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The Laplace Transform: Definition, Properties, Application Examples

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Transforms: an Introduction

What is a transform?

- A **mapping** of a mathematical function from one domain to another.
- A change in **perspective**, not a change of the function.

Why use transforms?

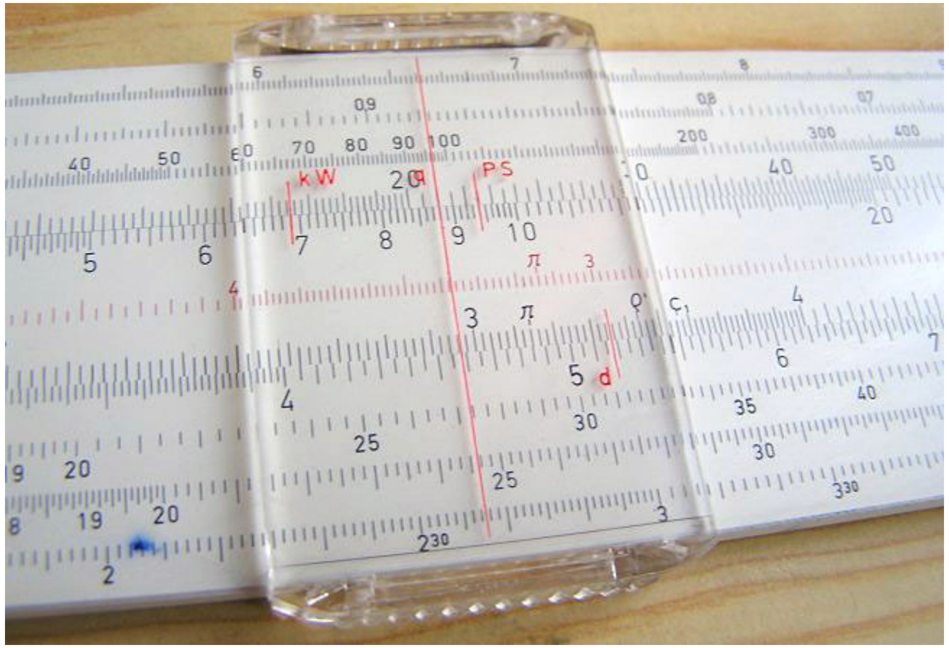
- Some mathematical **problems** are **difficult to solve** in their natural domain.
- Transform to and solve in a **new domain**, where **the problem is simplified**.
- Solved the problem in the new domain, transform the solution back into the original domain.

- Trade off the extra effort of transforming/inverse-transforming for simplification of the solution procedure.

A Simple Transform Example – Slide Rule

- Slide rules make use of a logarithmic transform.

- Multiplication/division of large numbers is difficult:
  - **Transform** the numbers to the logarithmic domain
  - Add/subtract (**easy**) in the log domain to multiply/divide (difficult) in the linear domain
  - Apply the **inverse transform** to get back to the original domain



- Extra effort is required, but **the problem is simplified!**

The Laplace Transform

Motivation and Definition

Laplace Transform

- An **integral transform** mapping functions from the **time domain** to the **Laplace domain** or s-domain.

$$g(t) \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad G(s)$$

- Time-domain functions are functions of time, t

- Laplace-domain functions are functions of *s*

- *s* is a complex variable $s = \sigma + j\,\omega$

$$\sigma = \mathrm{Re}(s) = \Re s$$

$$\omega = \mathrm{Im}(s) = \Im s$$

Laplace Transform - Motivation

- We’ll use Laplace transforms **to solve differential equations**.
  - **Differential equations** in the **time domain**: difficult to solve.
  - Apply the Laplace transform: **transform to the s-domain.**
  - Differential equations become **algebraic equations**: easy to solve!
  - **Transform** the s-domain solution **back to the time domain**.
- Transforming back and forth requires extra effort, but the solution is greatly simplified!

Causal Time-Domain Functions

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- We are interested in functions in the time domain defined for t ≥ 0, assuming they are null for all negative times, i.e.

