LET US CONTINUE TO SEE SOME TOPICS IN DYNAMICS

ASM2, 21/4/26

## – ORTHOGONALITY OF VIBRATION MODES

$\boxed{i \neq j}$ $\quad \underline{\phi}^{(i)} \cdot \underset{\sim}{M} \underline{\phi}^{(j)} = \underline{\phi}^{(i)} \cdot \underset{\sim}{K} \underline{\phi}^{(j)} = 0$

$\quad (*) \qquad (**)$

**PROOF**

$(\underset{\sim}{K} - \omega_i^2 \underset{\sim}{M}) \underline{\phi}^{(i)} = \underline{0} \Rightarrow \underset{\sim}{K} \underline{\phi}^{(i)} = \omega_i^2 \underset{\sim}{M} \underline{\phi}^{(i)}$

$(\mathrm{I}) \quad \underline{\phi}^{(j)} \cdot \underset{\sim}{K} \underline{\phi}^{(i)} = \underline{\phi}^{(j)} \cdot \underset{\sim}{M} \underline{\phi}^{(i)} \omega_i^2$

$(\underset{\sim}{K} - \omega_j^2 \underset{\sim}{M}) \underline{\phi}^{(j)} = \underline{0} \Rightarrow \underline{\phi}^{(i)} \cdot \underset{\sim}{K} \underline{\phi}^{(j)} = \underline{\phi}^{(i)} \cdot \underset{\sim}{M} \underline{\phi}^{(j)} \omega_j^2 \quad (\mathrm{II})$

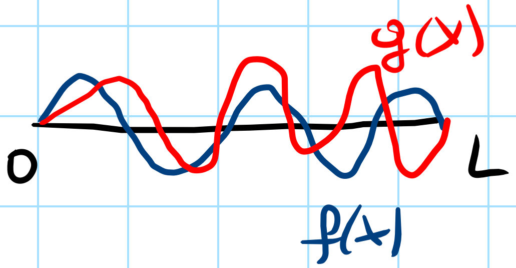
AS $\underset{\sim}{K}, \underset{\sim}{M}$ ARE SYMMETRIC, WE CAN SUBTRACT (II) TO (I)

OBTAINING

$0 = (\omega_i^2 - \omega_j^2)\, \underline{\phi}^{(i)} \cdot \underset{\sim}{M} \underline{\phi}^{(j)} \Rightarrow$ IF $\omega_i^2 \neq \omega_j^2$

THEN $\underline{\phi}^{(i)} \cdot \underset{\sim}{M} \underline{\phi}^{(j)} = 0 \quad (*)$ | IF I PUT $(*)$ IN (I) $\Rightarrow \underline{\phi}^{(j)} \cdot \underset{\sim}{K} \underline{\phi}^{(i)} = 0 \quad (**)$ ∎

FOR FUNCTIONS




$f(x), g(x)$ ARE ORTHOG. IF

$\int_0^L fg = 0$

CONCEPT OF ORTHOGONALITY

2 VECTORS $\underline{u}, \underline{v}$

$\Rightarrow \underline{u} \cdot \underline{v} = 0 \qquad (\underline{u} \perp \underline{v})$

$(\underline{u} \cdot \underset{\sim}{I}\, \underline{v} = 0)$

IDENTITY MATRIX

GENERALIZATION:

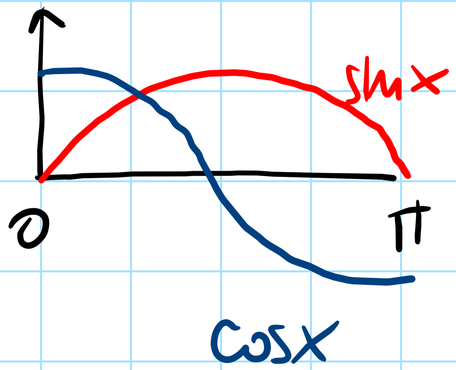
2 VECTORS ARE $\underset{\sim}{A}$-ORTHOG.

IF $\underline{u} \cdot \underset{\sim}{A} \underline{v} = 0 \qquad (\underset{\sim}{A} = \underset{\sim}{A}^T)$

$(\underset{\sim}{A} \underline{u} \cdot \underline{v} = 0)$

$\underline{u} \overset{A}{\perp} \underline{v}$

EX




$\int_0^\pi \sin x \cos x \, dx = 0$

sin x, cos x are ORTH.

