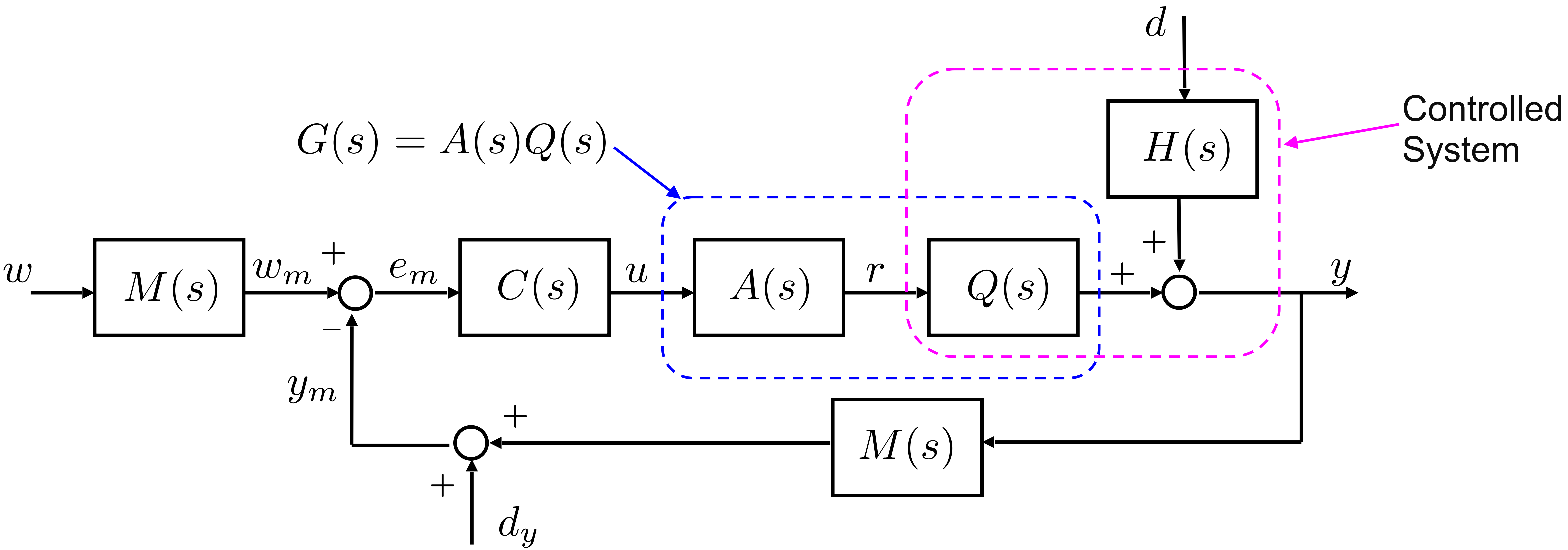
034IN - FONDAMENTI DI AUTOMATICA - FUNDAMENTALS OF AUTOMATIC CONTROL
A.Y. 2025-2026
Part X: Design of Feedback Control Systems

Gianfranco Fenu, Thomas Parisini

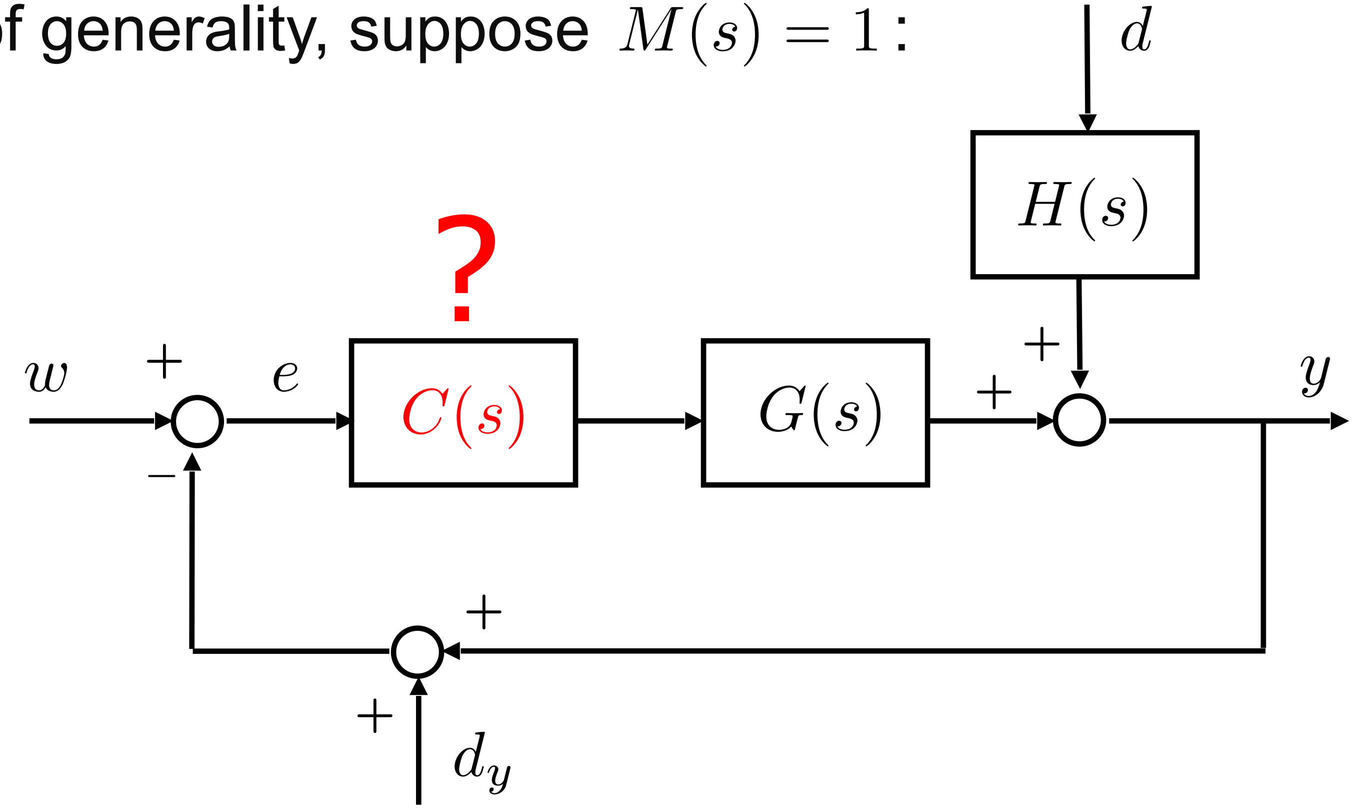
Department of Engineering and Architecture

Recall from Part 9, slide 10 (adding the output disturbance d_y):



Design of the Controller

Without loss of generality, suppose $M(s) = 1$:



- **Controller Design**: Determine the transfer function $C(s)$ of the controller such that the **closed-loop requirements are satisfied**
- **Very different** from the analysis dealt with in Part 9: now the open-loop transfer function $L(s) = C(s)G(s)$ is **unknown** because $C(s)$ is **unknown**!

