034IN - FONDAMENTI DI AUTOMATICA - FUNDAMENTALS OF AUTOMATIC CONTROL A.Y. 2025-2026

Part I: Motivations, Concepts, Examples

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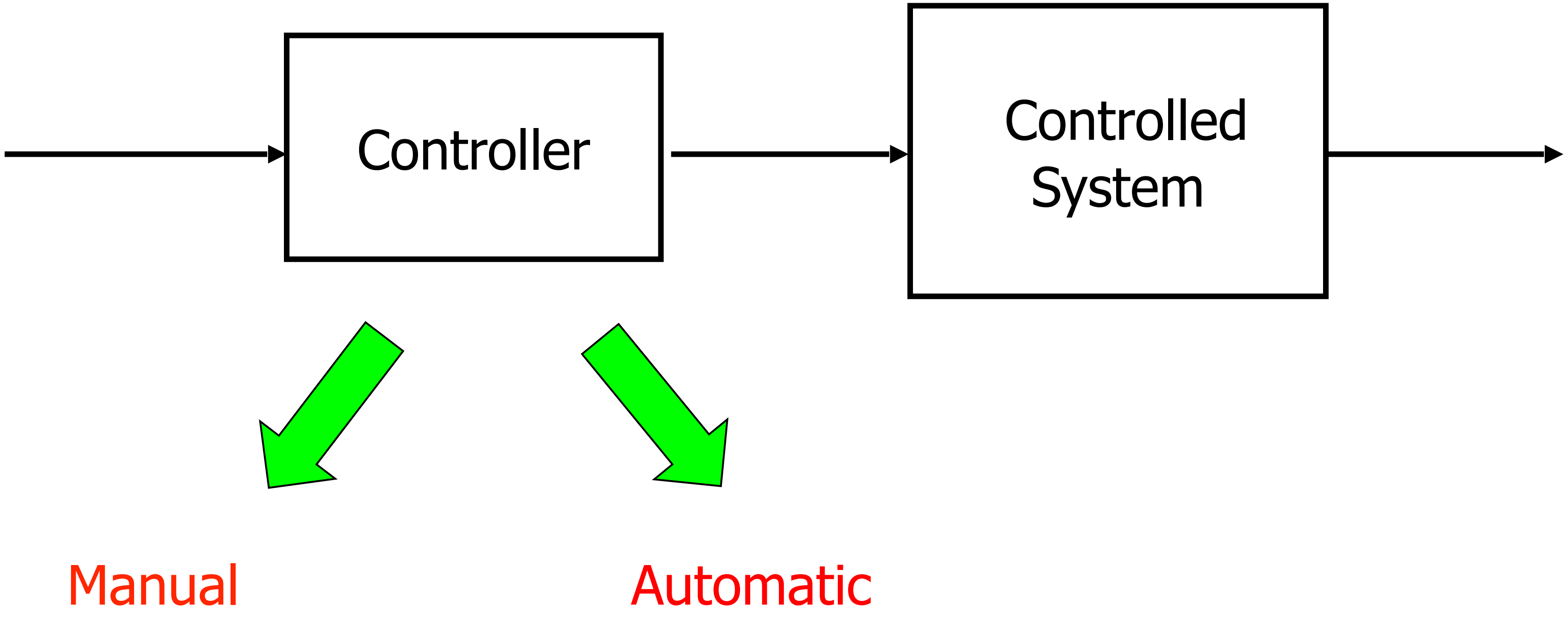
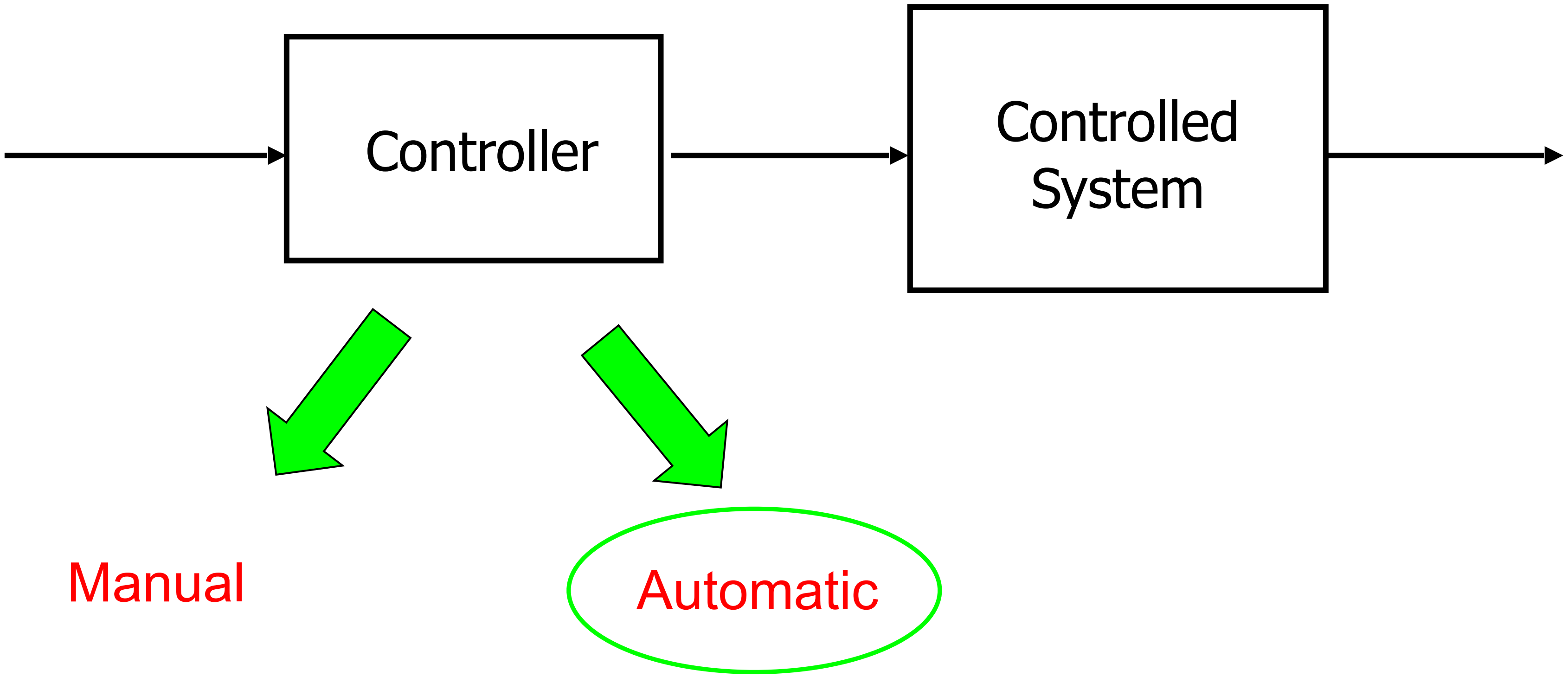
Content of the Module? → Methods and Tools to analyse and design automatic control systems of practical relevance

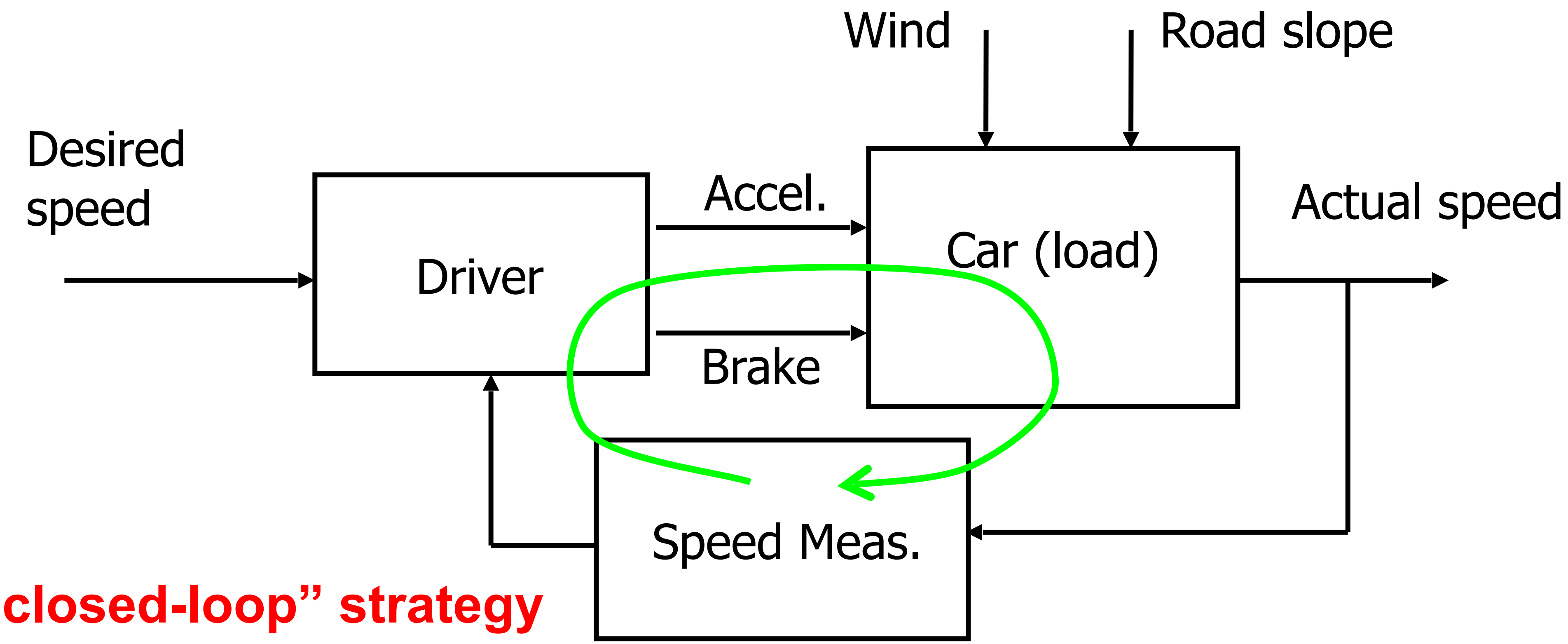


The Control Problem

To impose a given time-trajectory to one or more specific variables of an engineering system by acting on other variables that influence the behaviour of the system itself