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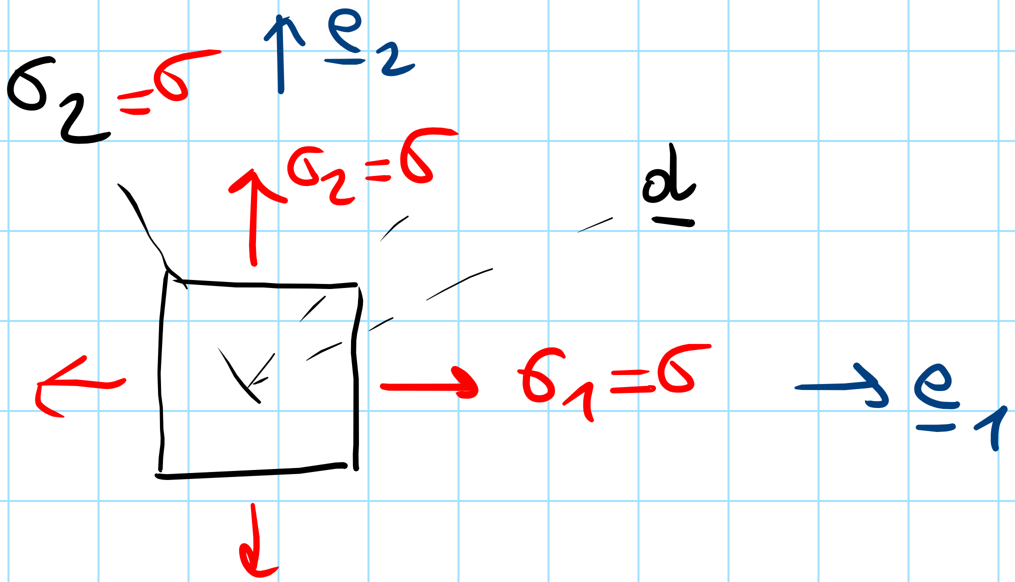
EX: 2 EIGENV. COINCIDENT

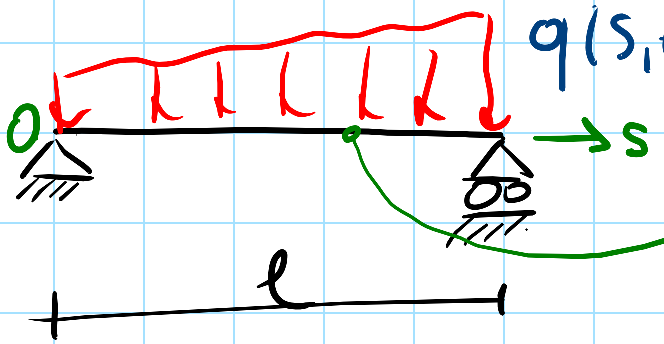
(2 EQUAL PRINCIPAL STRESSES)

$\sigma_1 = \sigma_2 = \sigma$



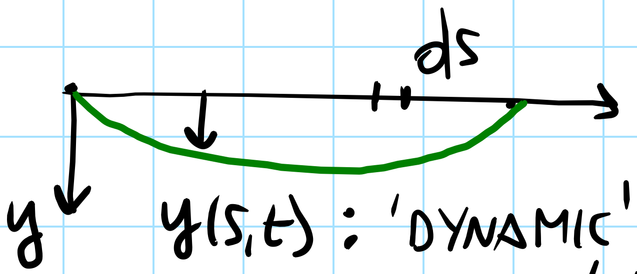

NOT ONLY $\underline{e}_1$ AND $\underline{e}_2$ ARE PRINCIPAL DIRECTIONS, BUT ALL DIRECTIONS $\underline{d} = \alpha \underline{e}_1 + \beta \underline{e}_2$ ARE PRINCIPAL DIR.