Closed-Loop Qualitative Requirements on Bode Diagrams

- Closed-loop asymptotic stability $\Rightarrow$ $\begin{cases} \mu > 0 \\ \varphi_m > 0 \end{cases}$

- Static precision $\Rightarrow$ $\begin{cases} g > 0 & \text{(pole(s) in 0)} \\ & \text{and/or} \\ \mu & \text{“large enough”} \end{cases}$

- Dynamic precision
  - speed of the closed-loop response $\Rightarrow$ ω_c “large enough”
  - closed-loop damping ratio $\Rightarrow$ φ_m “large enough”

Closed-Loop Qualitative Requirements on Bode Diagrams (contd.)

UNIVERSITÀ DEGLI STUDI DI TRIESTE

- Disturbance rejection on direct path $\Rightarrow$ $\begin{cases} \omega_c \text{ “large enough”} \\ |L(j\omega)|_{\mathrm{dB}}\,,\ \omega < \omega_c \text{ “large enough”} \end{cases}$

- Disturbance rejection on feedback path $\Rightarrow$ $\begin{cases} \omega_c \text{ “not too large”} \\ |L(j\omega)|_{\mathrm{dB}}\,,\ \omega < \omega_c \text{ “small”} \end{cases}$

- Robust stability $\Rightarrow$ $\begin{cases} \varphi_m \\ K_m \end{cases}$ “large enough”

UNIVERSITÀ DEGLI STUDI DI TRIESTE

Starting from closed-loop requirements, precise specifications on Bode diagrams are devised

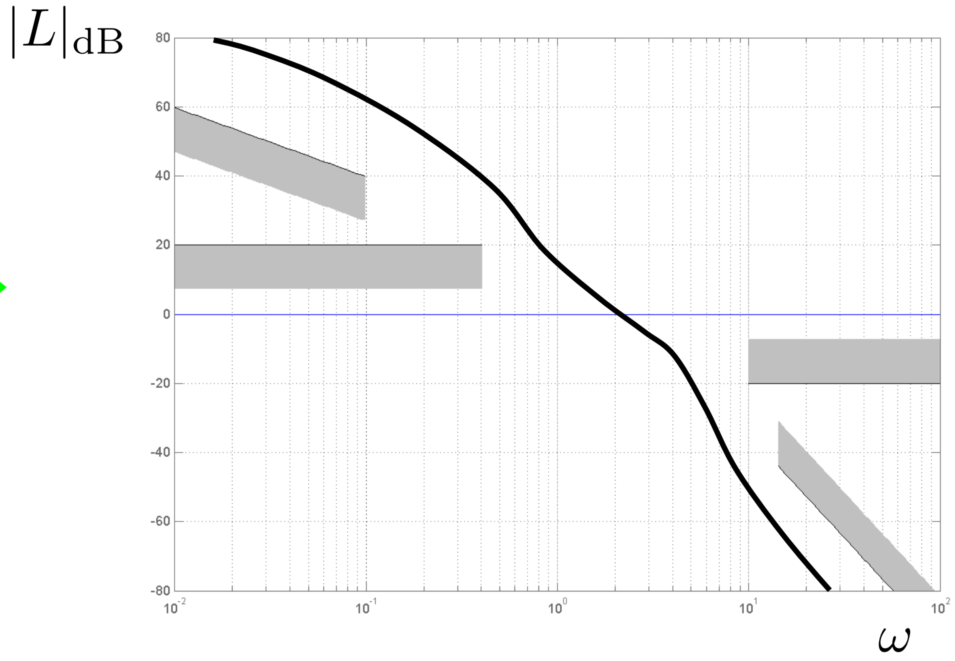
Example

- Static Specifications

$$|e(\infty)| \leq \bar{e}\,, \text{ with } w, d \text{ specified}$$

- Dynamic Specifications
  - $\omega_{\min} \leq \omega_c \leq \omega_{\max}$
  - $\varphi_m \geq \bar{\varphi}_m$
  - $K_m \geq \bar{K}_m$

