}\right)$$

$$e^{r_1 + \ldots + r_k} \sim \log N\left(k\mu, k\sigma^2\right)$$

$$E\left(e^{r_1 + \ldots + r_k}\right) = e^{k\mu + k\sigma^2/2}$$

$$E(X_k) = X_0 \cdot e^{k\mu + k\sigma^2/2}$$