(PRINCIPAL COORDINATES AND DECOUPLING) ASM2, 31/3/26

--- $\ddot{z}_1 + \omega_1^2 z_1 = 0 \longrightarrow \omega_1^2 = \dfrac{\hat{k}_1}{\hat{m}_1} = \dfrac{\underline{\phi}^{(1)} \cdot \underset{\sim}{K}\,\underline{\phi}^{(1)}}{\underline{\phi}^{(1)} \cdot \underset{\sim}{M}\,\underline{\phi}^{(1)}}$

RAYLEIGH RATIO $\Rightarrow$ IMPORTANT BECAUSE IT CAN EXPLOITED TO ESTIMATE THE NATURAL CIRCULAR FREQUENCY IN EXPERIMENTAL APPROACHES.

$\underline{u}(t) = \underset{\sim}{\Phi}\,\underline{z}(t)$

PRINCIPAL COORDINATES AND DECOUPLING WITH EXTERNAL FORCES (FORZANTI ESTERNE)

$$\underset{\sim}{M}\,\underline{\ddot{u}}(t) + \underset{\sim}{K}\,\underline{u}(t) = \overbrace{\underline{F}(t)}^{\text{GIVEN}}$$

LET US FOCUS ON $\underline{F}(t) = \begin{bmatrix} F_1 \\ \vdots \\ F_N \end{bmatrix} \sin \bar{\omega} t$

$$\underset{\sim}{M}\,\underset{\sim}{\Phi}\,\underline{\ddot{z}} + \underset{\sim}{K}\,\underset{\sim}{\Phi}\,\underline{z} = \underline{F}(t)$$

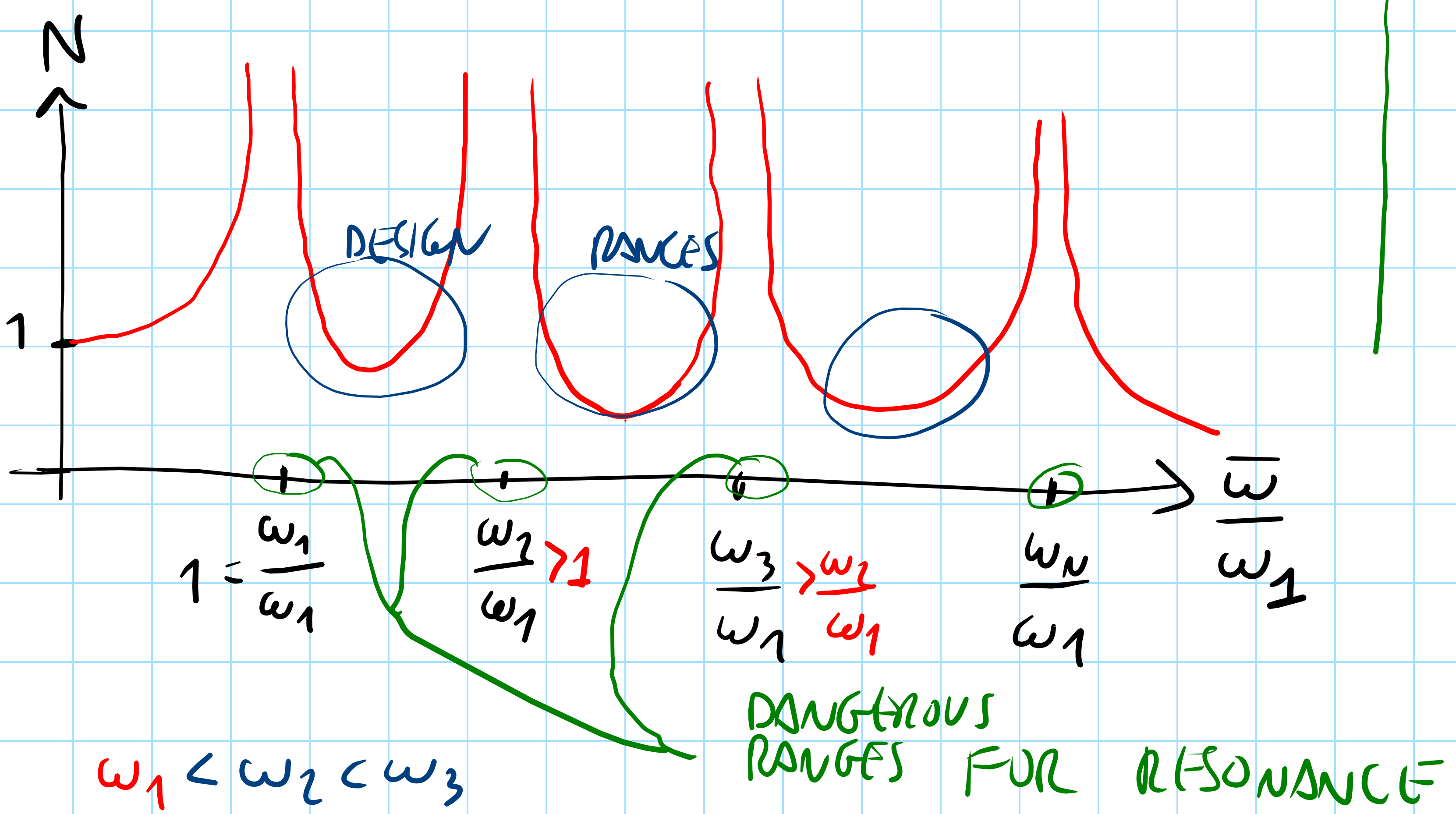
$$\underbrace{\underset{\sim}{\Phi}^T \underset{\sim}{M}\,\underset{\sim}{\Phi}}_{\hat{\underset{\sim}{M}}}\,\underline{\ddot{z}} + \underbrace{\underset{\sim}{\Phi}^T \underset{\sim}{K}\,\underset{\sim}{\Phi}}_{\hat{\underset{\sim}{K}}}\,\underline{z} = \overbrace{\underset{\sim}{\Phi}^T \underline{F}(t)}^{\hat{\underline{F}}} \longrightarrow \hat{\underset{\sim}{M}}\,\underline{\ddot{z}} + \hat{\underset{\sim}{K}}\,\underline{z} = \hat{\underline{F}}(t)$$