Constraints on **open-loop** Bode diagrams of $|L(j\omega)|_{\mathrm{dB}}$



Design Specifications on Bode Diagrams: Example

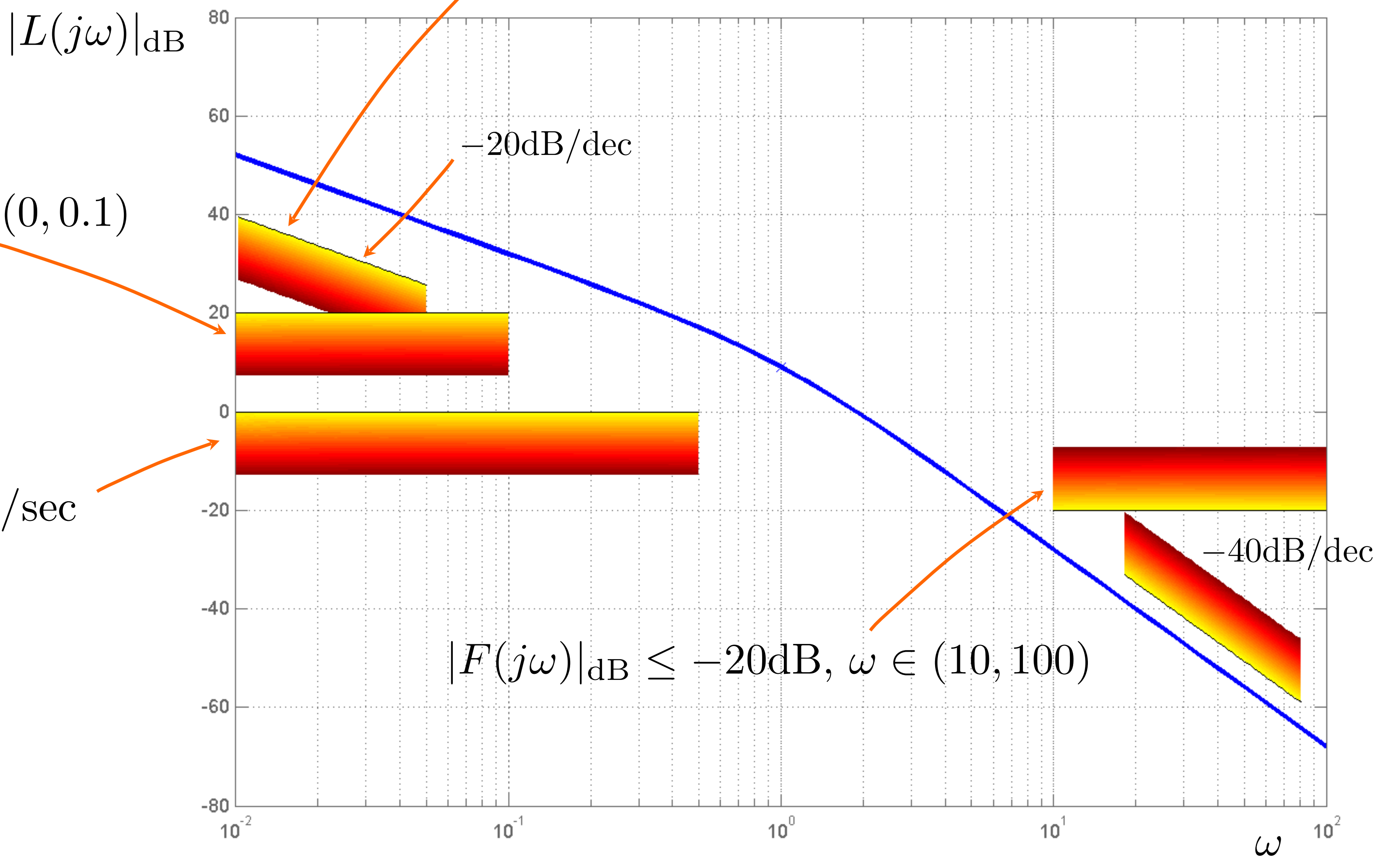
$$L(s) = \frac{4}{s(s+1)}$$

$|e(\infty)| = 0, \ \text{with } w(t) = A \cdot 1(t), \ \forall A$

$|S(j\omega)|_{\text{dB}} \leq -20\text{dB}, \ \omega \in (0, 0.1)$

$\Delta\% \leq 3 \ \Rightarrow \ \xi \geq 0.75$

$t_s \leq 10\,\text{sec} \ \Rightarrow \ \omega_c \geq 0.5\text{rad/sec}$



Design Specifications on Root Locus

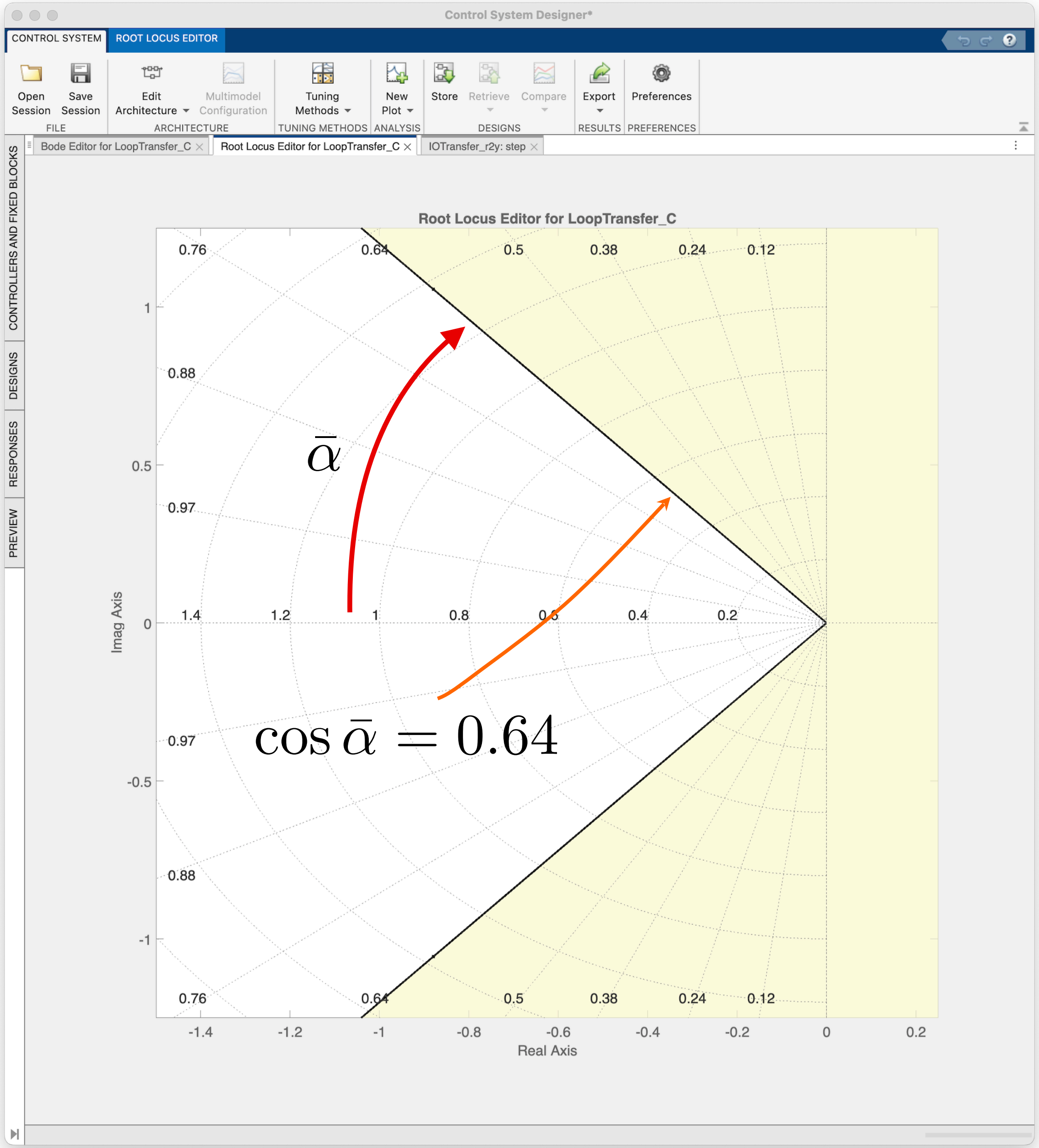
Closed-loop requirement:

- the **damping ratio** ξ has to satisfy the constraint

$$\xi \geq \bar{\xi}$$

- The region of the complex plane where the constraint $\xi \geq \bar{\xi}$ is satisfied can be drawn as a **graphical constraint** on the RL:

$$s \in \mathbb{C} : \; \xi = \cos\alpha \geq \bar{\xi} = \cos\bar{\alpha}$$



Design Specifications on Root Locus (contd.)