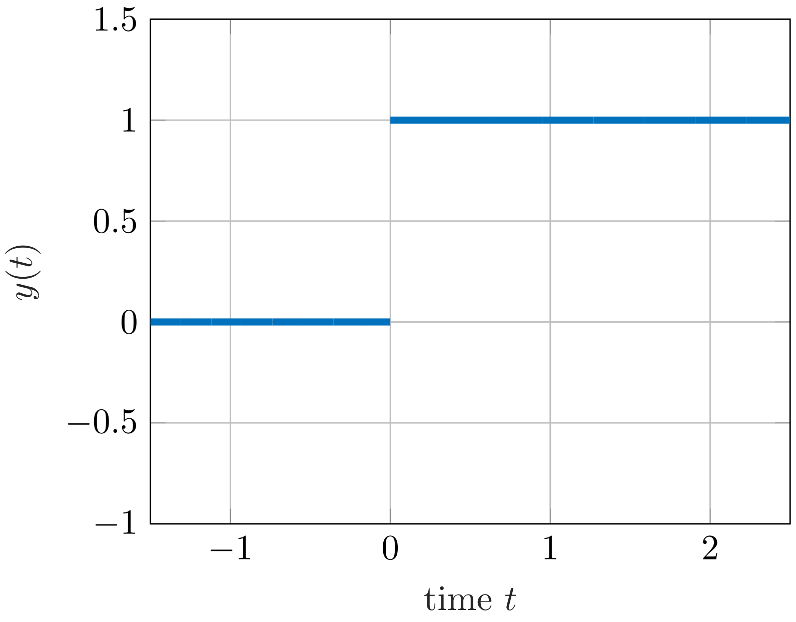
$$f(t) \equiv 0, \quad \forall t < 0$$

- Time domain functions that have this property are called **causal functions**.

Examples

- The **unit step function** or Heaviside step function

$$1(t) := \begin{cases} 0\,, & \forall t < 0 \\ 1\,, & \forall t \geq 0 \end{cases}$$



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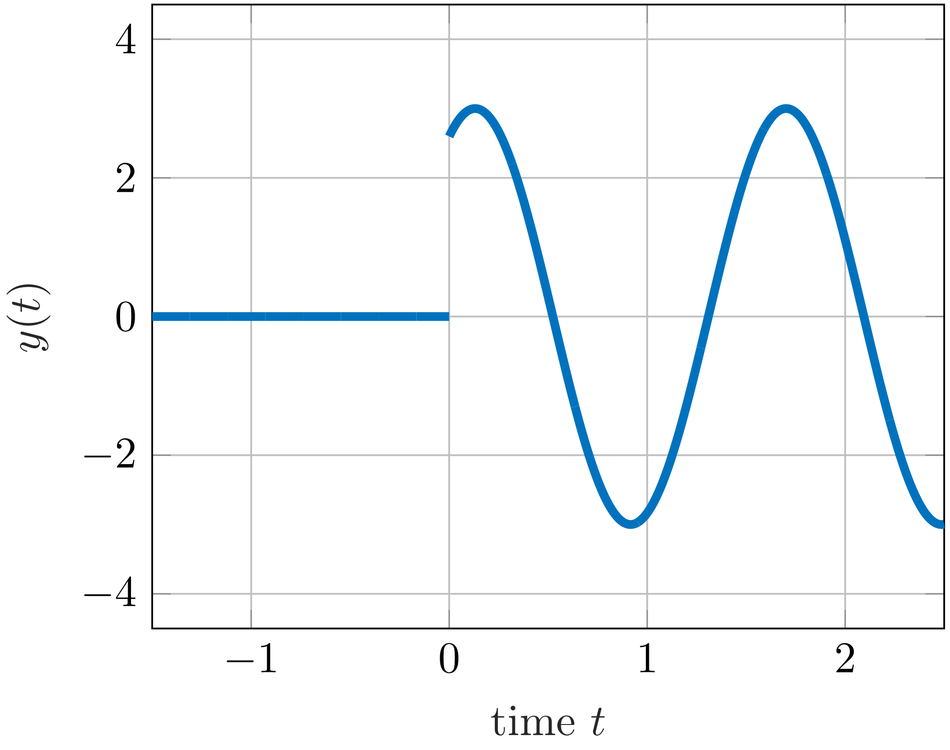
Causal Time-Domain Functions: Notation

- We'll use the Heaviside function to **emphasise** that we are considering a causal function

$$
\begin{aligned}
f(t) &= \left[3\sin\left(4t+\frac{\pi}{3}\right)\right]\cdot 1(t) \\
&= \begin{cases} 0\,, & \forall t < 0 \\ 3\sin\left(4t+\frac{\pi}{3}\right)\,, & \forall t \geq 0 \end{cases}
\end{aligned}
$$



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Laplace Transform - The Unilateral or One-sided Transform

- Given a **causal function** $f(t)$, the Laplace transform of $f(t)$ is the function $F(s) = \mathcal{L}(f(t))$ defined by

$$F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

for those $s \in \mathbb{C}$ for which the integral makes sense.

- *F* is a complex-valued function of complex numbers
- s is called the (*complex*) *frequency variable*, with units sec^{-1}; t is called the *time variable* (in sec); st is unit-less
- Lower limit of integration is $t = 0$: the transform is called the **unilateral** or **one-sided Laplace transform** (Laplace transform for short).