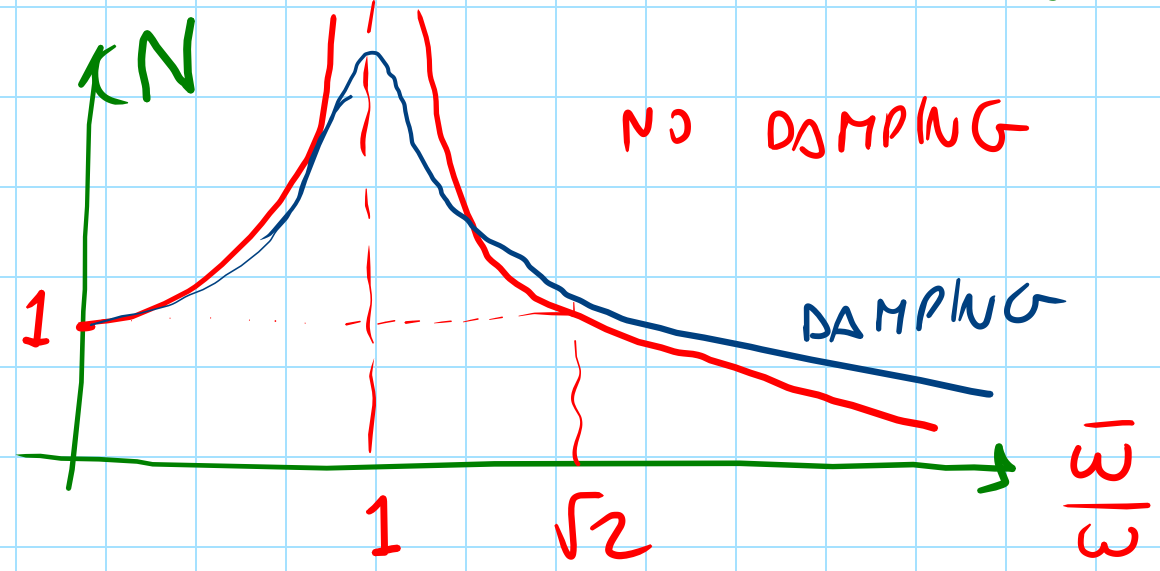
$$\begin{cases} \hat{m}_1 \ddot{z}_1 + \hat{k}_1 z_1 = \hat{F}_1(t) \\ \vdots \\ \hat{m}_N \ddot{z}_N + \hat{k}_N z_N = \hat{F}_N(t) \end{cases}$$

N INDEPENDENT RESONATORS

EACH EQ CAN BE STUDIED INDEPENDENTLY ; CHECK THE RESONANCE PROBLEM IN S.D.O.F OSCILLATOR.