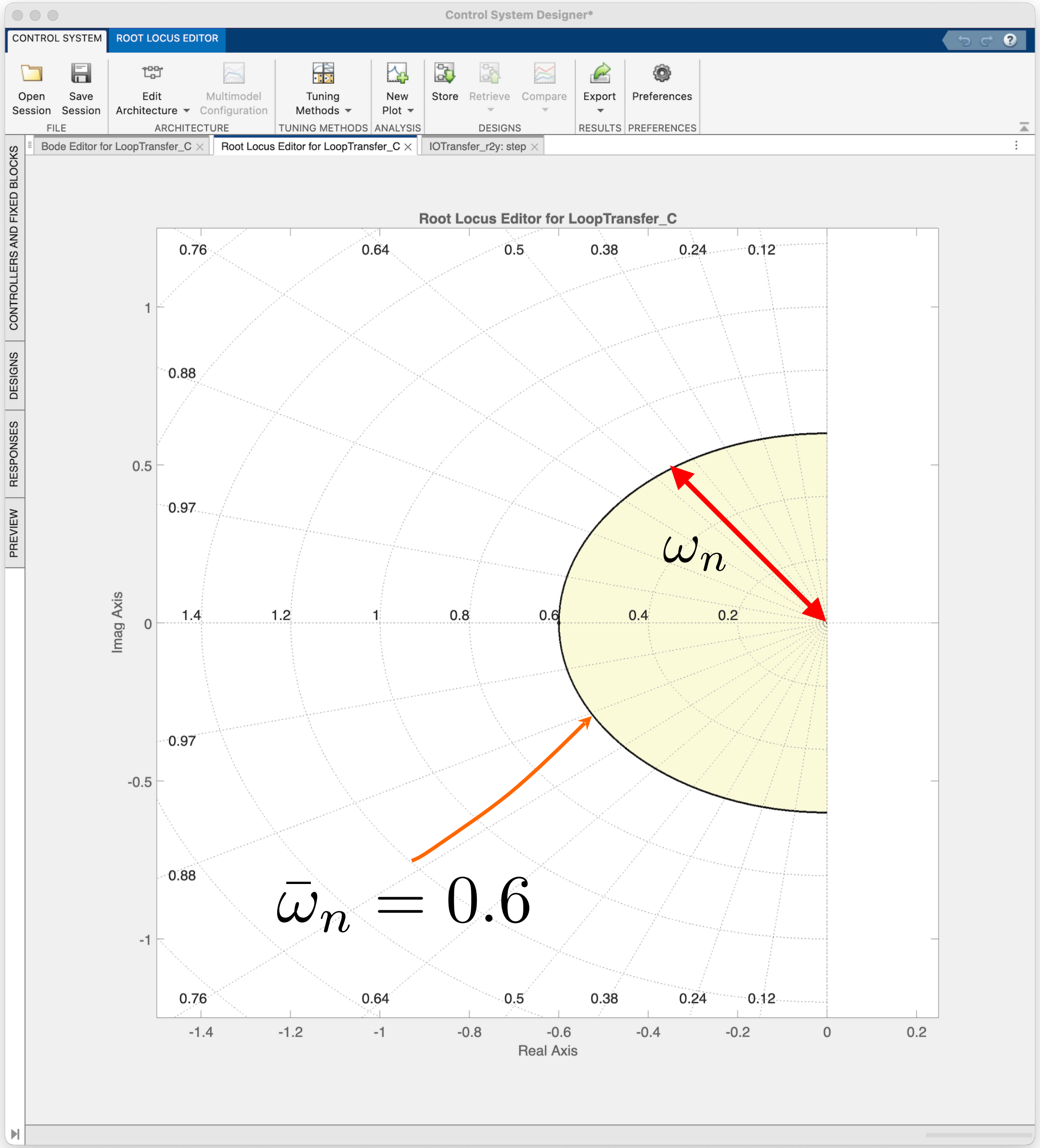
Closed-loop requirement:

- the **natural angular frequency** ω_n has to satisfy the constraint

$$\omega_n \geq \bar{\omega}_n$$

- The region of the complex plane where the constraint $\omega_n \geq \bar{\omega}_n$ is satisfied can be drawn as a **graphical constraint** on the RL:

$$s \in \mathbb{C} : \omega_n \geq \bar{\omega}_n$$



Design Specifications on Root Locus (contd.)

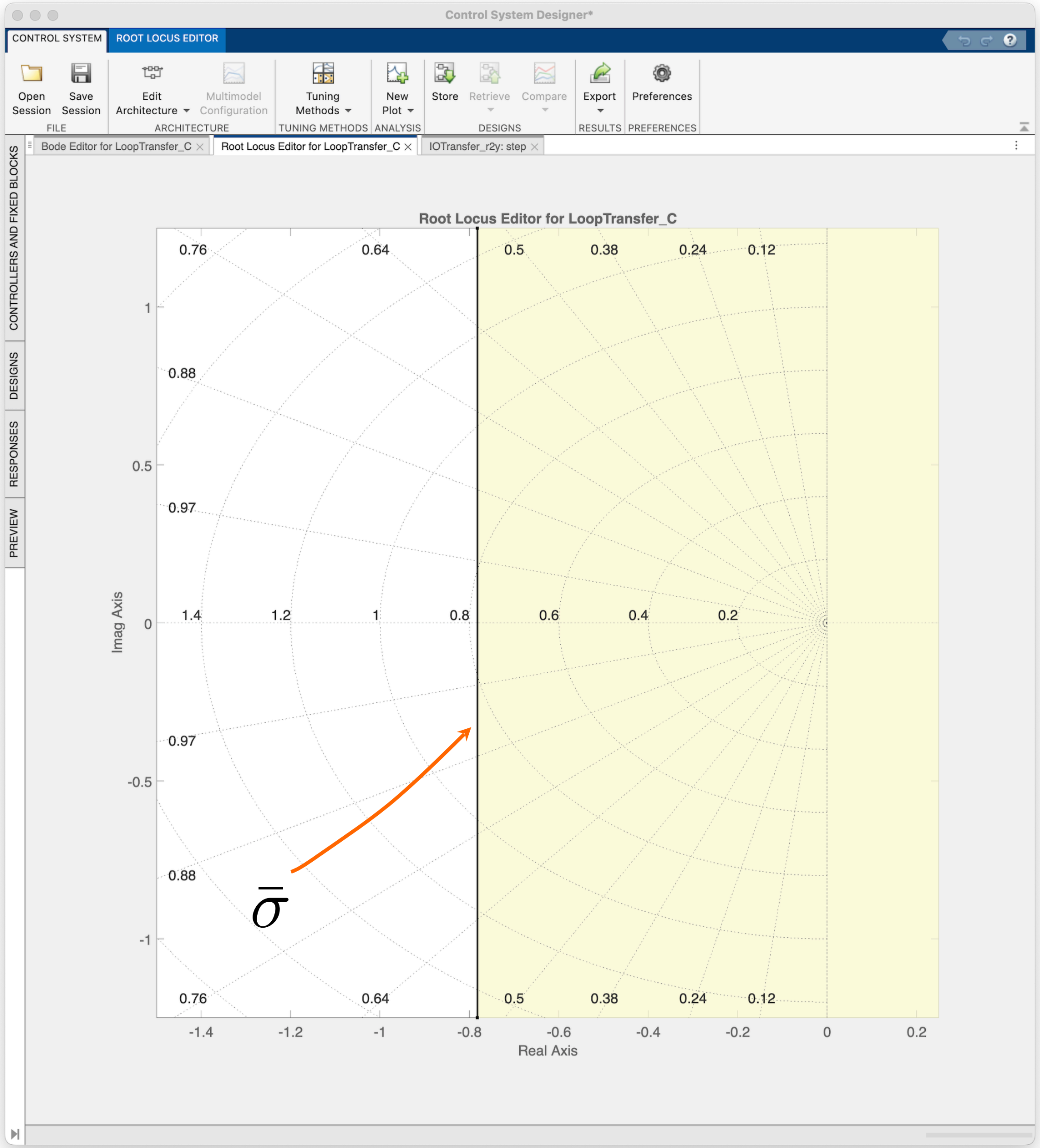
Closed-loop requirement:

- the (negative) **real part of the poles** σ has to satisfy the constraint

$$\sigma \leq \bar{\sigma}$$

- The region of the complex plane where the constraint $\sigma \leq \bar{\sigma}$ is satisfied can be drawn as a **graphical constraint** on the RL:

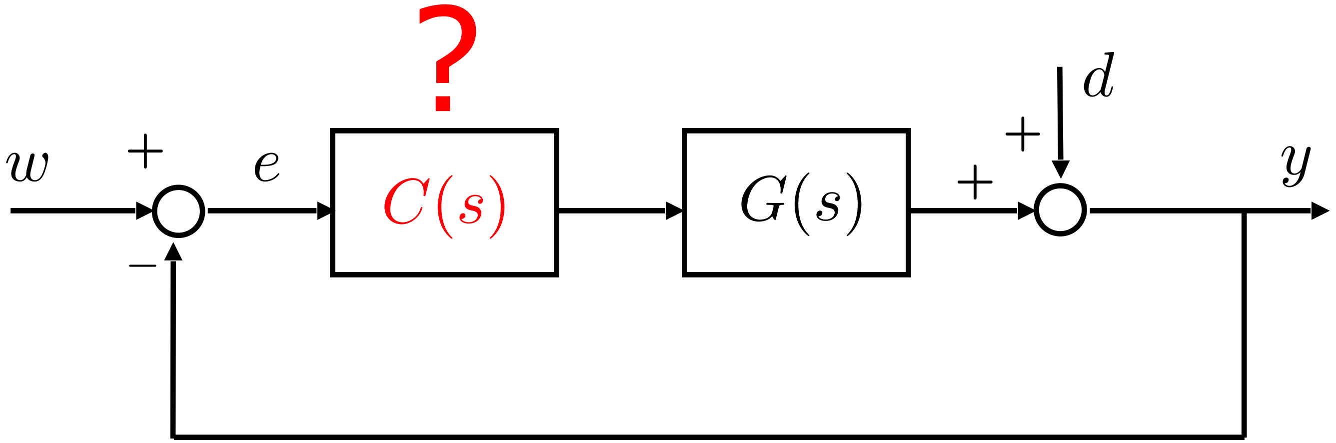
$$s \in \mathbb{C} : \ \sigma = \mathrm{Re}(s) \leq \bar{\sigma}$$



Loop-Shaping Iterative Controller Design

- Analyse the **closed-loop requirements** and translate them to **design specifications on the Bode diagrams**
- Select an **initial attempt** for the controller $C(s)$ using a very simple structure. For example, a typical initial choice is an algebraic controller $C_0(s) = \mu_C$
- Check whether the chosen controller satisfies **all specifications**. If not, consider another attempt using a more complex structure of the controller and proceed with the design trying to meet all specifications
- **Iterate** the procedure till when all specifications are met

Loop-Shaping Iterative Controller Design: Example 1



$$G(s) = \frac{10}{(1+10s)(1+5s)(1+s)}$$

- **Design specifications**:
  - $|e(\infty)| \leq 0.1 \quad \text{with} \begin{cases} w(t) = A \cdot 1(t), & |A| \leq 1 \\ d(t) = B \cdot 1(t), & |B| \leq 5 \end{cases}$
  - $\omega_c \geq 0.2$
  - $\varphi_m \geq 60°$
- **Controller Structure**:

used to meet **static** specifications

used to meet **dynamic** specifications

$$C(s) = C_1(s) \cdot C_2(s) \quad \text{where} \;\; C_1(s) = \frac{\mu_C}{s^g}\,; \qquad C_2(s) = \frac{\prod (1+sT_i)}{\prod (1+s\tau_i)}$$

Loop-Shaping Iterative Controller Design: Example 1 (contd.)

$$L(s) = C(s)G(s) = C_1(s)C_2(s)\frac{10}{(1+10s)(1+5s)(1+s)}$$

Static Design:

Since $C_2(0) = 1$ the **static design is not influenced by**

gain: $10\mu_C > 0$

type: g

Due to the **linearity** of the closed-loop system:

$$e(\infty) = e_w(\infty) + e_d(\infty)$$

where:

$$e_w(t) = e(t)\,|_{d(t)=0} \;; \quad e_d(t) = e(t)\,|_{w(t)=0}$$

Hence:

$$e(\infty) = e_w(\infty) + e_d(\infty)$$

$$|e(\infty)| \leq |e_w(\infty)| + |e_d(\infty)|$$

and:

$$|e_w(\infty)| = \begin{cases} \dfrac{|A|}{1+10\mu_C} \leq \dfrac{1}{1+10\mu_C}\,, & \text{if } g = 0 \\ 0\,, \ \text{if } g > 0 \end{cases}$$

$$|e_d(\infty)| = \begin{cases} \dfrac{|B|}{1+10\mu_C} \leq \dfrac{5}{1+10\mu_C}\,, & \text{if } g = 0 \\ 0\,, \ \text{if } g > 0 \end{cases}$$

Loop-Shaping Iterative Controller Design: Example 1 (contd.)