Laplace Transform - Notation Convention

$$F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

- **common notation convention**: lower case letter denotes the time-domain function $f(t)$; capital letter denotes its Laplace transform $F(s)$

- for example

$$U(s) = \mathcal{L}\left(u(t)\right) \qquad V_{\text{in}}(s) = \mathcal{L}\left(v_{\text{in}}(t)\right)$$

Laplace Transform - Existence

$$f(t) \quad \stackrel{?}{\longleftrightarrow} \quad F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

Which class or classes of functions possess a Laplace transform?

- **Fact 1**: $f(t)$ is **continuous for** $t = 0$; if not, $f(t)$ must contain **no impulses** at $t = 0$.
- **Fact 2**: any <u>continuous</u>, <u>bounded</u>, <u>causal</u> function $f(t)$ admits Laplace transform.
- **Fact 3**: any **piecewise continuous**, **causal** function $f(t)$ admits Laplace transform.

Piecewise continuous function $\longleftrightarrow$ jump discontinuities and no vertical asymptotes

Laplace Transform - Example: Exponential Function

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- Let's find the Laplace transform of the causal function $f(t) = e^t \cdot 1(t)$

$$F(s) = \int_0^\infty e^t\, e^{-st}\, dt = \int_0^\infty e^{(1-s)t}\, dt = \frac{1}{1-s}\, e^{(1-s)t}\Big|_0^\infty = \frac{1}{s-1}$$

provided we can say $e^{(1-s)t} \to 0$ as $t \to \infty$, which is true for $\mathrm{Re}(s) > 1$

$$\left| e^{(1-s)t} \right| = \left| e^{-j\omega t} \right| \cdot \left| e^{(1-\sigma)\, t} \right| = e^{(1-\sigma)\, t}$$

- The integral defining F makes sense for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ (the 'region of convergence' of F)
- The resulting formula for F makes sense for all $s \in \mathbb{C}$, except $s = 1$

Laplace Transform - Example: Unit Step Function

- Let's find the Laplace transform of the causal function $f(t) = 1(t)$

$$F(s) = \int_0^\infty 1(t)\, e^{-st}\, dt = \int_0^\infty e^{-st}\, dt = -\frac{1}{s} e^{-st}\Big|_0^\infty = \frac{1}{s}$$

- The integral defining F makes sense for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ (the 'region of convergence' of F)
- The resulting formula for F makes sense for all $s \in \mathbb{C}$, except $s = 0$
- We'll ignore these (sometimes important) details and just write $\mathcal{L}\left(1(t)\right) = \frac{1}{s}$

Laplace Transform - Properties: Linearity

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- The Laplace transform is **linear**

- If $f(t)$ and $g(t)$ are any causal functions, for which the Laplace transform exists, and a is any scalar, we have

$$\mathcal{L}\left(a\, f(t)\right) = aF(s) \qquad \mathcal{L}\left(f(t) + g(t)\right) = F(s) + G(s)$$

i.e., homogeneity and superposition hold.

Example

$$\mathcal{L}\left(3 \cdot 1(t) - 2e^{t} \cdot 1(t)\right) = 3 \cdot \frac{1}{s} - 2 \cdot \frac{1}{s-1} = \frac{s-3}{s\,(s-1)}$$

Laplace Transform - Example: Trigonometric Functions

- Let's find the Laplace transform of the causal function $f(t) = [\cos(\omega t)] \cdot 1(t)$

- First step: express $f(t)$ as $$f(t) = \left[\frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t}\right] \cdot 1(t)$$

- Second step: according to the *linearity* property

$$\begin{aligned} F(s) = \mathcal{L}(f(t)) &= \frac{1}{2}\mathcal{L}\left(e^{j\omega t} \cdot 1(t)\right) + \frac{1}{2}\mathcal{L}\left(e^{-j\omega t} \cdot 1(t)\right) \\ &= \frac{1}{2}\int_0^{\infty} e^{(-s+j\omega)t}\, dt + \frac{1}{2}\int_0^{\infty} e^{(-s-j\omega)t}\, dt \\ &= \frac{1}{2}\frac{1}{s-j\omega} + \frac{1}{2}\frac{1}{s+j\omega} = \frac{s}{s^2+\omega^2} \end{aligned}$$

Laplace Transform - Example: Trigonometric Functions

$$f(t) = [\cos(\omega t)] \cdot 1(t) \qquad \Longleftrightarrow \qquad F(s) = \mathcal{L}(f(t)) = \frac{s}{s^2 + \omega^2}$$

- How do you compute this Laplace transform using MATLAB?

```
clear variables
syms t s
syms omega
sympref('HeavisideAtOrigin', 1);
f(t) = cos(omega*t)*heaviside(t);

F(s ) = laplace(f, t, s)
```

```
F(s) =
```

$$\frac{s}{\omega^2 + s^2}$$

Laplace Transform - One-to-One Property

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- The Laplace transform is one-to-one:

$$f(t) = g(t) \longleftrightarrow \mathcal{L}\left(f(t)\right) = \mathcal{L}\left(g(t)\right)$$