Example: Speed Control

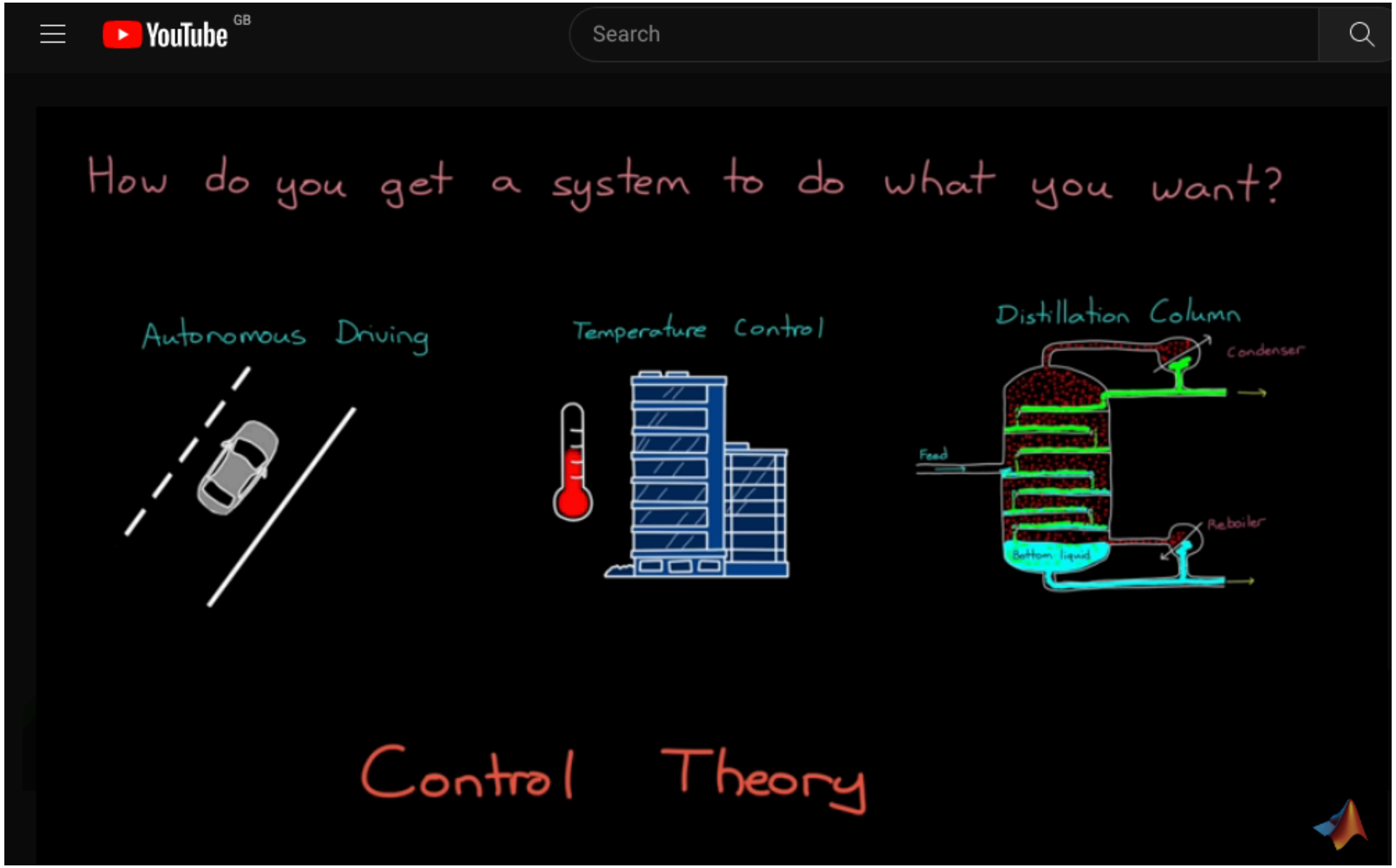


"closed-loop" strategy

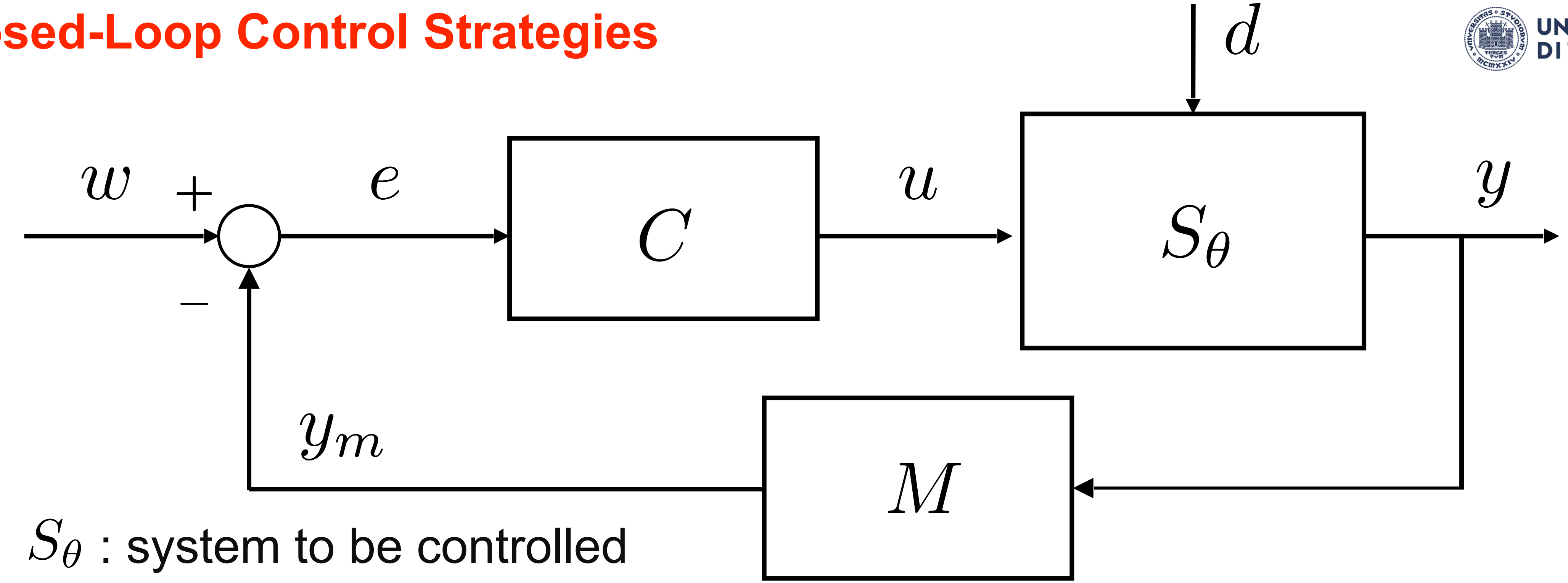
Effective in the presence of uncertainty

Control Theory Brief Tour (Youtube)

https://www.youtube.com/watch?v=lBC1nEq0_nk



Closed-Loop Control Strategies



- S_θ : system to be controlled
- C : controller
- θ : system's parameters
- y : controlled variable (output)
- u : control variable (accessible to the controller)
- d : disturbance
- w : reference variable (set-point)

- e : error
- y_m : measured output

Requirements for an Analog Continuous-Time Control System

The basic requirement is:

$$y(t) \simeq w(t)$$

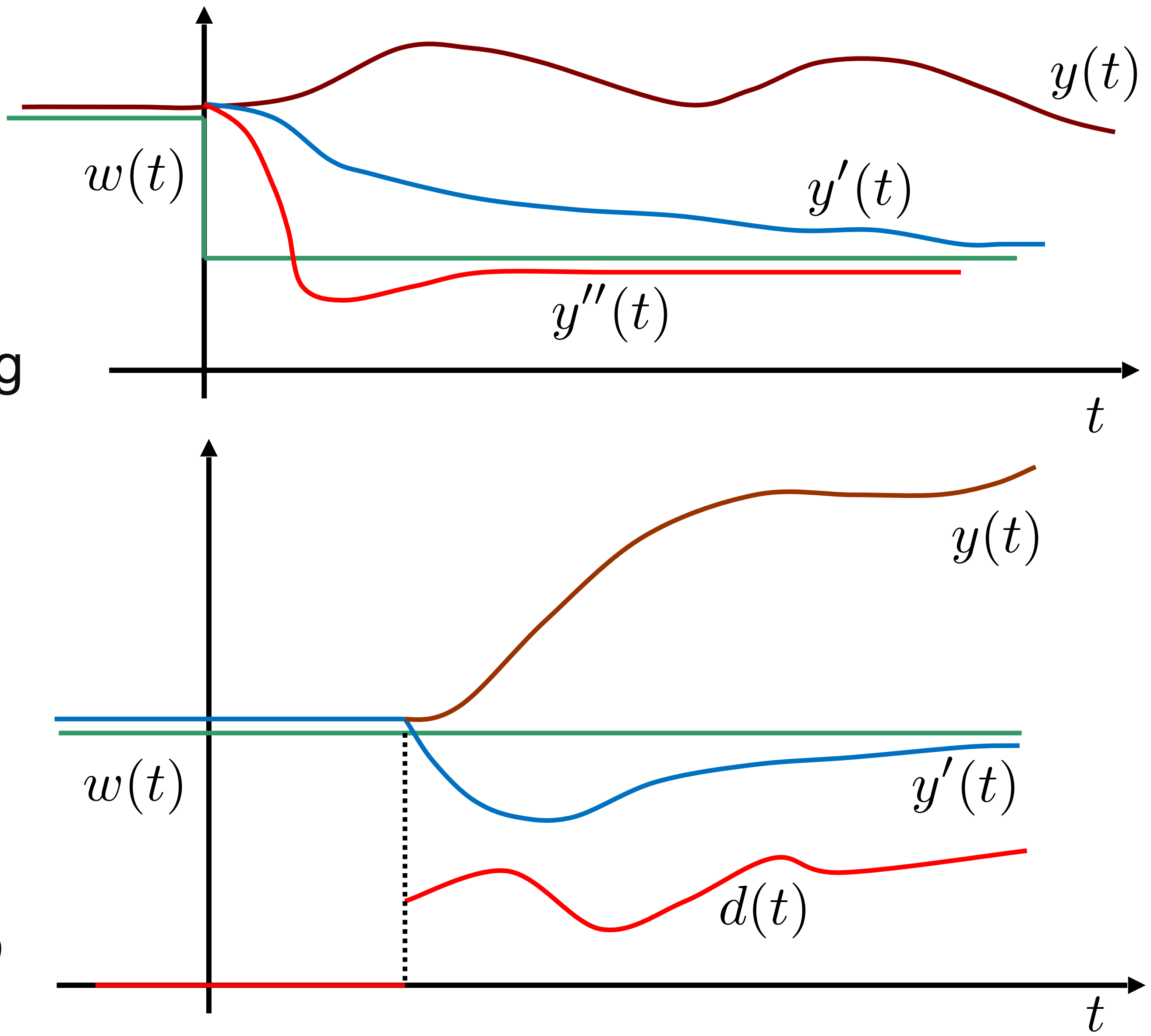
Introducing the error variable $e(t) = w(t) - y(t)$, the above requirement becomes:

$e(t) \simeq 0$ in all situations "of interest"

Without loss of generality, we focus on the continuous-time case. The discrete-time case is perfectly analogous

General Requirements of Control Systems

- (A) Static precision
  - ✦ In equilibrium conditions
- (B) Dynamic precision
  - ✦ Speed of response
  - ✦ Damping of possible oscillations
  - ✦ Capability of tracking fast-varying reference variables $w(t)$
- (C) Insensitivity to disturbances
  - ✦ Capability of rejecting a disturbance $d(t)$
- (D) Robustness
  - ✦ (A), (B), (C) in the presence of uncertain system's parameters θ



Example 2: Position Control

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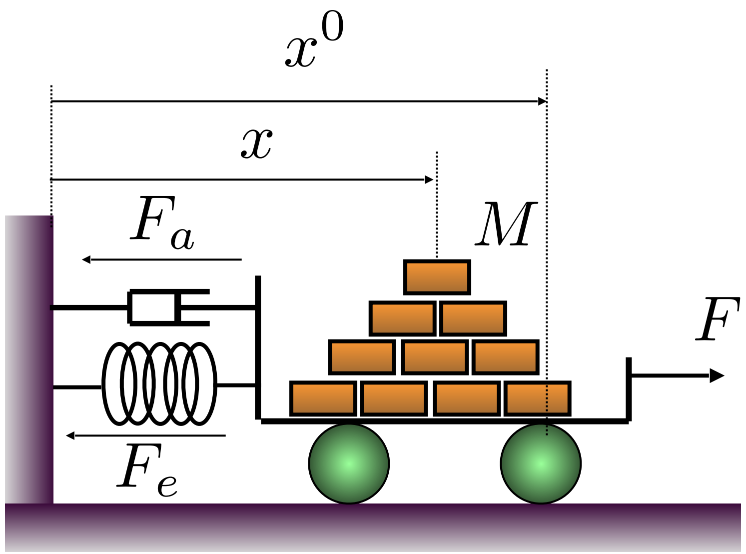
Mechanical system



- Control input: external force F
- Controlled output: position of the cart x
- Reference output: desired position $w = x^0$
- Elastic spring force: $F_e = kx$
- Damper viscous friction force: $F_a = h\dot{x}$

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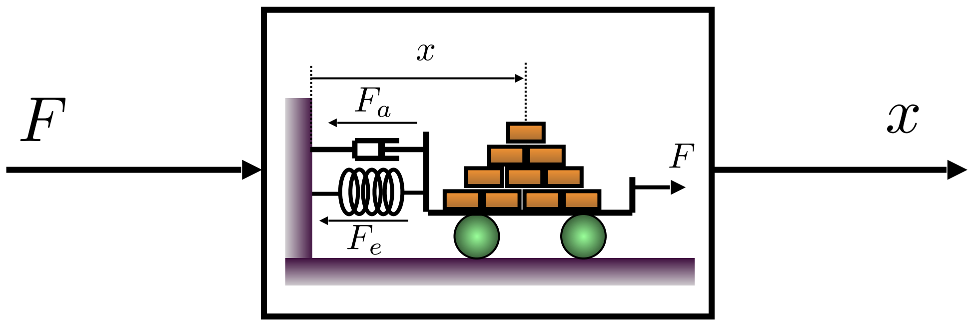
$$F = kx$$

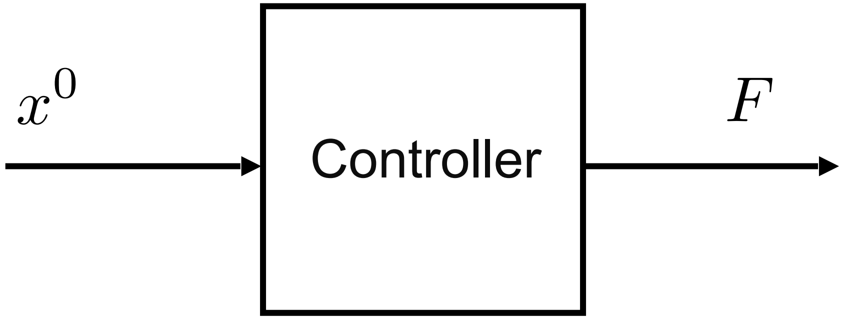
Equilibrium of the forces in **static** conditions



$$x = \frac{1}{k} F$$

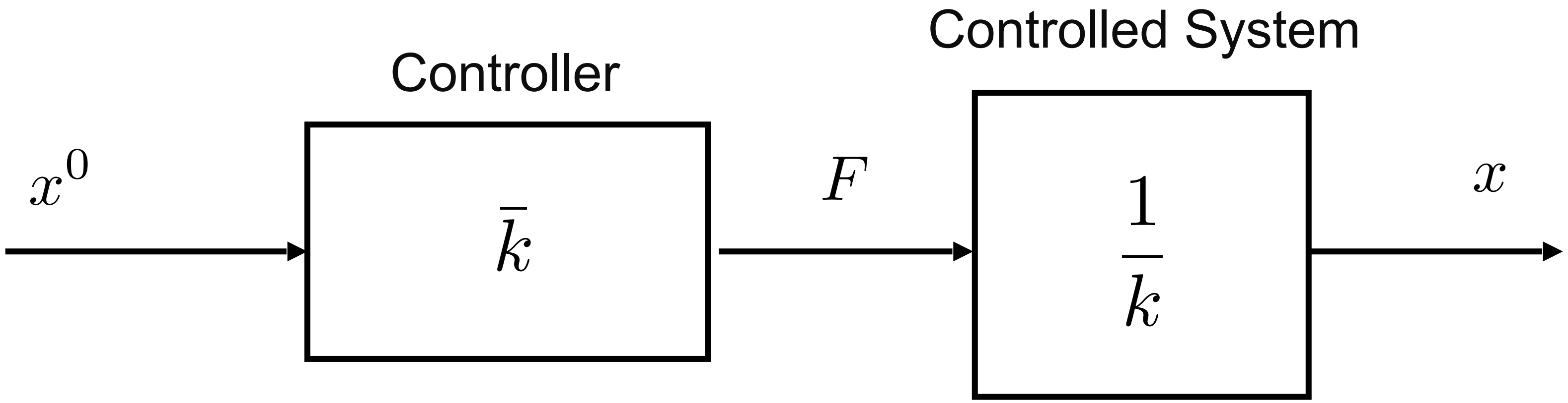
output input

Open-Loop Control Strategy



$$F = \bar{k} x^o$$

$\bar{k}$ nominal value of the spring constant



$$x = \frac{\bar{k}}{k} x^o \quad \Rightarrow \quad e = x^o - x = x^o \left(1 - \frac{\bar{k}}{k}\right)$$

Open-Loop Control Strategy (contd.)