UNIVERSITÀ DEGLI STUDI DI TRIESTE

- **Option 1**: static design with $g = 0$ (no poles in the origin introduced by $C_1(s)$)

$$|e(\infty)| \leq \frac{1}{1+10\mu_C} + \frac{5}{1+10\mu_C} = \frac{6}{1+10\mu_C}$$

Imposing $|e(\infty)| \leq 0.1$:

$$\frac{6}{1+10\mu_C} \leq 0.1 \quad \Longrightarrow \quad \mu_C \geq 5.9$$

Possible choice: $\boxed{C_1(s) = 8}$

- **Option 2**: static design with $g = 1$ (one pole in the origin introduced by $C_1(s)$)

$$e(\infty) = 0\,, \quad \forall\, \mu_C \quad \Longrightarrow \quad \boxed{C_1(s) = \frac{\mu_C}{s}}$$

Dynamic Design, Option (A):

Pick the static controller: $C_1(s) = 8$

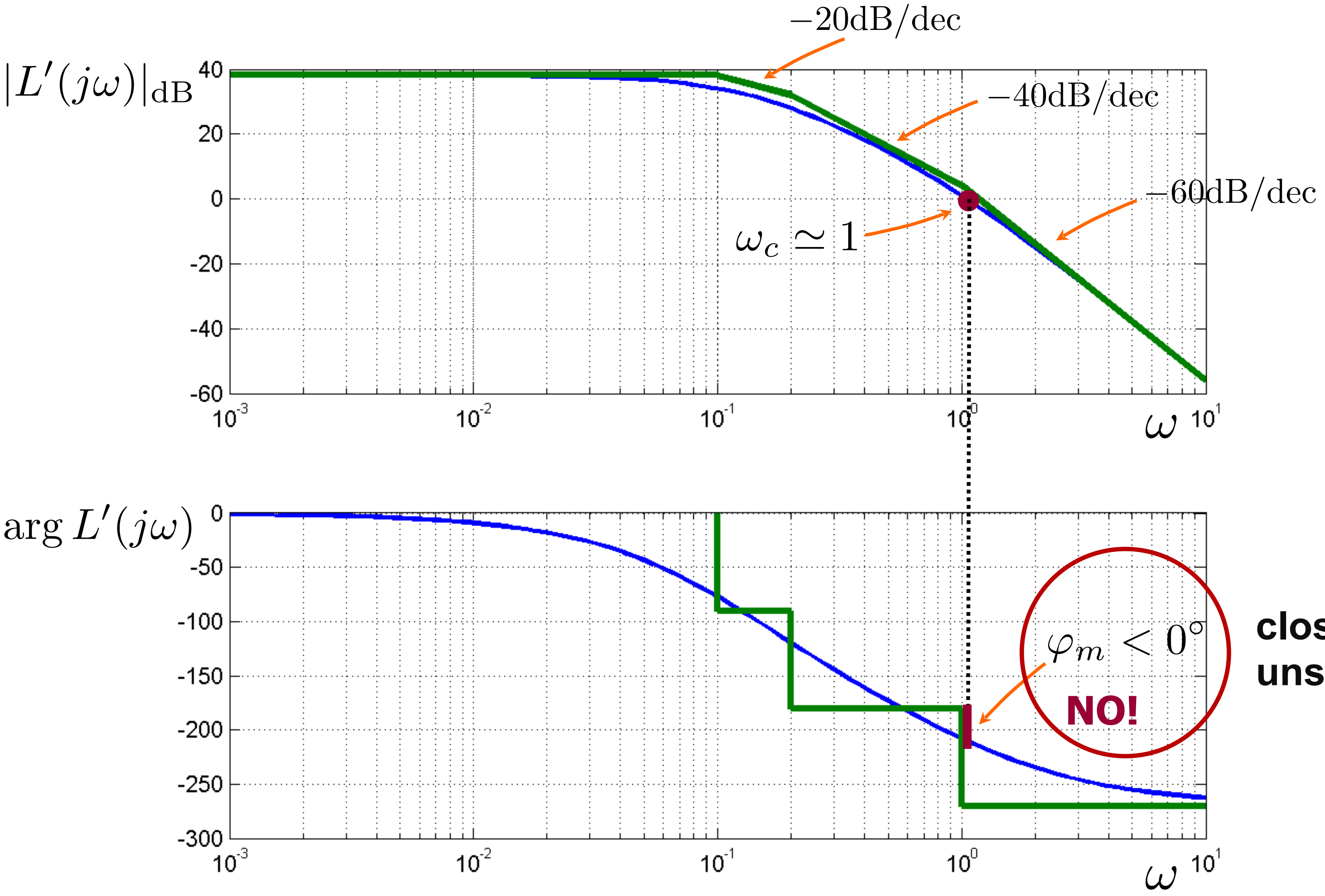
$$L(s) = 8\,G(s)\,C_2(s) = L'(s)\,C_2(s)$$

$$\text{with} \quad L'(s) = \frac{80}{(1+10s)(1+5s)(1+s)}$$

We proceed by a **sequence of attempts** with increasing complexity of the structure of controller $C_2(s)$ by following the **iterative procedure** outlined in Slide 8:

Loop-Shaping Iterative Controller Design: Example 1 (contd.)