- $F(s)$ determines $f(t)$
- **Inverse Laplace transform** is well defined

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Example (previous page)

$$\mathcal{L}^{-1}\left[\frac{s-3}{s\,(s-1)}\right] = \mathcal{L}^{-1}\left[\frac{3}{s} - \frac{2}{s-1}\right] =$$

$$= \mathcal{L}^{-1}\left[\frac{3}{s}\right] - \mathcal{L}^{-1}\left[\frac{2}{s-1}\right] \quad = 3 \cdot 1(t) - 2e^{t} \cdot 1(t)$$

Laplace Transform - Transform of a Derivative

- If the function $f(t)$ is continuous at $t = 0$, then $\mathcal{L}\left(\dot{f}(t)\right) = s\,F(s) - f(0)$

- **time-domain differentiation** becomes **multiplication by frequency variable** s (as with phasors)
- plus a term that includes initial condition, i.e. $-f(0)$

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Example

$$f(t) = 2\,e^{t} \cdot 1(t) \implies \dot{f}(t) = 2e^{t} \cdot 1(t)$$

$$\mathcal{L}\left(\dot{f}(t)\right) = s\left(\frac{2}{s-1}\right) - f(0) = \frac{2s}{s-1} - 2 = \frac{2}{s-1}$$

Laplace Transform - Transform of a Derivative (cont.)

$$\mathcal{L}\left(\dot{f}\right) = sF(s) - f(0)$$

- higher order derivatives: applying derivative formula twice yields

$$\begin{aligned}\mathcal{L}\left(\ddot{f}\right) &= s\,\mathcal{L}\left(\dot{f}\right) - \dot{f}(0)\\ &= s\,\left(s\,F(s) - f(0)\right) - \dot{f}(0)\\ &= s^2F(s) - s\,f(0) - \dot{f}(0)\end{aligned}$$

- a similar formula holds for $$\mathcal{L}\left(f^{(k)}\right) = s^kF(s) - \sum_{n=0}^{k-1} s^{k-n-1}\,f^{(n)}(0)$$

Laplace Transform - Transform of an Integral

- Let g be the running integral of a causal function f

$$g(t) = \int_0^t f(\tau)\, d\tau$$

- then

$$G(s) = \frac{1}{s} F(s)$$

time-domain integral becomes **division by frequency variable** s

Laplace Transform - Transform of an Integral (cont.)

$$g(t) = \int_0^t f(\tau)\, d\tau \iff G(s) = \frac{1}{s} F(s)$$

Example

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$$f(t) = 1(t) \iff F(s) = \frac{1}{s}$$

$$g(t) = \int_0^t 1(\tau)\, d\tau = t\,,\ t \geq 0 \iff G(s) = \frac{1}{s} F(s) = \frac{1}{s^2}$$

Laplace Transform - Multiplication by t

- Let $f(t)$ be a causal function and define $$g(t) = t \cdot f(t)$$

- Then we have $$G(s) = -\frac{d}{ds} F(s)$$

- In general $$g(t) = t^k \cdot f(t)$$

$$\Updownarrow$$

$$G(s) = \left(-\frac{d}{ds}\right)^k F(s) = \underbrace{-\frac{d}{ds}\left[-\frac{d}{ds}\left[-\frac{d}{ds}\left[\cdots - \frac{d}{ds}F(s)\right]\right]\right]}_{k \text{ times}}$$

k times

Laplace Transform - Multiplication by t (cont.)

- **Examples**

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$$f(t) = e^t \cdot 1(t)\,,\;\; g(t) = t\,e^t \cdot 1(t)$$

$$\mathcal{L}\left(g(t)\right) = -\frac{d}{d\,s}\left(\frac{1}{s-1}\right) = \frac{1}{(s-1)^2}$$

$$f(t) = t\,e^t \cdot 1(t)\,,\;\; g(t) = t^2\,e^t \cdot 1(t)$$

$$\mathcal{L}\left(g(t)\right) = -\frac{d}{d\,s}\left(\frac{1}{(s-1)^2}\right) = \frac{2}{(s-1)^3}$$

Laplace Transform - Multiplication by t (cont.)