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- In nominal conditions $k = \bar{k}$ ⟹ $e = 0$

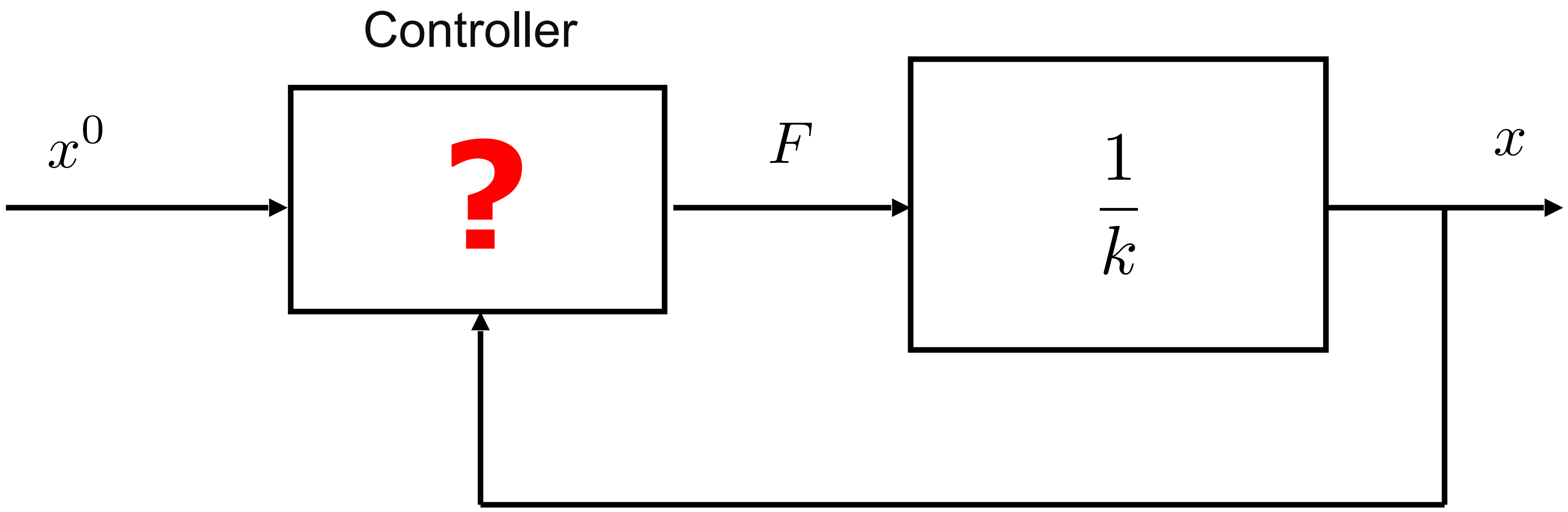
- In uncertain conditions $k \neq \bar{k}$ ⟹ $e = x^o \dfrac{\Delta k}{k} \neq 0$

$$\Delta k = k - \bar{k} \neq 0$$

uncertainty (Δk)

↳ **There is no way to compensate for the uncertainty by an open-loop control strategy**

Closed-Loop Control Strategy



Let us opt for a **proportional control** strategy:

$$F = \alpha \underbrace{(x^o - x)}_{e}, \; \alpha > 0$$

Closed-Loop Control Strategy (contd.)

Suppose $x^o \neq 0$:

$$x = \frac{1}{k}F = \frac{1}{k}\alpha\left(x^o - x\right) \Rightarrow x\left(1 + \frac{\alpha}{k}\right) = \frac{\alpha}{k}x^o$$

$$\Rightarrow x = \frac{\alpha/k}{1+\alpha/k}x^o \Rightarrow \boxed{e = x^o - x = \frac{1}{1+\alpha/k}x^o}$$

- In nominal conditions $k = \bar{k}$ $\Rightarrow$ $e \neq 0$

- In uncertain conditions $k \neq \bar{k}$ $\Rightarrow$ $e \simeq 0 \quad \text{if} \quad \alpha \gg k_{\max}$

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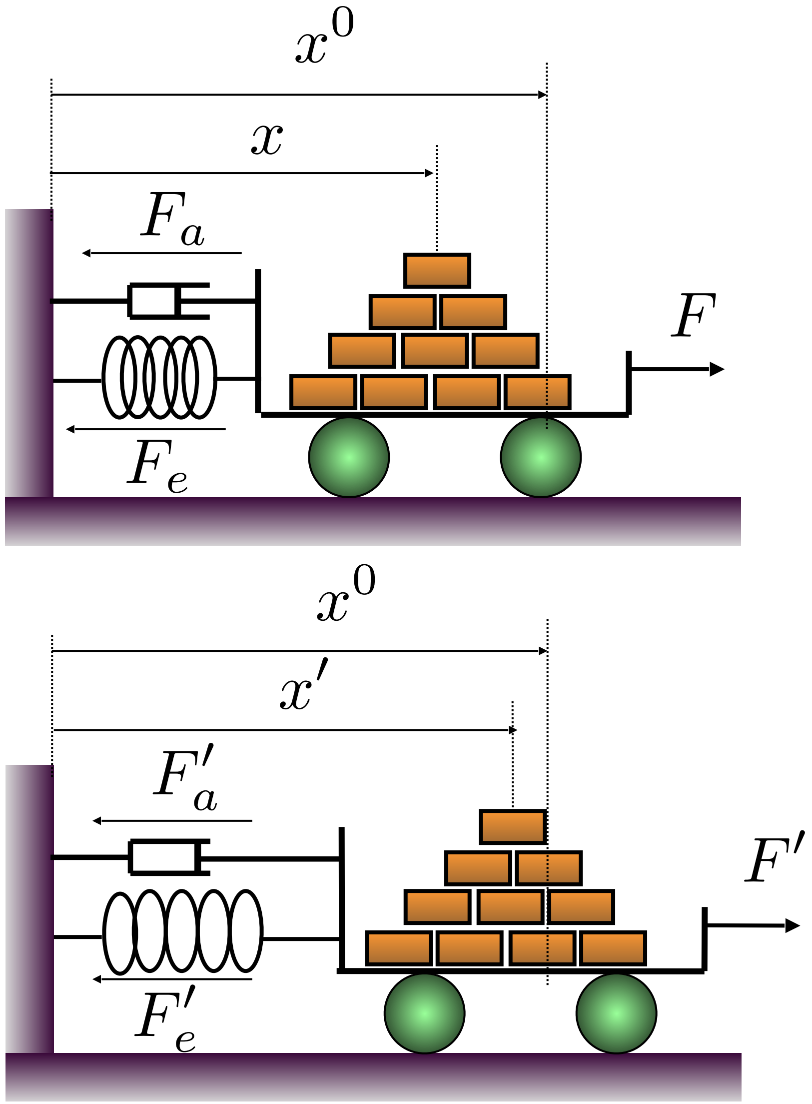


$$F = \alpha \underbrace{(x^o - x)}_{e}, \ \alpha > 0$$

The effects of mass, spring, and damper are not negligible, and dynamic behaviours such as oscillations may occur

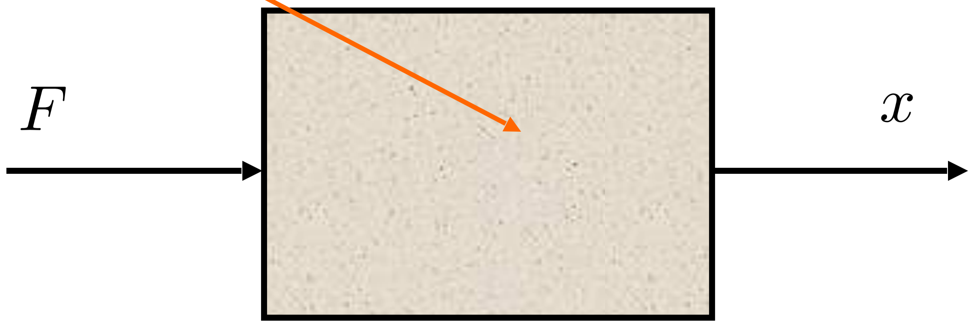
↳ **Models capturing the dynamic modes of behaviour are clearly necessary**

Dynamic Modelling (contd.)

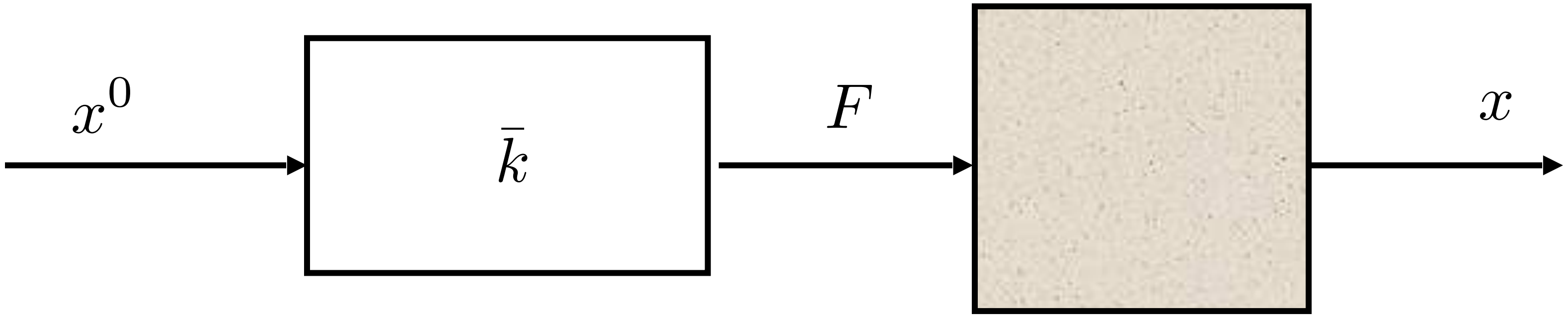
$$\text{sum of all external forces} = M\ddot{x}$$

$$F - kx - h\dot{x} = M\ddot{x}$$

$$M\ddot{x} + h\dot{x} + kx = F$$



Open-Loop Control Strategy



$$M\ddot{x} + h\dot{x} + kx = \bar{k}x^o$$

- $M\ddot{x} + h\dot{x}$: dynamic terms
- $\bar{k}x^o$: constant
- $kx = \bar{k}x^o$: condition for static equilibrium

$x(0)$, $\dot{x}(0)$: initial conditions

Open-Loop Control Strategy (contd.)