Attempt 1: $C_1(s) = 8,\ C_2(s) = 1$



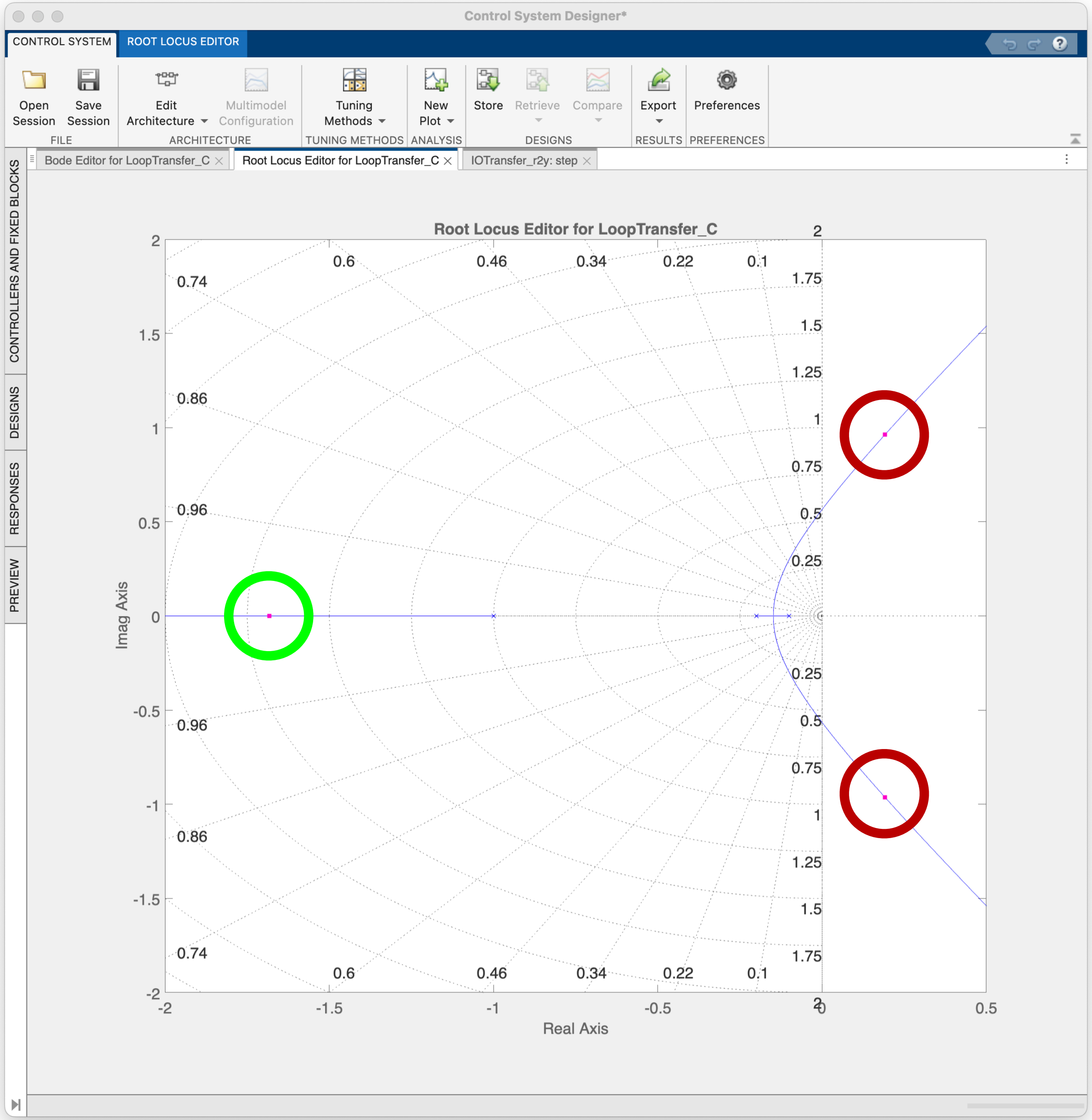
closed-loop unstable

Attempt 1: $C_1(s) = 8,\ C_2(s) = 1$

unstable **closed-loop** pole

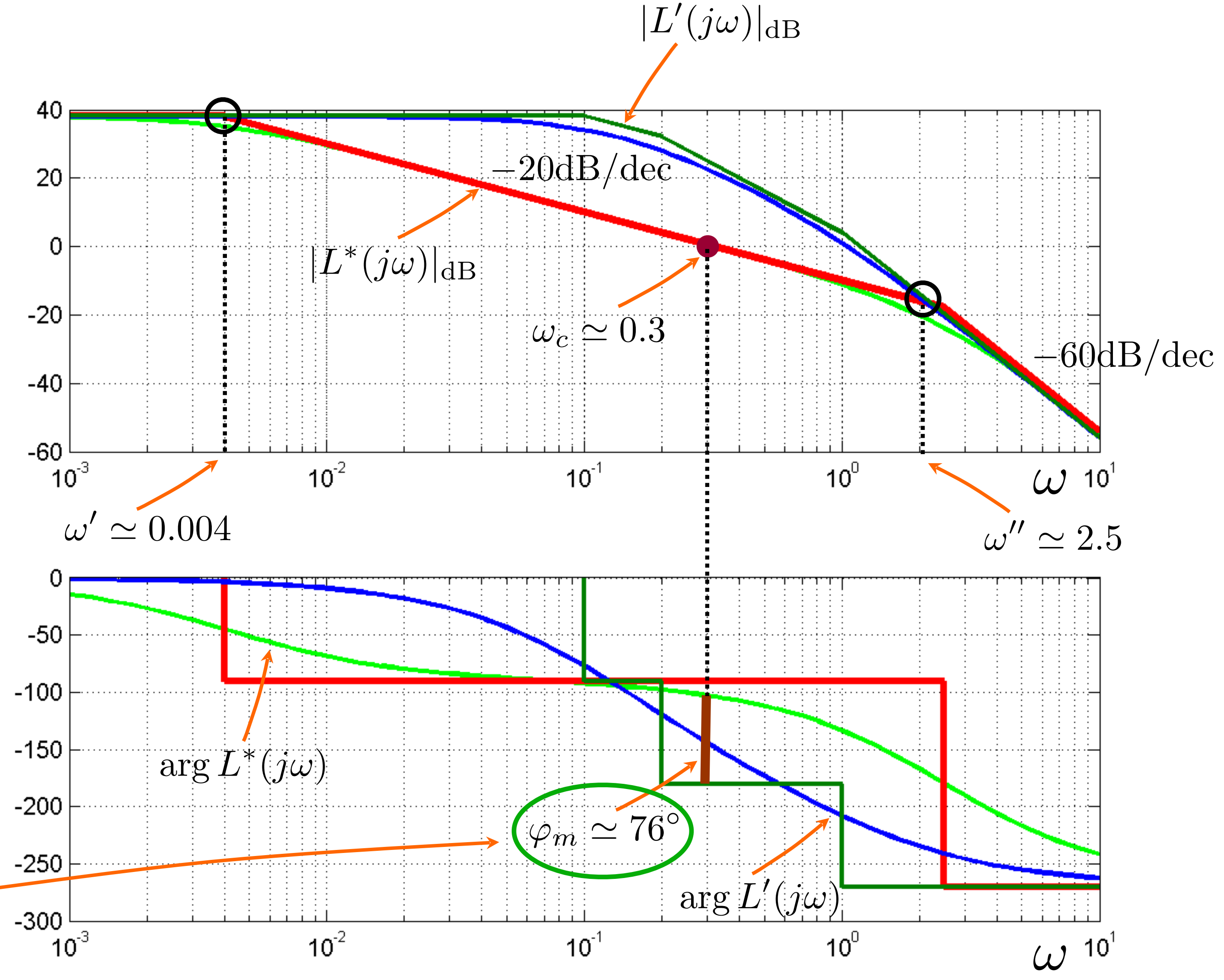
asymptotically stable **closed-loop** pole

- The choice of controller $C_1(s) = 8$ to satisfy the static requirement is **not feasible** because the closed-loop system would be **unstable**
- Decreasing the gain in $C_1(s)$ would "*move*" the close-loop poles into the open left-plane but this way the static requirement would **not** be satisfied



Loop-Shaping Iterative Controller Design: Example 1 (contd.)