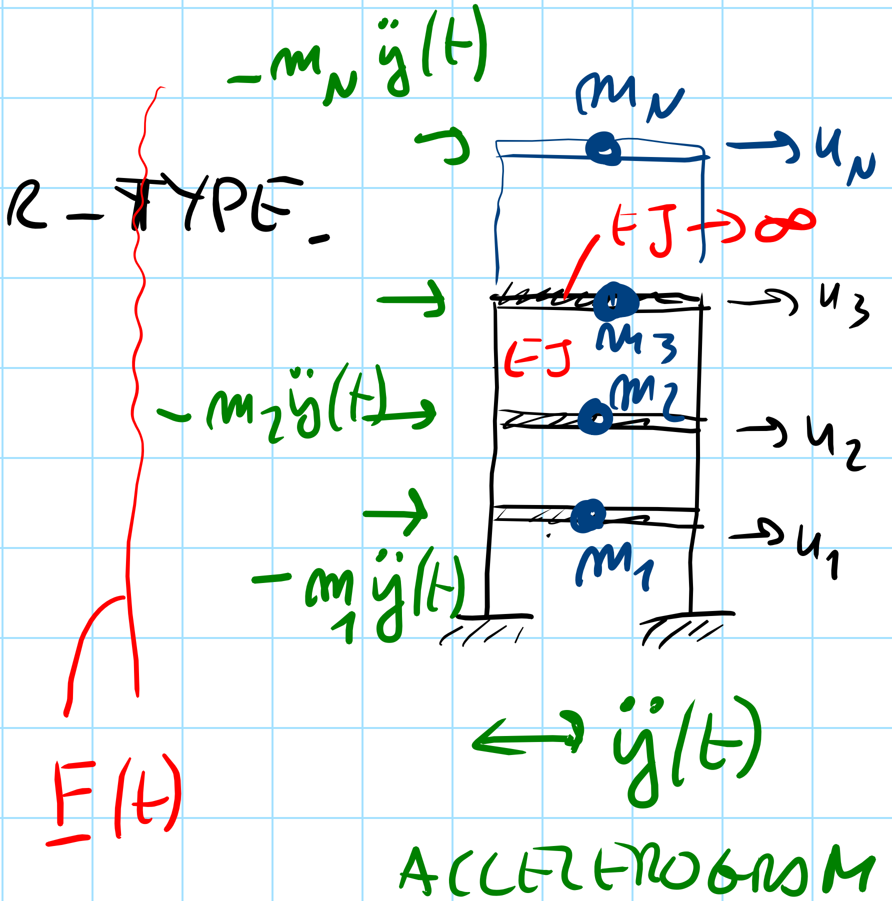
RECALL AMPLIFICATION DIAGRAM IN S.D.O.F. RESONANCE PROBLEM





– IMPOSED GROUND MOTION

IMAGINE, AS AN INTRODUCTION, THAT THE FRAME IS SHEAR-TYPE.
(N: RIGID FLOOR) N: DOFS (ONLY DISPLACEMENTS)



$$\hat{\underline{F}}(t) = -\underset{\sim}{\underline{\Phi}}^T \underset{\sim}{M}\, \underline{1}\, \ddot{y}(t)$$

$$\underline{F}(t) = -\underset{\sim}{M}\, \underline{1}\, \ddot{y}(t)$$

UNITARY VECTOR

IN THE SYSTEM EXPRESSED IN TERMS OF PRINCIP. COORD. $\underline{Z}(t)$:

$$\underset{\sim}{M} = \begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_N \end{bmatrix}$$

$$\hat{m}_1 \ddot{z}_1 + \hat{K}_1 z_1 = -\underline{\phi}^{(1)} \cdot \underset{\sim}{M}\, \underline{1}\, \ddot{y}(t)$$

$$\vdots$$

$$\hat{m}_N \ddot{z}_N + \hat{K}_N z_N = -\underline{\phi}^{(N)} \cdot \underset{\sim}{M}\, \underline{1}\, \ddot{y}(t)$$

$$m(\ddot{x} + \ddot{y}) + Kx = 0$$

$$m\ddot{x} + Kx = \underbrace{-m\ddot{y}(t)}_{F(t)}$$

$$\ddot{z}_1 + \omega_1^2 z_1 = -\frac{\underline{\phi}^{(1)} \cdot \underset{\sim}{M}\, \underline{1}}{\hat{m}_1}\, \ddot{y}(t)$$