# DYNAMICS OF BEAMS AS A GENERALIZED 1 DOF SYSTEM




t: TIME, q(s,t): EXTERNAL FORCE

EJ, μ: MASS PER UNIT LENGTH




y(s,t): 'DYNAMIC' ELASTIC LINE (UNKNOWN!)

$$y(s,t) = \underbrace{\psi(s)}_{\text{SHAPE FUNCTION}} \, \underbrace{x(t)}_{\text{TIME VARIABLE}}$$

$$\dot{y}(s,t) = \psi(s)\,\dot{x}(t) \quad \text{(MAP OF VELOCITY)}$$

$$\kappa(s,t) = -y''(s,t) = -\psi''(s)\,x(t) \quad \text{(CURVATURE)}$$

$$M(s,t) = -EJ\,y'' = -EJ\,\psi''(s)\,x(t) \quad \text{(BENDING MOM)}$$

FUND. EQ. TO CONSIDER FOR THE PR.

– BALANCE OF POWERS

$$\underbrace{\frac{d}{dt}\left[E_{kin} + E_{elast.}\right]}_{\text{INTERNAL POWER}} = \underbrace{\text{POWER OF THE LOAD}}_{\text{EXTERNAL POWER}}$$

$$E_{kin} = \frac{1}{2}\int_0^l \mu\,ds\,\dot{y}^2 = \frac{1}{2}\underbrace{\int_0^l \mu\,\psi^2(s)\,ds}_{\text{EQUIV. MASS OF THE BEAM}: m_{eq}}\,\dot{x}^2(t)$$

$$E_{elast.} = \frac{1}{2}\int_0^l M\kappa\,ds = \frac{1}{2}\int_0^l (-EJ\psi''x)(-\psi''x)\,ds$$

$$= \frac{1}{2}\underbrace{\int_0^l EJ\,\psi''(s)^2\,ds}_{K_{eq}: \text{ EQUIV. STIFFNESS}}\,x^2(t)$$

POWER OF THE LOAD ( FORCE F AND VELOCITY v : POWER Fv)

$$\text{POWER}(q(s,t)) = \int_0^\ell q \, ds \, \dot{y} = \underbrace{\int_0^\ell q(s,t)\,\psi(s)\,ds}_{F_{eq}\ :\ \text{EQUIV. FORCE } (F_{eq}(t))} \ \dot{x}(t)$$

BALANCE OF POWERS:

$$\frac{d}{dt}\left[\frac{1}{2} m_{eq}\,\dot{x}(t)^2 + \frac{1}{2} K_{eq}\, x(t)^2\right] = F_{eq}\,\dot{x}(t)$$

$$\cancel{\frac{1}{2}} m_{eq}\, \cancel{2}\, \cancel{\dot{x}(t)}\, \ddot{x}(t) + \cancel{\frac{1}{2}} K_{eq}\, \cancel{2} x(t)\, \cancel{\dot{x}(t)} = F_{eq}\, \cancel{\dot{x}(t)}$$

$$\boxed{m_{eq}\,\ddot{x}(t) + K_{eq}\,x(t) = F_{eq}(t)}$$

EQ. OF 1 DOF SYSTEM ($x(t)$ IS THE UNKNOWN !)

IN ORDER TO GET $x(t)$ : 1°: CHOOSE $\psi(s)$, 2°: COMPUTE $m_{eq}, K_{eq}, F_{eq}$,
3°: SOLVE FOR $x(t)$, 4°: SOLUTION $y(s,t) = \psi(s)\,x(t)$

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IN THE CASE OF FREE OSCILLATIONS, $F_{eq}=0$

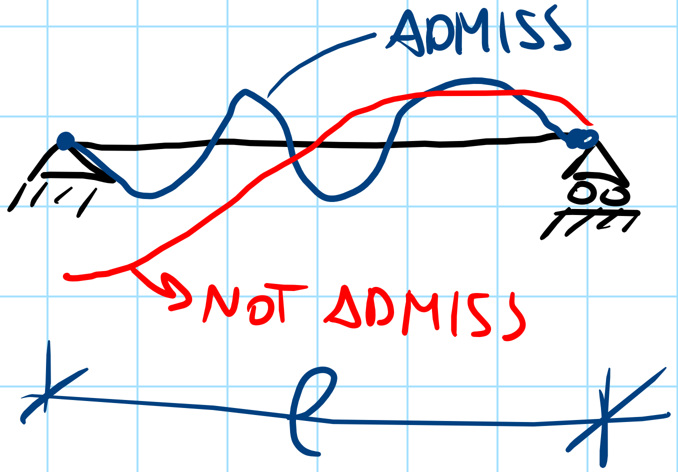
$$m_{eq}\ddot{x}(t)+K_{eq}\,x(t)=0 \longrightarrow \omega=\sqrt{\frac{K_{eq}}{m_{eq}}}=\frac{\sqrt{\int_0^\ell EJ\,\psi''^2\,ds}}{\sqrt{\int_0^\ell \mu\,\psi^2\,ds}}$$