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Livescripts in MS Teams: see Part 1: CART_OPENLOOP_CONTROL

From differential equations theory:

$$M\ddot{x} + h\dot{x} + kx = \bar{k}x^o$$

constant term due to open-loop control

$$M\lambda^2 + h\lambda + k = 0 \quad \Rightarrow \quad \lambda_1, \lambda_2$$

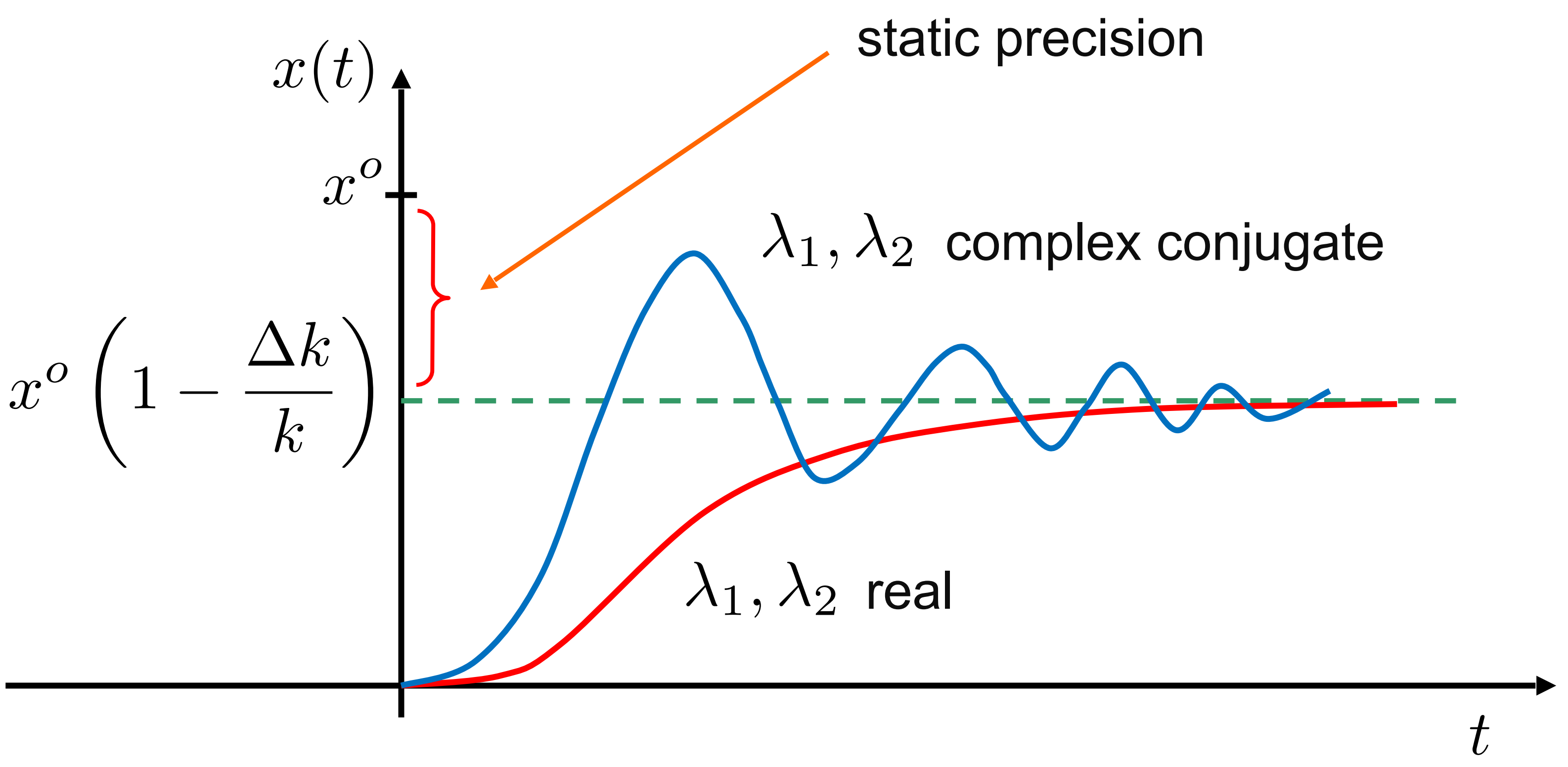
polynomial algebraic equation — roots

- If λ_1, λ_2 are real $\Rightarrow$ $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3$

- If λ_1, λ_2 are complex conjugate, that is $\lambda_1 = \sigma + j\omega,\ \lambda_2 = \sigma - j\omega$

$$x(t) = c_4 e^{\sigma t} \cos(\omega t + c_5) + c_6$$

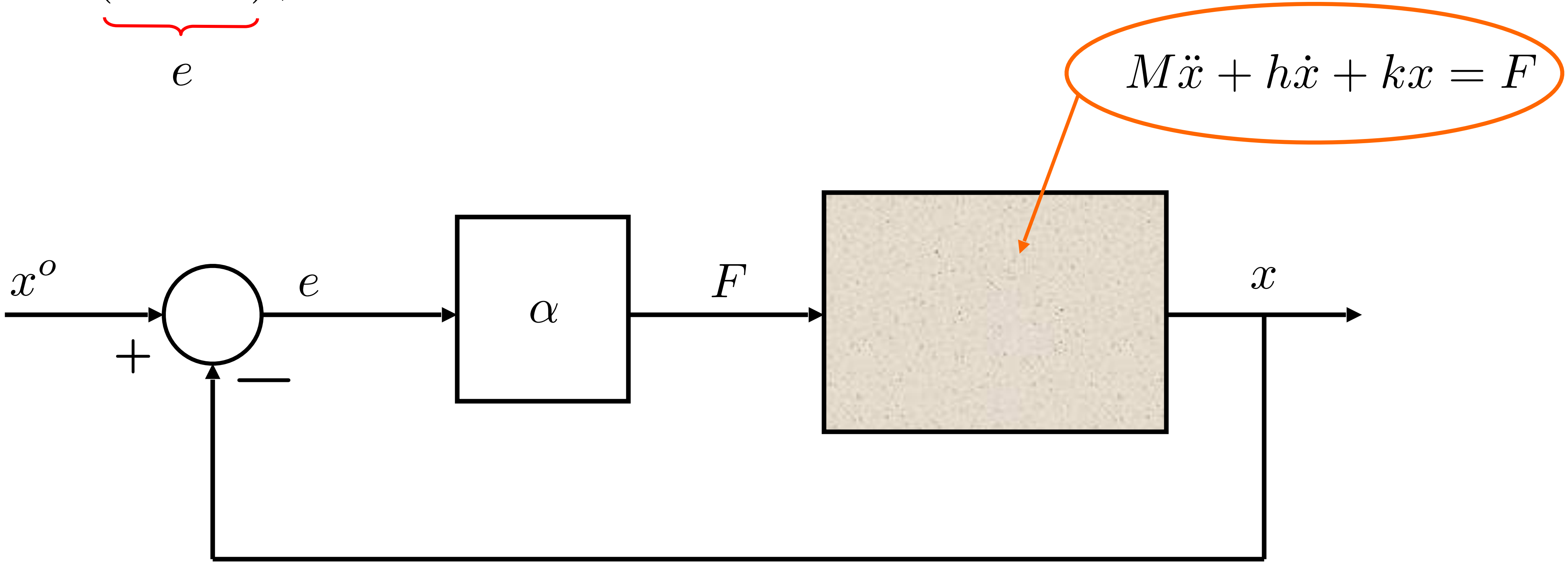
Open-Loop Control Strategy (contd.)



By the open-loop control strategy the dynamic behaviour cannot be modified since it only depends on the system's parameters M, k, h and not on the controller. Hence <u>dynamic precision cannot be modified by the controller</u>

Let us use again a **proportional closed-loop control** strategy:

$$F = \alpha \underbrace{(x^o - x)}_{e}, \quad \alpha > 0$$



Closed-Loop Control Strategy (contd.)

Livescripts in MS Teams: see Part 1: CART_CLOSEDLOOP_P_CONTROLLER

Plugging in the proportional control scheme we get:

$$M\ddot{x} + h\dot{x} + kx = \alpha(x^o - x)$$

$$M\ddot{x} + h\dot{x} + (k+\alpha)x = \alpha x^o$$

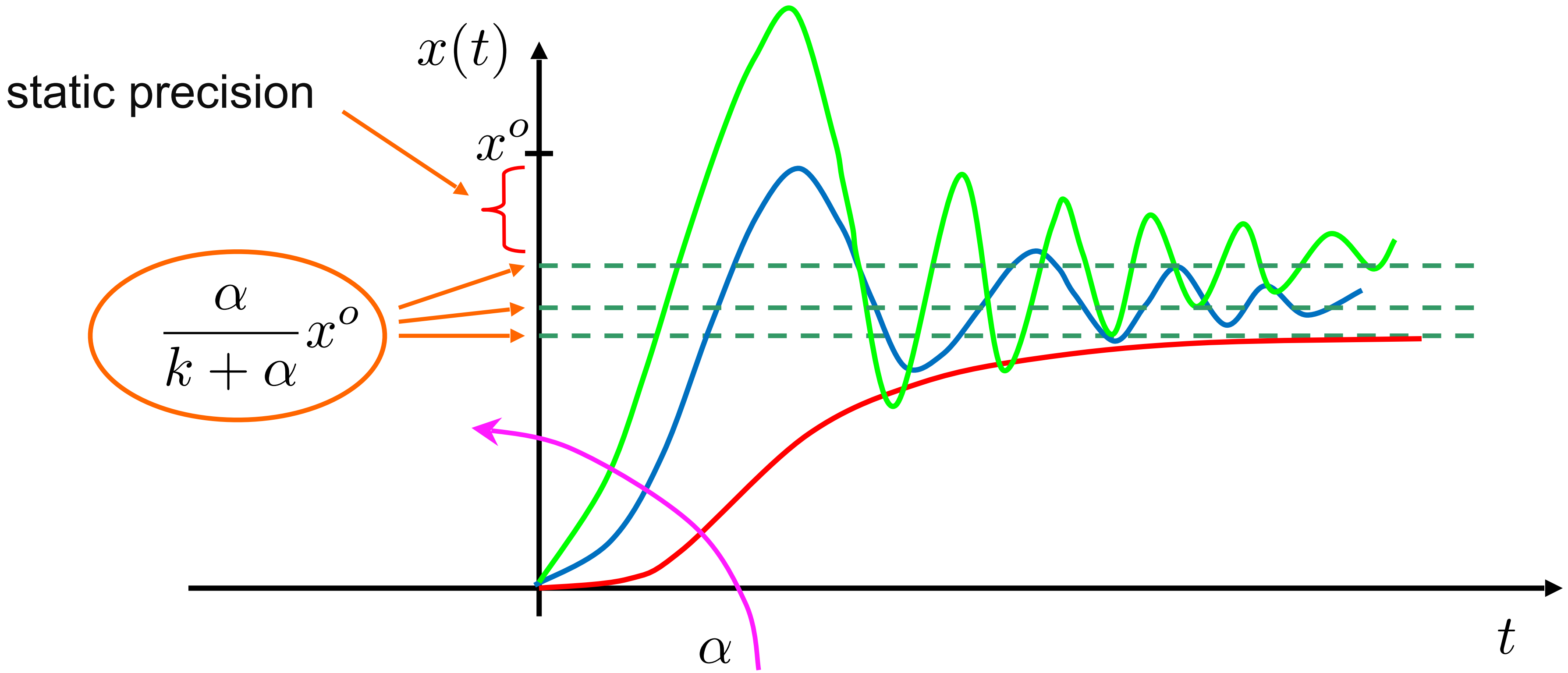
$$M\lambda^2 + h\lambda + (k+\alpha) = 0 \quad \Rightarrow \quad \lambda_1, \lambda_2$$

polynomial algebraic equation — roots

the constant term of the algebraic equation is **influenced by the controller gain** α

The roots λ_1, λ_2 can be modified by choosing α !!