MODAL PARTICIPATION FACTOR ($g_1, \Gamma_1 \dots$) OF THE 1^{ST} MODE

: WEIGHT THE "MASS" CONTRIBUTION OF THE 1^{ST} MODE TO THE OSCILLATION

NOTE

$$\underset{\sim}{\Phi}^T \underset{\sim}{M}\, \underline{1} \rightarrow \begin{bmatrix} \phi_1^{(1)} & \phi_2^{(1)} & \cdots & \phi_N^{(1)} \\ & & & \end{bmatrix} \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \phi_1^{(1)} & \phi_2^{(1)} & \cdots & \phi_N^{(1)} \\ & & & \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix}$$

→ THE FIRST COMPONENT $\hat{F}_1 \sim \phi_1^{(1)} m_1 + \phi_2^{(1)} m_2 + \cdots + \phi_N^{(1)} m_N = \underline{\phi}^{(1)} \cdot \underset{\sim}{M}\, \underline{1}$

$\hat{F}_2 \qquad = \underline{\phi}^{(2)} \cdot \underset{\sim}{M}\, \underline{1}$

LET US INVESTIGATE MORE ABOUT g_i ($\underset{\sim}{M}$: DIAGONAL)

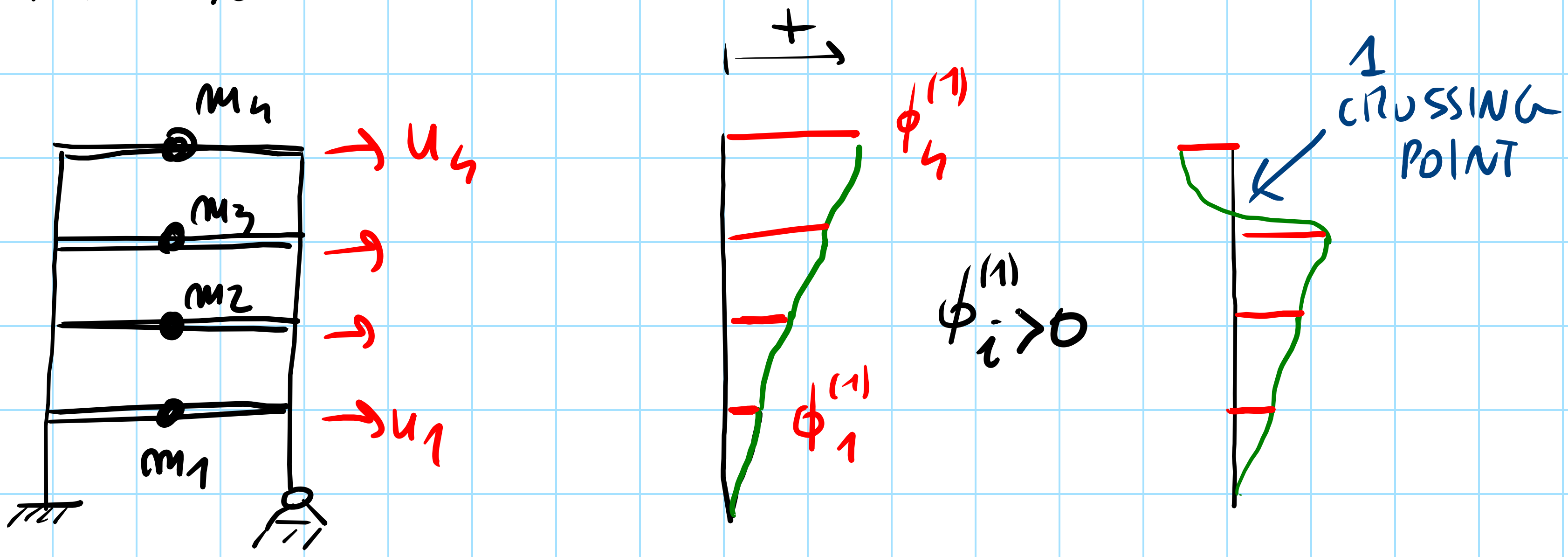
$\phi_i^{(1)} \gtrless 0$

$$g_1 = \frac{m_1 \phi_1^{(1)} + m_2 \phi_2^{(1)} + \cdots + m_N \phi_N^{(1)}}{m_1 \phi_1^{(1)} \phi_1^{(1)} + m_2 \phi_2^{(2)} \phi_2^{(2)} + \cdots + m_N \phi_N^{(1)} \phi_N^{(1)}} \quad > 0$$

g_i DEPENDS STRONGLY ON THE SIGN OF THE COMPONENTS OF $\underline{\phi}^{(1)}$