- **Examples**

  - In general

$$f(t) = e^t \cdot 1(t)\,,\;\; g(t) = t^k\, e^t \cdot 1(t)$$

$$\mathcal{L}\big(g(t)\big) = \mathcal{L}\left(t^k e^t \cdot 1(t)\right) = \frac{(k-1)!}{(s-1)^{k+1}}$$

Laplace Transform - Time Scaling

- Let $f(t)$ be a causal function, admitting Laplace transform, and let $g(t)$ be defined as

$$g(t) = f\,(a\,t)\ , \quad a > 0$$

then

$$G(s) = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Remark: this makes sense! If time is scaled by a, then frequency s is scaled by $\frac{1}{a}$

Laplace Transform - Time Scaling (cont.)

$$g(t) = f\,(a\,t)\;, \quad a > 0 \quad \Longleftrightarrow \quad G(s) = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Example

starting from $\mathcal{L}\left(e^{t} \cdot 1(t)\right) = \dfrac{1}{s-1}$

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we obtain

$$\mathcal{L}\left(e^{at} \cdot 1(t)\right) = \left(\frac{1}{a}\right) \frac{1}{\left(\frac{s}{a}\right) - 1} = \frac{1}{s-a}\;, \quad a > 0$$

Laplace Transform - Exponential Scaling (Frequency Shifting)

- Let $f(t)$ be a causal function, with Laplace transform $F(s)$, and let $g(t)$ be defined as (a is a scalar value)

$$g(t) = e^{a\,t} f\,(t)$$

then

$$G(s) = F\,(s - a)$$

Remark: a frequency shift corresponds to a multiplication with an exponential term in the time domain!

Laplace Transform - Exponential Scaling (cont.)

Example: given $$\mathcal{L}\left[(\cos(\omega t)) \cdot 1(t)\right] = \frac{s}{s^2+\omega^2}$$

then

$$\mathcal{L}\left[\left(e^{-t}\cos(t)\right) \cdot 1(t)\right] = \frac{(s+1)}{(s+1)^2+1}$$

$$= \frac{s+1}{s^2+2s+2}$$

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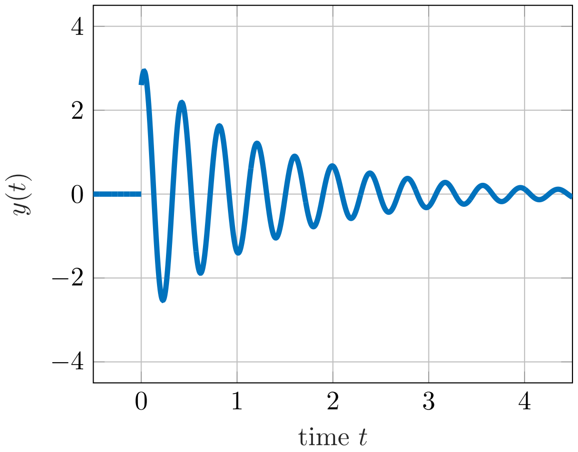
Laplace Transform - Time Shifting (Time Delay)

- Let $f(t)$ be a causal function, with Laplace transform $F(s)$, and $T > 0$
- Define $g(t)$ as

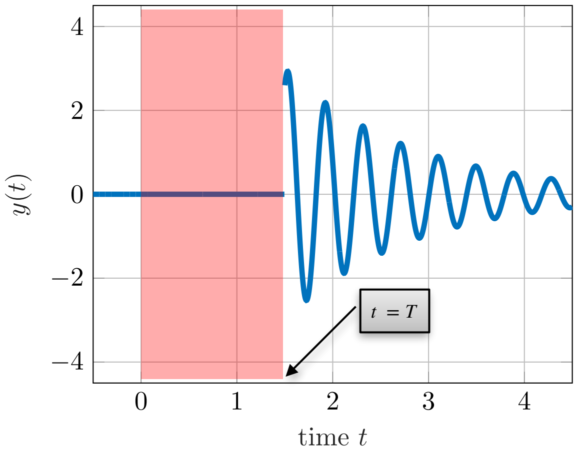
$$g(t) = \begin{cases} 0 & 0 \leq t < T \\ f(t-T) & t \geq T \end{cases} \quad \longleftrightarrow \quad g(t) = f(t-T) \cdot 1(t-T)$$

i.e., g is f, delayed by T seconds and 'zero-padded' up to T



$$f(t) = \left[3e^{-0.75t} \sin\left(16t + \frac{\pi}{3}\right)\right] \cdot 1(t)$$

$$g(t) = f(t-1.5) \cdot 1(t-1.5)$$



Laplace Transform - Time Shifting (Time Delay) (cont.)

- Let $f(t)$ be a causal function, with Laplace transform $F(s)$, and $T > 0$
- Define $g(t)$ as

$$g(t) = f(t - T) \cdot 1(t - T)$$

then

$$G(s) = e^{-sT} F(s)$$

Remark: a time shift corresponds to a multiplication with an exponential term in the frequency domain!

Laplace Transform - Time Shifting (Time Delay) (cont.)