Closed-Loop Control Strategy (contd.)



- **The dynamic precision also depends on the choice of the control gain**
- **Static and dynamic requirements are in contrast with each other: better static precision implies worse dynamic precision**

A Different Closed-Loop Control Strategy

Let us now opt for a **proportional/derivative** control scheme:

Livescripts in MS Teams: see Part 1: CART_CLOSEDLOOP_PD_CONTROLLER

$$F = \alpha\left(x^o - x\right) + \beta \frac{d}{dt}\left(x^o - x\right), \;\; \alpha, \beta > 0$$

$$M\ddot{x} + h\dot{x} + kx = \alpha(x^o - x) - \beta\dot{x}$$

$$M\ddot{x} + (h+\beta)\dot{x} + (k+\alpha)x = \alpha x^o$$

the first-order term of the algebraic equation is influenced by the **derivative controller gain** β

$$M\lambda^2 + (h+\beta)\lambda + (k+\alpha) = 0 \quad \Rightarrow \quad \lambda_1, \lambda_2$$

polynomial algebraic equation

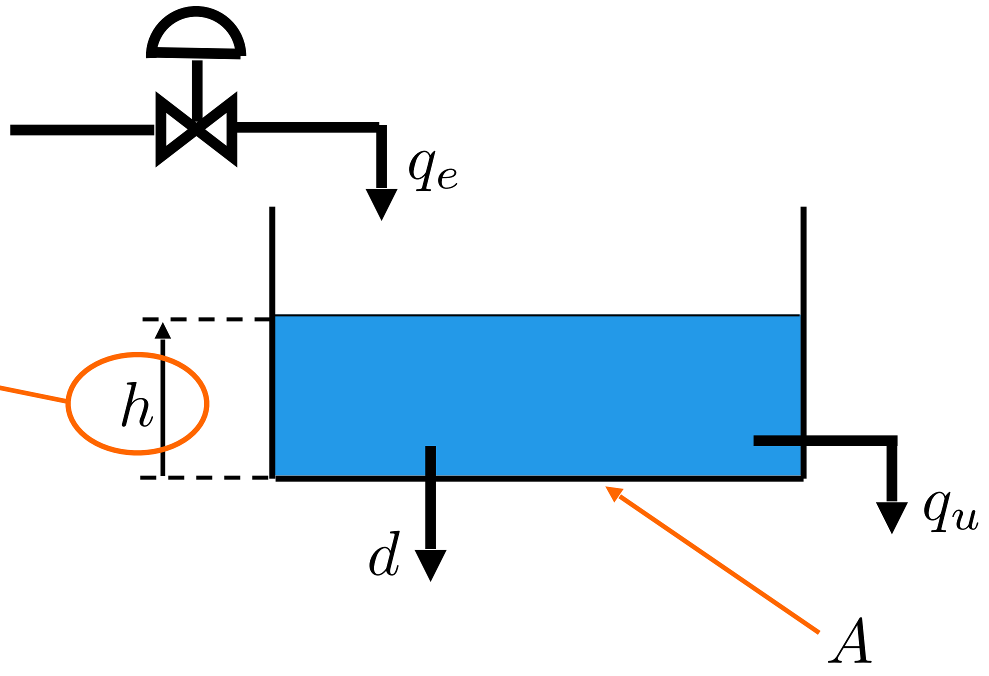
roots

the constant term of the algebraic equation is influenced by the **proportional controller gain** α

The roots λ_1, λ_2 can be modified by choosing α, β !!

Example 3: Tank Level Closed-Loop Control

Hydraulic system



- Control input: input flow-rate q_e
- Controlled output: level of liquid in the tank h
- Reference output: constant desired level in the tank $w = h^o$
- Out-flow rate: $q_u = kh$
- Disturbance: d

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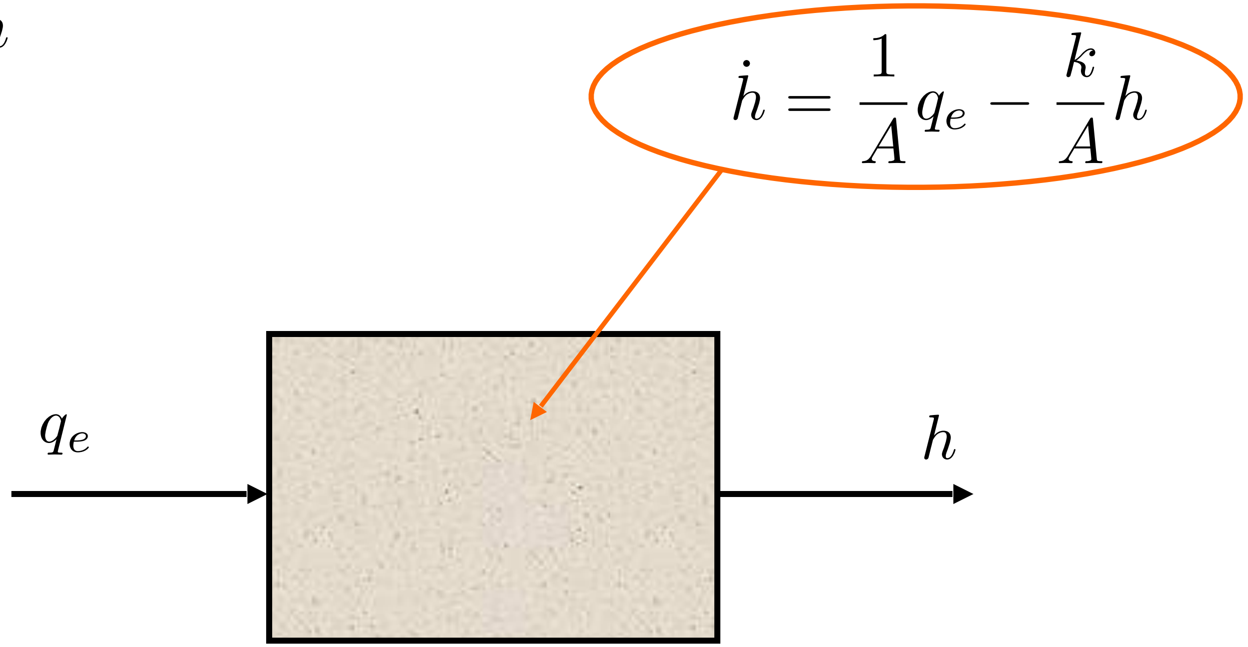
By the usual assumption of infinite-height of the tank and supposing (for the moment) that $d = 0$:

$$A\dot{h} = q_e - kh$$

conservation equation:

volume time-variation = input flow-rate - out flow-rate

$$\dot{h} = \frac{1}{A} q_e - \frac{k}{A} h$$

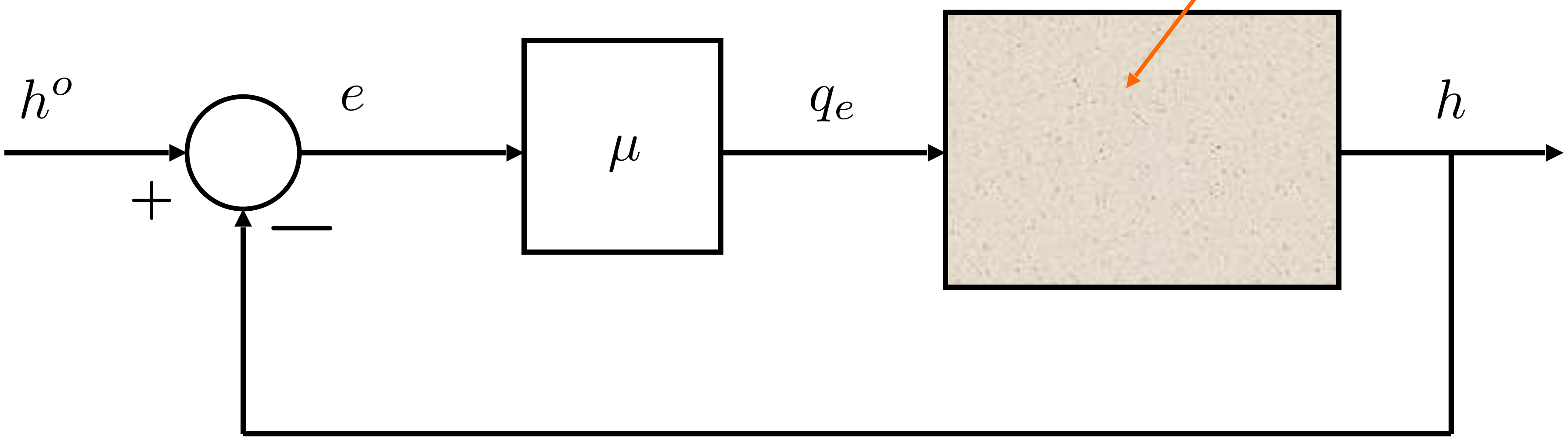


Closed-Loop Control Strategy

Let us opt for a **proportional control** strategy:

$$q_e = \mu \underbrace{(h^o - h)}_{e}, \;\; \mu > 0$$

$$\dot{h} = \frac{1}{A} q_e - \frac{k}{A} h$$



Closed-Loop Control Strategy (contd.)