IN A STRUCTURE, IT IS IMPORTANT THE COMPARISON BETWEEN ALL g_i

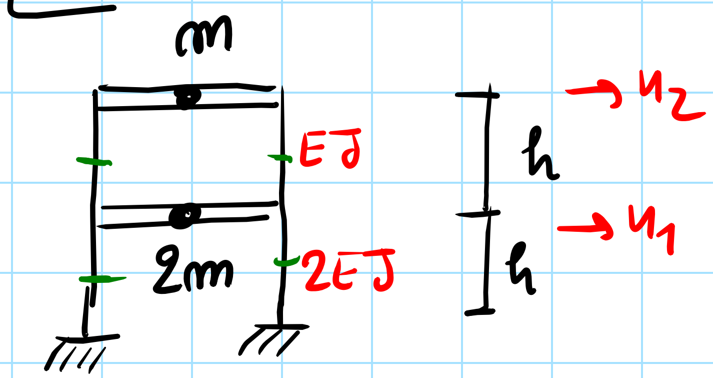
VIBRATION MODES OF SHEAR-TYPE FRAMES



$\underline{\phi}^{(1)}, \omega_1$ $\underline{\phi}^{(2)}, \omega_2$ $\underline{\phi}^{(3)}, \omega_3$ $\underline{\phi}^{(4)}, \omega_4$

$g_1 > g_2 > g_3 > g_4$

EX



$$\underset{\sim}{M} = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}$$

$$\underset{\sim}{K} = \overset{k}{\frac{24EJ}{h^3}} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

EIGENVECTOR ANALYSIS

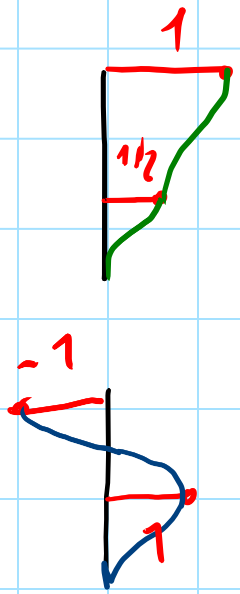
$$\left(\underset{\sim}{K} - \omega^2 \underset{\sim}{M}\right) \underline{\phi} = \underline{0}$$

$$\begin{cases} \omega_1^2 = \frac{1}{2} \frac{k}{m} \\ \omega_2^2 = 2 \frac{k}{m} \end{cases}$$

NATURAL CIRC. FREQ.

$\Rightarrow$ 1ST MODE $\underline{\phi}^{(1)} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

$\Rightarrow$ 2ND MODE $\underline{\phi}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

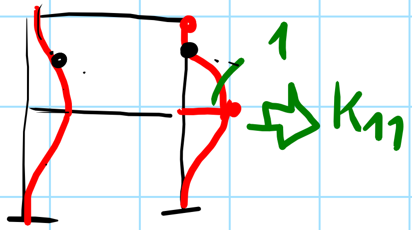


EIGENVECTORS NORMALIZED WITH MAX $\phi_j = 1$

$$\underset{\sim}{\Phi} = \begin{bmatrix} 1/2 & 1 \\ 1 & -1 \end{bmatrix} \quad \left(= \underset{\sim}{\Phi}^T\right)$$