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Example: let's find the Laplace transform of a rectangular pulse signal

$$f(t) = \begin{cases} 1 & a \geq t \geq b \\ 0 & \text{otherwise} \end{cases}$$

where $0 < a < b$

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Write $f(t)$ as a unit step function delayed by a seconds, minus a unit step function delayed by b seconds

$$f(t) = 1(t-a) - 1(t-b) \longleftrightarrow F(s) = \mathcal{L}\left(1(t-a)\right) - \mathcal{L}\left(1(t-b)\right)$$

$$= \frac{e^{-a\,s} - e^{-b\,s}}{s}$$

Laplace Transform - Convolution (in the Time Domain)

- The **convolution** of causal functions f and g is the causal function h

$$h := f * g \;\longleftrightarrow\; h(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

- **Note**: same as $h(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$, i.e. $f * g = g * f$

$$H(s) = \mathcal{L}\,(f * g) = F(s)\,G(s)$$

Remark: Laplace transform turns convolution in time domain into multiplication in the frequency domain!

Laplace Transform of Common Functions

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$f(t)$	$F(s)$
$1(t)$	$\frac{1}{s}$
$t \cdot 1(t)$	$\frac{1}{s^2}$
$t^2 \cdot 1(t)$	$\frac{2}{s^3}$
$e^{a\,t} \cdot 1(t)$	$\frac{1}{s-a}$
$te^{a\,t} \cdot 1(t)$	$\frac{1}{(s-a)^2}$
$[\sin(\omega t)] \cdot 1(t)$	$\frac{\omega}{s^2+\omega^2}$

$f(t)$	$F(s)$
$[\cos(\omega t)] \cdot 1(t)$	$\frac{s}{s^2+\omega^2}$
$[t\sin(\omega t)] \cdot 1(t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
$[t\cos(\omega t)] \cdot 1(t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$[e^{\sigma t}\sin(\omega t)] \cdot 1(t)$	$\frac{\omega}{(s-\sigma)^2+\omega^2}$
$[e^{\sigma t}\cos(\omega t)] \cdot 1(t)$	$\frac{s-\sigma}{(s-\sigma)^2+\omega^2}$
$[t\,e^{\sigma t}\sin(\omega t)] \cdot 1(t)$	$\frac{2\omega(s-\sigma)}{[(s-\sigma)^2+\omega^2]^2}$
$[t\,e^{\sigma t}\cos(\omega t)] \cdot 1(t)$	$\frac{(s-\sigma)^2-\omega^2}{[(s-\sigma)^2+\omega^2]^2}$

Inverse Laplace Transform

- In principle we can recover $f(t)$ from $F(s)$ via

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} \, ds$$

where σ is large enough that $F(s)$ is defined for $\mathrm{Re}(s) \geq \sigma$

- Surprisingly, this formula isn't really useful!

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Inverse Laplace Transform (cont.)

- The **functions** in the **time domain** $f(t)$, with which we will deal, will always be **causal**, **piecewise continuous**, and they contain **no impulses at the time** $t = 0$

- Thus, the corresponding **Laplace transforms** F(s) will **always** be **rational polynomial functions**, i.e., the quotient of two polynomials

$$F(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)} \qquad \begin{array}{ll} \forall j & b_j \in \mathbb{R} \\ \forall i & a_i \in \mathbb{R} \end{array}$$

- Furthermore, the degree of the numerator polynomial will be less than that of the denominator polynomial:

$$\deg\left(N(s)\right) = m < n = \deg\left(D(s)\right)$$

Laplace Transform: Notation - Zeros and Poles

- Given a Laplace transform $F(s)$ as a rational polynomial function, taking in evidence the roots of both the polynomials, we define

$$F(s) = \frac{N(s)}{D(s)} = K\frac{(s-z_1)^{m_1}(s-z_2)^{m_2}\cdots(s-z_r)^{m_r}}{(s-p_1)^{n_1}(s-p_2)^{n_2}\cdots(s-p_q)^{n_q}}$$

zeros: the roots of the polynomial in the numerator $N(s)$

$$z_1,\, z_2,\, \ldots,\, z_r \in \mathbb{C}\,, \qquad \sum_{j=1}^{r} m_j = m = \deg\left(N(s)\right)$$

poles: the roots of the polynomial in the denominator $D(s)$

$$p_1,\, p_2,\, \ldots,\, p_q \in \mathbb{C}\,, \qquad \sum_{i=1}^{q} n_j = n = \deg\left(D(s)\right)$$

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Inverse Laplace Transform (cont.)

- Starting from a Laplace transform $F(s)$ that is a **strictly proper rational function**, it is always possible to express the function as a **sum of partial fractions**:

$$F(s) = \frac{C_{1,1}}{(s-p_1)} + \dots + \frac{C_{1,n_1}}{(s-p_1)^{n_1}} + \dots$$

$$\dots + \frac{C_{2,1}}{(s-p_2)} + \dots + \frac{C_{2,n_2}}{(s-p_2)^{n_2}} + \dots$$

$$\dots + \frac{C_{q,1}}{(s-p_q)} + \dots + \frac{C_{q,n_q}}{(s-p_q)^{n_q}}$$

$$C_{i,j} \in \mathbb{C}\,, \quad i = 1,\, 2,\, \dots,\, q\,, \quad j = 1,\, 2,\, \dots,\, n_i$$

Inverse Laplace Transform (cont.)