Livescripts in MS Teams: see Part 1: TANK_CLOSEDLOOP_P_CONTROLLER

Plugging-in the proportional control scheme we get:

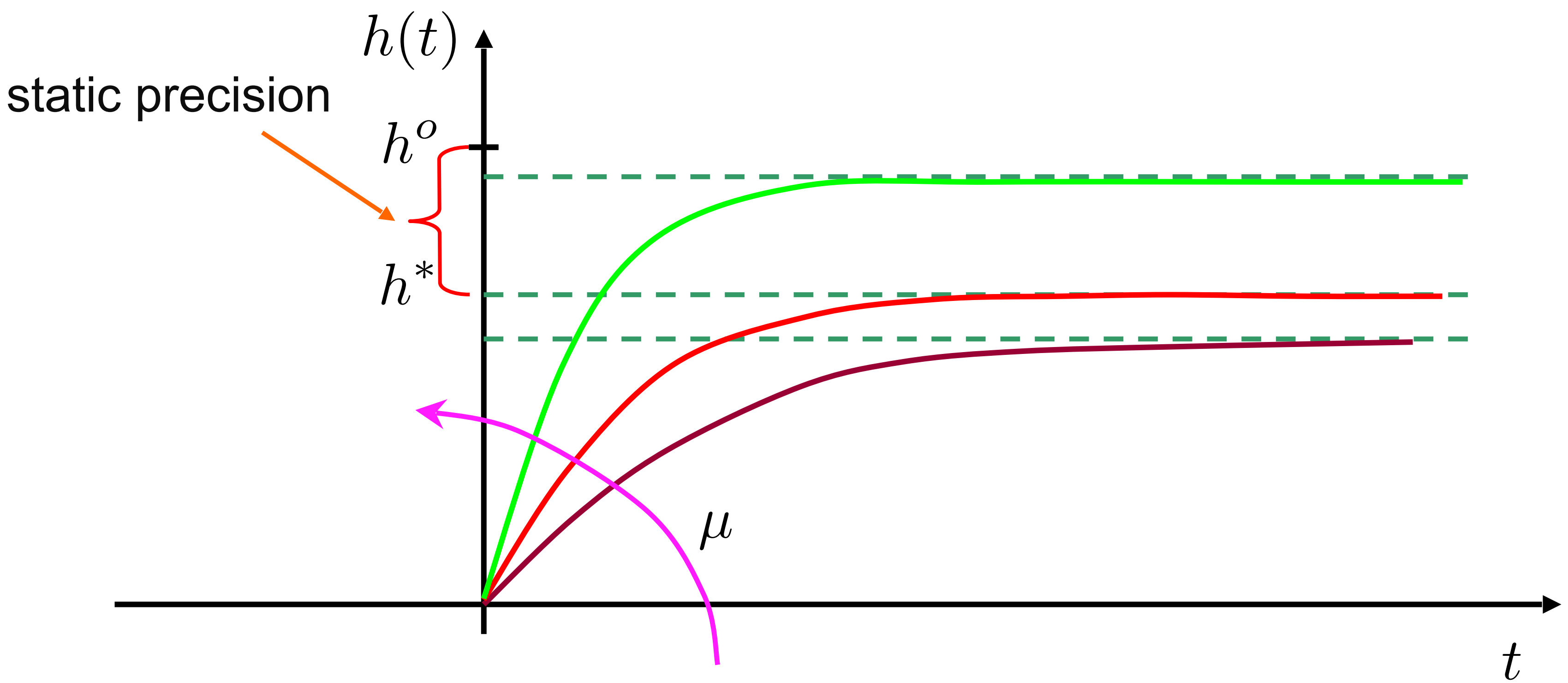
$$\dot{h} = \frac{1}{A}\mu\left(h^o - h\right) - \frac{k}{A}h$$

$$\dot{h} = \underbrace{-\frac{1}{A}\left(k+\mu\right)}_{\sigma > 0} h + \underbrace{\frac{\mu}{A}h^o}_{\text{constant}}$$

Assuming $h(0) = 0$ we get:

$$h(t) = \underbrace{\frac{\mu}{\mu + k}h^o}_{h^* = \lim_{t\to\infty} h(t)} \left(1 - e^{-\sigma t}\right), t \geq 0$$

Closed-Loop Control Strategy (contd.)



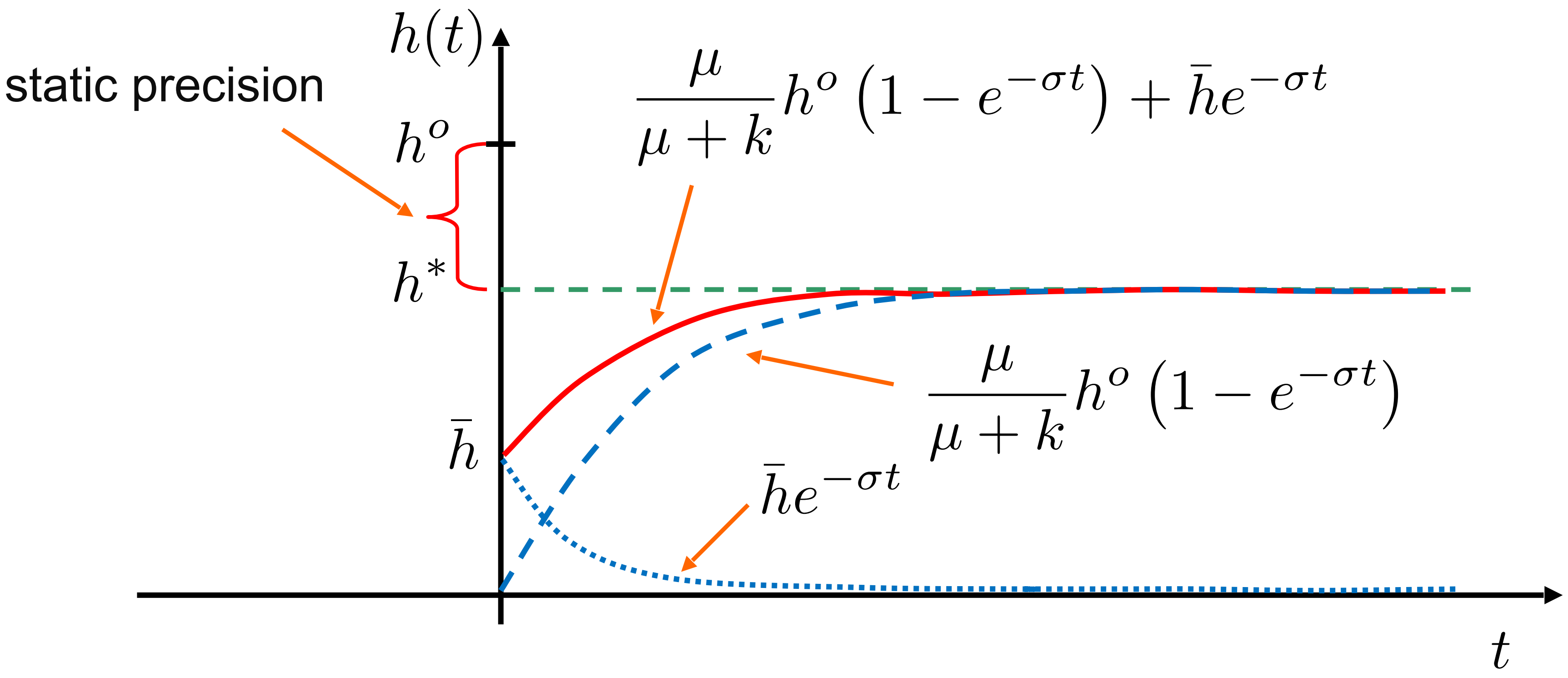
- **By increasing the control gain μ both the static and the dynamic precision improve**
- **In this hydraulic example, static and dynamic requirements are not in contrast with each other**

Non-Zero Initial Conditions

Assuming $h(0) = \bar{h} > 0$ we get:

$$h(t) = \underbrace{\frac{\mu}{\mu + k} h^o \left(1 - e^{-\sigma t}\right)}_{\text{effect of the control input}} + \overbrace{\bar{h} e^{-\sigma t}}^{\text{effect of the non-zero initial condition}}, t \geq 0$$

Non-Zero Initial Conditions (contd.)