$$\underline{u} = \underset{\sim}{\Phi}\, \underline{z}$$

k_{11}



$$= 12\frac{EJ}{h^3} \cdot 2 + 12 \cdot \frac{2EJ}{h^3} \cdot 2$$

$$= 24\frac{EJ}{h^3} \cdot 3$$

$$\hat{\underset{\sim}{M}} = \underset{\sim}{\Phi}^T \underset{\sim}{M} \underset{\sim}{\Phi} = \begin{bmatrix} 1/2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 1/2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} m & 2m \\ m & -m \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}m + m & 0 \\ 0 & 2m + m \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2}m & 0 \\ 0 & 3m \end{bmatrix}$$

$$\hat{\underset{\sim}{K}} = \underset{\sim}{\Phi}^T \underset{\sim}{K}\, \underset{\sim}{\Phi} = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 4 \\ +1/2 & -2 \end{bmatrix} k$$

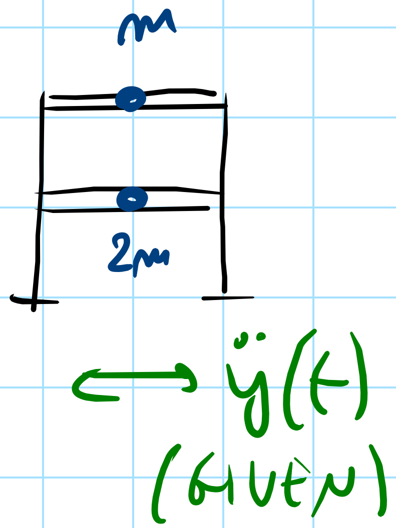
$$= \begin{bmatrix} 3/4 & 0 \\ 0 & 6 \end{bmatrix} k$$

SYSTEM WITH PRINCIP. COORDINATES:

$$\begin{cases} \overbrace{\frac{3}{2}m}^{\hat{m}_1} \ddot{z}_1 + \overbrace{\frac{3}{4}k}^{\hat{k}_1} z_1 = 0 & \Longrightarrow \omega_1^2 \; ?? \\ \underbrace{3m}_{\hat{m}_2} \ddot{z}_2 + \underbrace{6k}_{\hat{k}_2} z_2 = 0 & \Longrightarrow \omega_2^2 \; ?? \end{cases}$$

$$\Omega_1^2 = \frac{\hat{k}_1}{\hat{m}_1} = \frac{3}{4}k \cdot \frac{2}{3m} = \frac{1}{2}\frac{k}{m} \boxed{\overset{!}{=} \omega_1^2}$$

$$\Omega_2^2 = \frac{\hat{k}_2}{\hat{m}_2} = \frac{6}{3}\frac{k}{m} = 2\frac{k}{m} \boxed{\overset{!}{=} \omega_2^2}$$



$$\underline{F}(t) = -\begin{bmatrix} 2 \\ 1 \end{bmatrix} m\,\ddot{y}(t) \quad ; \quad \hat{\underline{F}}(t) = \underset{\sim}{\Phi}^T \underline{F}(t) = -\begin{bmatrix} \frac{1}{2} & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} m\,\ddot{y}(t)$$

$$= -\begin{bmatrix} 2 \\ 1 \end{bmatrix} m\,\ddot{y}(t)$$

IN THE PRINCIPAL SYSTEM:

$$\begin{cases} \frac{3}{2} m \ddot{z}_1 + \frac{3}{4} K z_1 = -2m\,\ddot{y}(t) \\ 3m\ddot{z}_2 + 6K z_2 = -m\ddot{y}(t) \end{cases} \Longrightarrow g_1, g_2 \, ?$$