- Owing to the linearity of the inverse Laplace transform, the time domain function we are looking for will be the linear combination of elementary exponential functions:

$$\mathcal{L}^{-1}[F(s)] = \sum_{i=1}^{q} \sum_{j=1}^{n_i} \mathcal{L}^{-1}\left[\frac{C_{i,j}}{(s-p_i)^j}\right]$$

$$\mathcal{L}^{-1}[F(s)] = \sum_{i=1}^{q} \sum_{j=1}^{n_i} \frac{C_{i,j}}{(j-1)!}\, t^{(j-1)}\, e^{p_i t}\, 1(t)$$

Inverse Laplace Transform (cont.)

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- How to calculate the $C_{i,\,j}$ coefficients?

- There is a not very easy formula:

$$C_{i,\,j} = \frac{1}{(n_i - j)!} \lim_{s \to p_i} \left\{ \frac{d^{(n_i - j)}}{d\, s^{(n_i - j)}} \left[F(s)\,(s - p_i)^{n_i} \right] \right\}$$

- We can exploit the **identity theorem for polynomials**.

- We will see both techniques in action by solving a linear differential equation using the Laplace transform.

Example: Solution of a Linear Differential Equation

- We want to solve the ODE equation:

$$\frac{d^2\, y(t)}{d\,t^2} + 3\frac{d\,y(t)}{d\,t} + 2y(t) = [1 + 3t] \cdot 1(t)$$

with initial conditions $y(0) = 1 \qquad \left.\frac{dy}{dt}\right|_{t=0} = 0$

- Applying the properties analysed so far, we obtain

$$\left\{s^2\,Y(s) - s\,y(0) - \dot{y}(0)\right\} + \{3\,[s\,Y(s) - y(0)]\} + \ldots$$

$$\ldots + 2\,Y(s) = \frac{1}{s} + \frac{3}{s^2}$$

Example: Solution of a Linear Differential Equation (cont.)

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- Substituting the values of the initial conditions gives:

$$\left\{s^2 Y(s) - s\,y(0) - \dot{y}(0)\right\} + \{3\,[s\,Y(s) - y(0)]\} + \ldots$$

$$\ldots + 2\,Y(s) = \frac{1}{s} + \frac{3}{s^2}$$

$$\left\{s^2 Y(s) - s\right\} + \{3\,[s\,Y(s) - 1]\} + 2Y(s) = \frac{1}{s} + \frac{3}{s^2}$$

$$(s^2 + 3\,s + 2)Y(s) - s - 3 = \frac{1}{s} + \frac{3}{s^2}$$

Example: Solution of a Linear Differential Equation (cont.)

- We have obtained an algebraic equation that has $Y(s)$ as its solution.

$$Y(s) = \frac{s+3}{s^2+3s+2} + \frac{1}{s^2+3s+2}\left(\frac{1}{s} + \frac{3}{s^2}\right)$$

- The first term of Y $Y(s)$ depends on the initial conditions (the so-called **free trajectory**), and the second term depends on the input u(t) (the so-called **forced trajectory**)

$$Y_l(s) = \frac{s+3}{s^2+3s+2}$$

$$Y_f(s) = \frac{s+3}{s^2\left(s^2+3s+2\right)}$$

Example: Solution of a Linear Differential Equation (cont.)

$$Y_l(s) = \frac{s+3}{s^2+3s+2} = \frac{C_1}{s+1} + \frac{C_2}{s+2}$$

- **First approach**: using the formula $C_{i,\,j} = \frac{1}{(n_i-j)!} \lim_{s\to p_i} \left\{ \frac{d^{(n_i-j)}}{d\,s^{(n_i-j)}} \left[F(s)\,(s-p_i)^{n_i} \right] \right\}$

$$C_1 = \lim_{s\to -1} \frac{s+3}{(s+1)(s+2)}(s+1) = 2$$

$$C_2 = \lim_{s\to -2} \frac{s+3}{(s+1)(s+2)}(s+2) = -1$$

$$Y_l(s) = \frac{2}{s+1} - \frac{1}{s+2}$$

Example: Solution of a Linear Differential Equation (cont.)