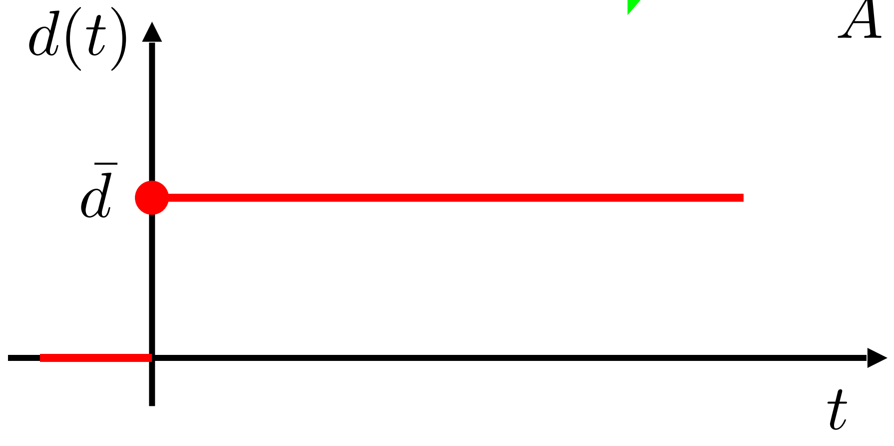


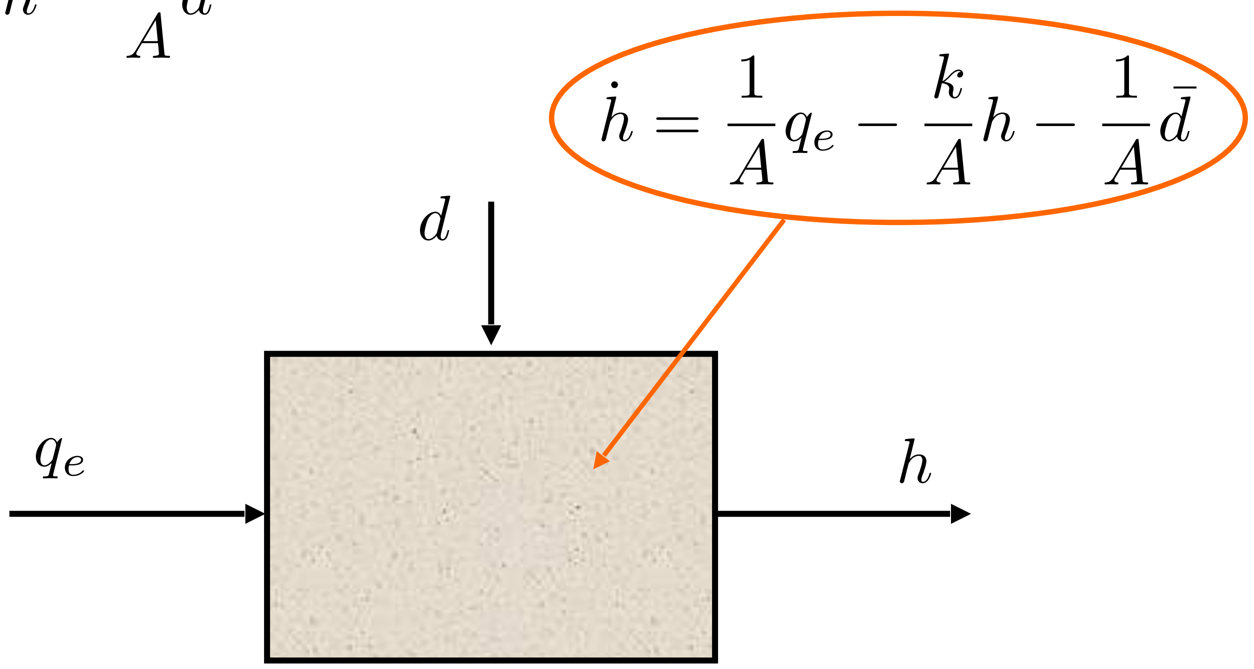
Dynamic Modelling with a Disturbance

By the usual assumption of infinite-height of the tank and supposing that a step-like disturbance $d(t) = \bar{d},\ t \geq 0$ (leakage) is acting on the tank:

$$A\dot{h} = q_e - kh - \bar{d}$$

$$\dot{h} = \frac{1}{A}q_e - \frac{k}{A}h - \frac{1}{A}\bar{d}$$





Closed-Loop Control Strategy

Plugging in the proportional control scheme we get:

$$q_e = \mu \left(h^o - h \right), \;\; \mu > 0$$

$$\dot{h} = \frac{1}{A} \mu \left(h^o - h \right) - \frac{k}{A} h - \frac{1}{A} \bar{d}$$

$$\dot{h} = \underbrace{-\frac{1}{A} \left(k + \mu \right)}_{\sigma > 0} h + \underbrace{\frac{\mu}{A} h^o - \frac{1}{A} \bar{d}}_{\text{constant}}$$

Closed-Loop Control Strategy (contd.)

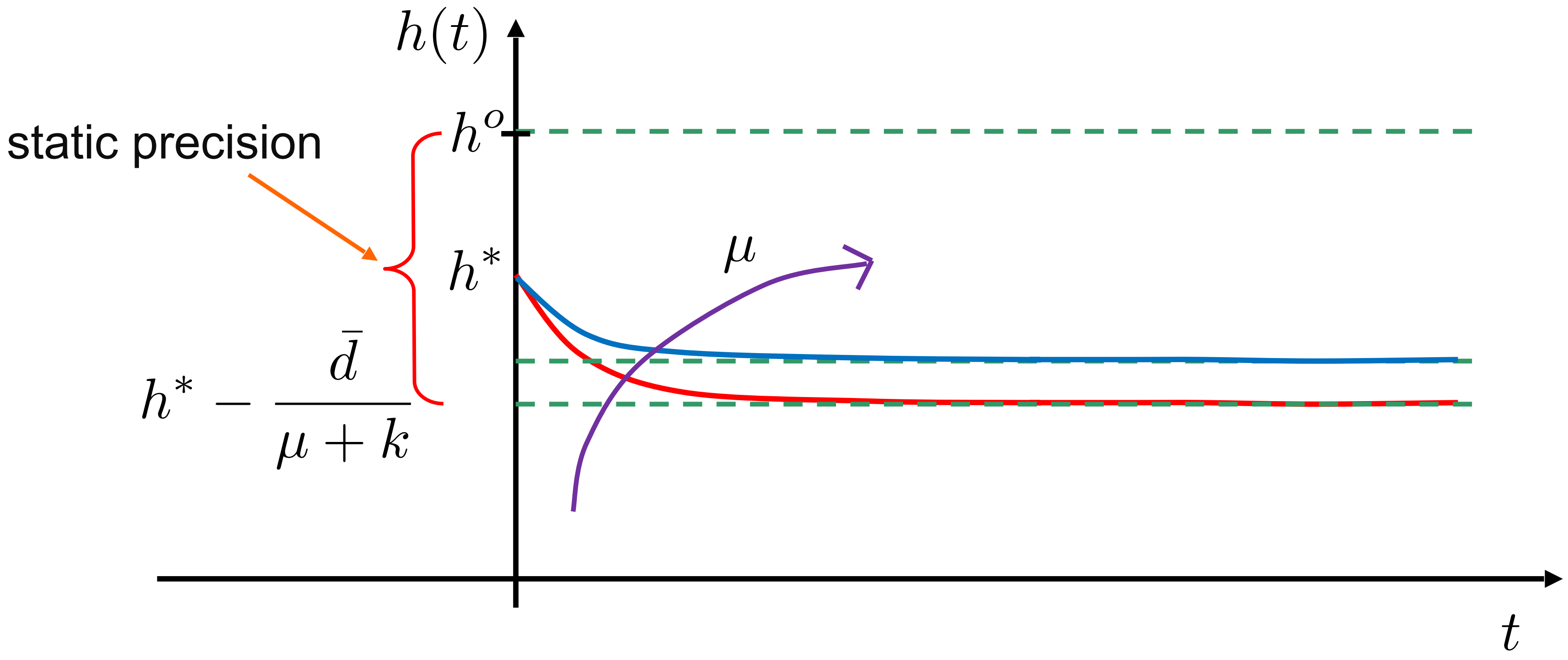
For simplicity suppose: $h(0) = h^* = \frac{\mu}{\mu + k} h^o$

$$h(t) = \frac{\mu}{\mu + k} h^o \left(1 - e^{-\sigma t}\right) + h(0) e^{-\sigma t} - \frac{\bar{d}}{\mu + k} \left(1 - e^{-\sigma t}\right)$$

$$= h^* - h^* e^{-\sigma t} + h^* e^{-\sigma t} - \frac{\bar{d}}{\mu + k} \left(1 - e^{-\sigma t}\right)$$

$$h(t) = h^* - \frac{\bar{d}}{\mu + k} \left(1 - e^{-\sigma t}\right), t \geq 0$$

$$\lim_{t \to \infty} h(t) = h^* - \frac{\bar{d}}{\mu + k}$$



Increasing the proportional gain μ improves the disturbance rejection performance

How to Improve the Static Precision?

A) Adding open-loop compensation actions:

$$q_e = \mu\left(h^o - h\right) + \overbrace{kh^o + \bar{d}}^{\text{open-loop action}}$$

$$\dot{h} = \frac{1}{A}\mu\left(h^o - h\right) + \frac{k}{A}h^o + \frac{1}{A}\bar{d} - \frac{k}{A}h - \frac{1}{A}\bar{d}$$

$$h(t) = h^o\left(1 - e^{-\sigma t}\right), t \geq 0$$

$$\lim_{t\to\infty} h(t) = h^o$$

But:

to implement the open-loop compensation actions, the knowledge of $k, \bar{d}$ is required

How to Improve the Static Precision?

B) Modifying the closed-loop controller by adding an **integral action** thus obtaining a **proportional/integral (PI) controller**:

Livescripts in MS Teams: see Part 1: TANK_CLOSEDLOOP_PI_CONTROLLER

$$q_e = \mu \left[h^o - h(t) \right] + \varphi \int_0^t \left[h^o - h(\tau) \right] d\tau \quad \mu, \varphi > 0$$

Justification:

$$q_e = \mu e(t) + \varphi \int_0^t e(\tau) d\tau \quad \Longrightarrow \quad \dot{q}_e = \mu \dot{e}(t) + \varphi e(t)$$

Equilibrium:

$$\begin{cases} e(t) = \text{cost} = \bar{e} \\ q_e(t) = \text{cost} \end{cases} \quad \Longrightarrow \quad 0 = \mu \cdot 0 + \varphi \cdot \bar{e}$$

$$\boxed{\bar{e} = 0}$$