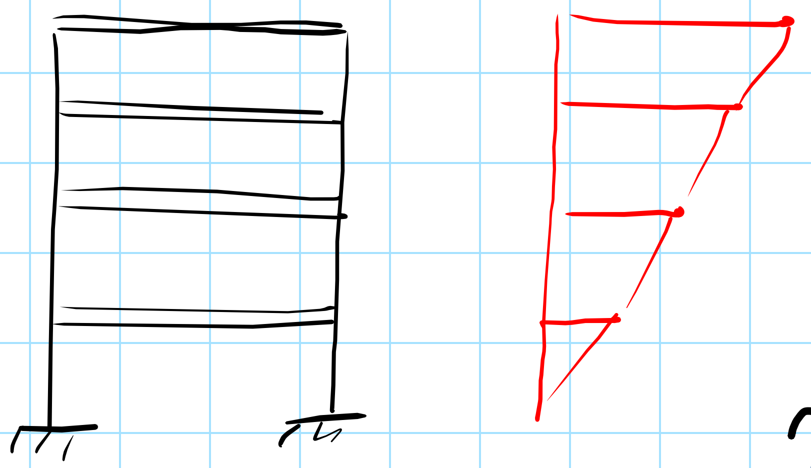
$$\begin{cases} \ddot{z}_1 + \omega_1^2 z_1 = -2m \cdot \frac{2}{3m}\,\ddot{y}(t) \\ \ddot{z}_2 + \omega_2^2 z_2 = -\frac{m}{3m}\,\ddot{y}(t) \end{cases} \Longrightarrow \begin{array}{l} \ddot{z}_1 + \omega_1^2 z_1 = -\underbrace{\frac{4}{3}}_{g_1}\,\ddot{y}(t) \\ \ddot{z}_2 + \omega_2^2 z_2 = -\underbrace{\frac{1}{3}}_{g_2}\,\ddot{y}(t) \end{array}$$

NOTE THESE EQS ARE INDEP. OF NORMALIZATION OF EIGENVECTORS

HOW TO USE RAYLEIGH RATIO "EXPERIMENTALLY"

$$\omega_1^2 = \frac{\underline{\phi}^{(1)} \cdot \underset{\sim}{K}\, \underline{\phi}^{(1)}}{\underline{\phi}^{(1)} \cdot \underset{\sim}{M}\, \underline{\phi}^{(1)}}$$

THIS RATIO CAN BE USED TO ESTIMATE ω_1 FOR A STRUCTURE IF WE "GUESS" THE COMPONENTS OF $\underline{\phi}^{(1)}$.

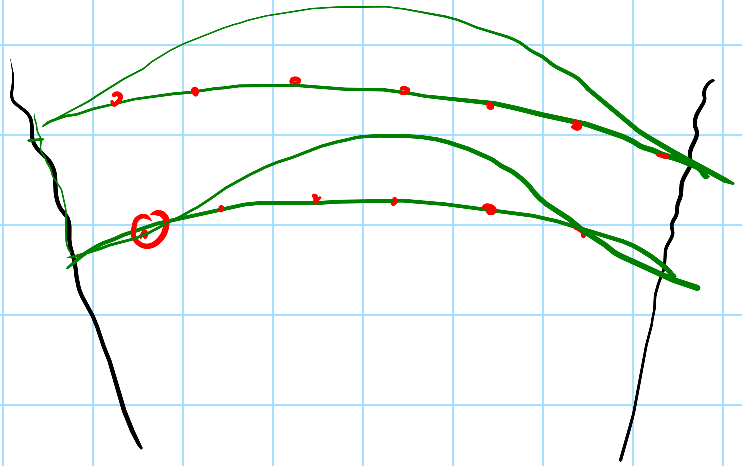


GUESS OF $\underline{\tilde{\phi}}^{(1)}$

(EXPERIENCE / EXPERIMENT. INVESTIGATION)

GUESS $\underset{\sim}{K}$, $\underset{\sim}{M}$

$$\omega_1^{APPROX^2} = \frac{\underline{\tilde{\phi}}^{(1)} \cdot \underset{\sim}{K}\, \underline{\tilde{\phi}}^{(1)}}{\underline{\tilde{\phi}}^{(1)} \cdot \underset{\sim}{M}\, \underline{\tilde{\phi}}^{(1)}}$$



$$\omega_1^{APPROX^2} = \frac{\tilde{\underline{\phi}}^{(1)} \cdot \underset{\sim}{K}\, \tilde{\underline{\phi}}^{(1)}}{\hat{\underline{\phi}}^{(1)} \cdot \underset{\sim}{M}\, \underline{\phi}^{(1)}}$$

DAM FROM THE TOP

⊙ SENSORS FEELING ENVIRONMENTAL VIBRATIONS

↳ FROM DATA ANALYSIS EXTRACT APPROX 1^{ST} VIBR MODE $\Rightarrow \tilde{\underline{\phi}}^{(1)}$

(OPERATIONAL MODAL ANALYSIS: OMA)

→ IN CONTRAST TO EXPERIMENTAL MODAL ANALYSIS (EMA) IN WHICH VIBRATIONS ARE FORCED THROUGH AN EXTERNAL MACHINE