RAYLEIGH RATIO

WE CAN ESTIMATE $\omega$ (NATURAL FREQUENCY) BY SELECTING A SUITABLE $\psi(s)$ (EXPERIENCE WILL LEAD TO THE CHOICE)

($\psi(s)$ PLAYS THE ROLE OF $\underline{\phi}^{(i)}$ IN THE "DISCRETE" VERSION OF THE RAYLEIGH RATIO)

$\psi(s)$ MUST SATISFY CONSTRAINTS IN THE STRUCTURE (GEOMETRICAL CONDITIONS)




$\psi(0)=\psi(\ell)=0$

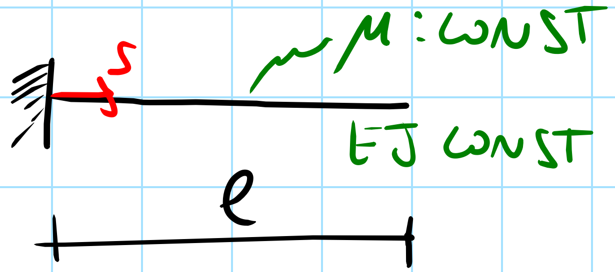
MUST BE SATISFIED

THE CONDITIONS

$\psi''(0)=\psi''(\ell)=0$ (ABSENCE OF CONCENTR MOMENT)

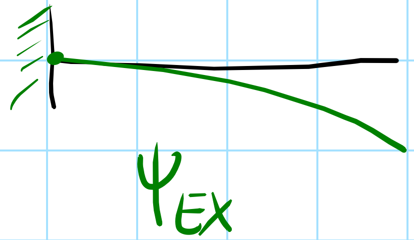
ARE NOT MANDATORY FOR SELECTION AN ADMISS $\psi(s)$

EX




($\omega$? : NATURAL FREQUENCY (LOW FREQUENCY))

EXACT SOLUT. $\omega_{EX} = \frac{3.516}{\ell^2}\sqrt{\frac{EJ}{\mu}}$




$\psi_{EX}$ : ACTUAL VIBRATION MODE

$\psi_{EX} = f\left(\cos\left(\frac{s}{\ell}\right), \sin(\ ), \cosh(\ ), \sinh(\ )\right)$

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1° APPLICATION OF THE GENERALIZED 1 DOF APPROACH

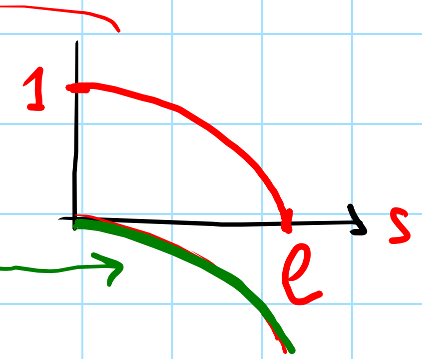
$\psi_1(s) = \frac{3s^2}{2\ell^2} - \frac{s^3}{2\ell^3}$ [IS $\psi_1(s)$ ADMISS ?] $\psi_1(0) = \psi_1'(0) \overset{?}{=} 0$

OK

$\psi_1'(s) = \frac{3s}{\ell^2} - \frac{3s^2}{2\ell^3} \Rightarrow \psi_1'(0) = 0$

$m_{eq}^{(1)} = 0.236\,\mu\ell$ ; $K_{eq}^{(1)} = \frac{3EJ}{\ell^3} \Longrightarrow \omega^{(1)} = \frac{3.568}{\ell^2}\sqrt{\frac{EJ}{\mu}}$ ERROR OF 1.5%