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$$Y_l(s) = \frac{s+3}{s^2+3s+2} = \frac{C_1}{s+1} + \frac{C_2}{s+2}$$

- **Second approach**: exploiting the **identity theorem for polynomials.**

$$\frac{s+3}{(s+1)(s+2)} = \frac{s(C_1+C_2)+(2C_1+C_2)}{(s+1)(s+2)}$$

$$\begin{cases} C_1 + C_2 = 1 \\ 2C_1 + C_2 = 3 \end{cases} \Longrightarrow \begin{cases} C_1 = 2 \\ C_2 = -1 \end{cases}$$

$$Y_l(s) = \frac{2}{s+1} - \frac{1}{s+2}$$

Example: Solution of a Linear Differential Equation (cont.)

$$Y_l(s) = \frac{2}{s+1} - \frac{1}{s+2}$$

$$\mathcal{L}^{-1}\left\{Y_l(s)\right\} = y_l(t) = \left[2\,e^{-t} \;-\; e^{-2t}\right] \cdot 1(t)$$

Example: Solution of a Linear Differential Equation (cont.)

$$Y_f(s) = \frac{s+3}{s^2\left(s^2+3s+2\right)} = \frac{C_{1,\,1}}{s+1} + \frac{C_{2,\,1}}{s+2} + \frac{C_{3,\,1}}{s} + \frac{C_{3,\,2}}{s^2}$$

- **First approach**: using the formula $C_{i,\,j} = \frac{1}{(n_i - j)!} \lim_{s \to p_i} \left\{ \frac{d^{(n_i - j)}}{d\,s^{(n_i-j)}} \left[F(s)\,(s-p_i)^{n_i}\right] \right\}$

$$C_{1,1} = \left.\frac{(s+3)}{(s^2)(s+2)}\right|_{s=-1} = 2$$

$$C_{3,1} = \frac{d}{ds}\left.\frac{(s+3)}{(s+1)(s+2)}\right|_{s=0} = -\frac{7}{4}$$

$$C_{2,1} = \left.\frac{(s+3)}{(s^2)(s+1)}\right|_{s=-2} = -\frac{1}{4}$$

$$C_{3,2} = \left.\frac{(s+3)}{(s+1)(s+2)}\right|_{s=0} = \frac{3}{2}$$

Example: Solution of a Linear Differential Equation (cont.)

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$$Y_f(s) = \frac{s+3}{s^2\left(s^2+3s+2\right)} = \frac{C_{1,\,1}}{s+1} + \frac{C_{2,\,1}}{s+2} + \frac{C_{3,\,1}}{s} + \frac{C_{3,\,2}}{s^2}$$

- **Second approach**: exploiting the **identity theorem for polynomials.**

$$C_{1,\,1}s^2(s+2)+C_{2,\,1}s^2(s+1)+C_{3,\,1}s(s+1)(s+2)+C_{3,\,2}(s+1)(s+2) = s+3$$

$$s=0 \;\longrightarrow\; 2C_{3,\,2}=3 \;\longrightarrow\; C_{3,\,2}=\frac{3}{2} \qquad\qquad s=-1 \;\longrightarrow\; C_{1,\,1}=2$$

$$s=-2 \;\longrightarrow\; -4C_{2,\,1}=1 \;\longrightarrow\; C_{2,\,1}=-\frac{1}{4}$$

$$s=+1 \;\longrightarrow\; 3C_{1,\,1}+2C_{2,\,1}+6C_{3,\,1}+6C_{3,2}=4 \;\longrightarrow\; C_{3,\,1}=-\frac{7}{4}$$

Example: Solution of a Linear Differential Equation (cont.)

$$Y_f = \frac{2}{(s+1)} - \frac{1}{4}\frac{1}{(s+2)} - \frac{7}{4s} + \frac{3}{2s^2}$$

$$\mathcal{L}\left(Y_f(s)\right) = y_f(t) = \left[\frac{3}{2}t - \frac{7}{4} + 2e^{-t} - \frac{1}{4}\,e^{-2t}\right] \cdot 1(t)$$

- The solution of the ODE equation is:

$$y(t) = y_l(t) + y_f(t)$$

The Laplace Transform:
Initial and Final Value Theorems

Can we determine the initial value of a time-domain function from its Laplace transform?

Can we determine the steady-state value of a time-domain function from its Laplace transform?

Initial Value Theorem

- Given a Laplace transform $F(s)$ that is a **strictly proper rational function**, the following property holds

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s\, F(s)$$

This is a useful theorem: one can immediately analyse the initial behaviour of $f(t)$ directly from its Laplace transform $F(s)$!

Final Value Theorem

- If all the **poles** of $F(s)$ are in the **left open half-plane**, with **at most** one **simple pole in the origin**, then **the two limits both exist finitely** and **assume equal values**.

$$\lim_{t \to +\infty} f(t) = \lim_{s \to 0} s \cdot F(s)$$

This is a useful theorem: one can immediately find the steady-state value of $f(t)$ directly from its Laplace transform $F(s)$!

Application Example

- Consider the Laplace transform $F(s)$:

$$F(s) = \frac{5s+3}{(s+1)(s+2)^2}$$

- Determine

$$f(0) = ?$$

$$\lim_{t\to\infty} f(t) = ?$$

$$\dot{f}(0) = ?$$