2° APPLICATION $\psi_2(s) = 1 - \cos\frac{\pi s}{2\ell}$




$\psi_2(0)=0$

$\psi_2' = \sin\frac{\pi s}{2\ell}\cdot\frac{\pi}{2\ell}\,,\quad \psi_2'(0)=0$

ADMISS CHECKED

$m_{eq}^{(2)} = 0.227\,\mu\ell$

$K_{eq}^{(2)} = 3.044\,\frac{EJ}{\ell^3}$

$\Rightarrow \omega^{(2)} = \frac{3.664}{\ell^2}\sqrt{\frac{EJ}{\mu}}$

ERROR 4%

NOTE: CHECK DIMENS OF $\omega$

$$[\omega] = \frac{1}{L^2}\sqrt{\frac{FL^2\,L}{M}} = \frac{1}{L^2}\sqrt{\frac{\cancel{F}L^3\,L}{\cancel{F}T^2}} = \frac{1}{L^2}\,\frac{L^2}{T} = \frac{1}{T}$$

OK

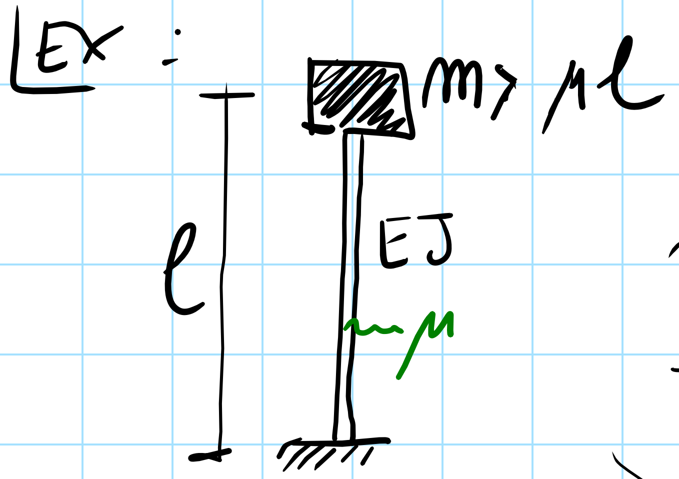
$F = M\frac{L}{T^2}\,;\quad M = \frac{F\,T^2}{L}$

NOTE: BOTH $\psi_1, \psi_2$ SATISFY $\psi_k''(\ell)=0$

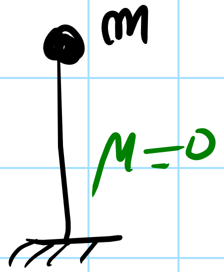
(NULL BENDING MOM AT THE FREE POINT OF THE CANTILEVER).

THIS CONDITION IS NOT STRICTLY NECESSARY (BUT WELCOME)

EX:



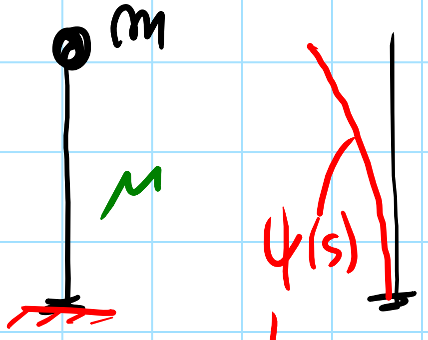

$\omega$?




$K = \frac{3EJ}{\ell^3}$

$\omega = \sqrt{\frac{K}{m}}$

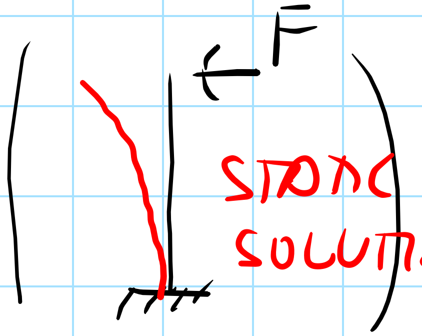
SIMPLE RESONATOR



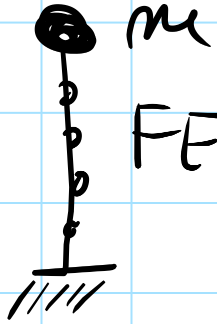

↳ MUST BE SELECTED

$\omega = \sqrt{\frac{K_{eq}}{m_{eq}}}$

GENERALIZED 1 DOF




(STATIC SOLUTION $\Rightarrow \psi(s)$)




FEM, NUMERIC APPROACHES