Attempt 2:



closed-loop asympt. stable

Procedure followed in Attempt 2 of Dynamic Design Option (A) :

- Set a satisfactory critical angular frequency: $\omega_c = 0.3 \geq 0.2$
- Draw a straight line on the magnitude Bode plot with slope -20dB/dec and passing through the point $(\omega_c, 0\text{dB})$; find the frequencies ω', ω'' where the straight line intersects the diagram of $|L'(j\omega)|_{\text{dB}}$
- Construct the asymptotic Bode diagram of $|L^*(j\omega)|_{\text{dB}}$ which coincides with the drawn straight line for $\omega \in [\omega', \omega'']$ and with the diagram of $|L'(j\omega)|_{\text{dB}}$ for $\omega < \omega'$ and $\omega > \omega''$
- The resulting asymptotic diagram of $|L^*(j\omega)|_{\text{dB}}$ gives:

$$L^*(s) = \frac{80}{(1+s/0.004)(1+s/2.5)^2} \Longrightarrow C_2(s) = \frac{L^*(s)}{L'(s)} = \frac{(1+10s)(1+5s)(1+s)}{(1+250s)(1+0.4s)^2}$$

UNIVERSITÀ DEGLI STUDI DI TRIESTE

Analysis of the controller obtained with Design Option (A):

guarantees the **required static precision**

zeros introduced to "cancel" the poles -0.1, -0.2, -1 thus **increasing** $\arg L(j\omega)$

$$C(s) = C_1(s)C_2(s) = 8\,\frac{\boxed{(1+10s)(1+5s)(1+s)}}{\boxed{(1+250s)}\boxed{(1+0.4s)^2}}$$

"moves" to low frequency the slope of $-20\mathrm{dB/dec}$ in order to cross the $0\mathrm{dB}$ axis at the desired critical frequency with a slope $-20\mathrm{dB/dec}$ to guarantee the desired phase margin (the system is minimum phase and hence **the Bode criterion can be used**)

high-frequency poles:

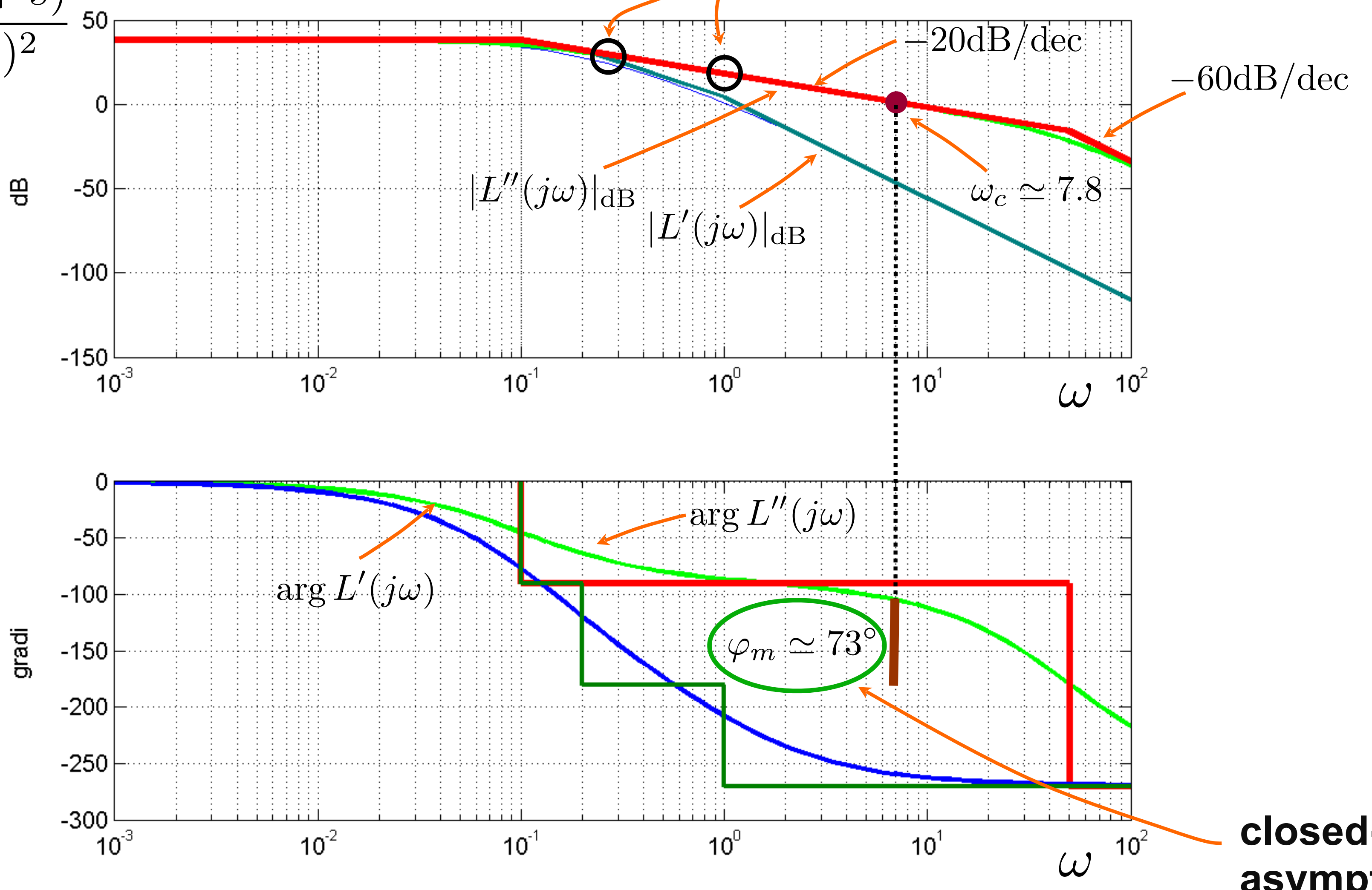
- physical realisability
- attenuation of high-frequency output disturbances

Loop-Shaping Iterative Controller Design: Example 1 (contd.)

Dynamic Design, Option (B):

$$C(s) = 8\,\frac{(1+10s)(1+5s)(1+s)}{(1+250s)(1+0.4s)^2}$$

“cancellation” of the two “faster” poles -0.2, -1



closed-loop asympt. stable

Loop-Shaping Iterative Controller Design: Example 1 (contd.)

Dynamic Design, Option (B):

$$C(s) = 8\,\frac{(1+10s)(1+5s)(1+s)}{(1+250s)(1+0.4s)^2}$$

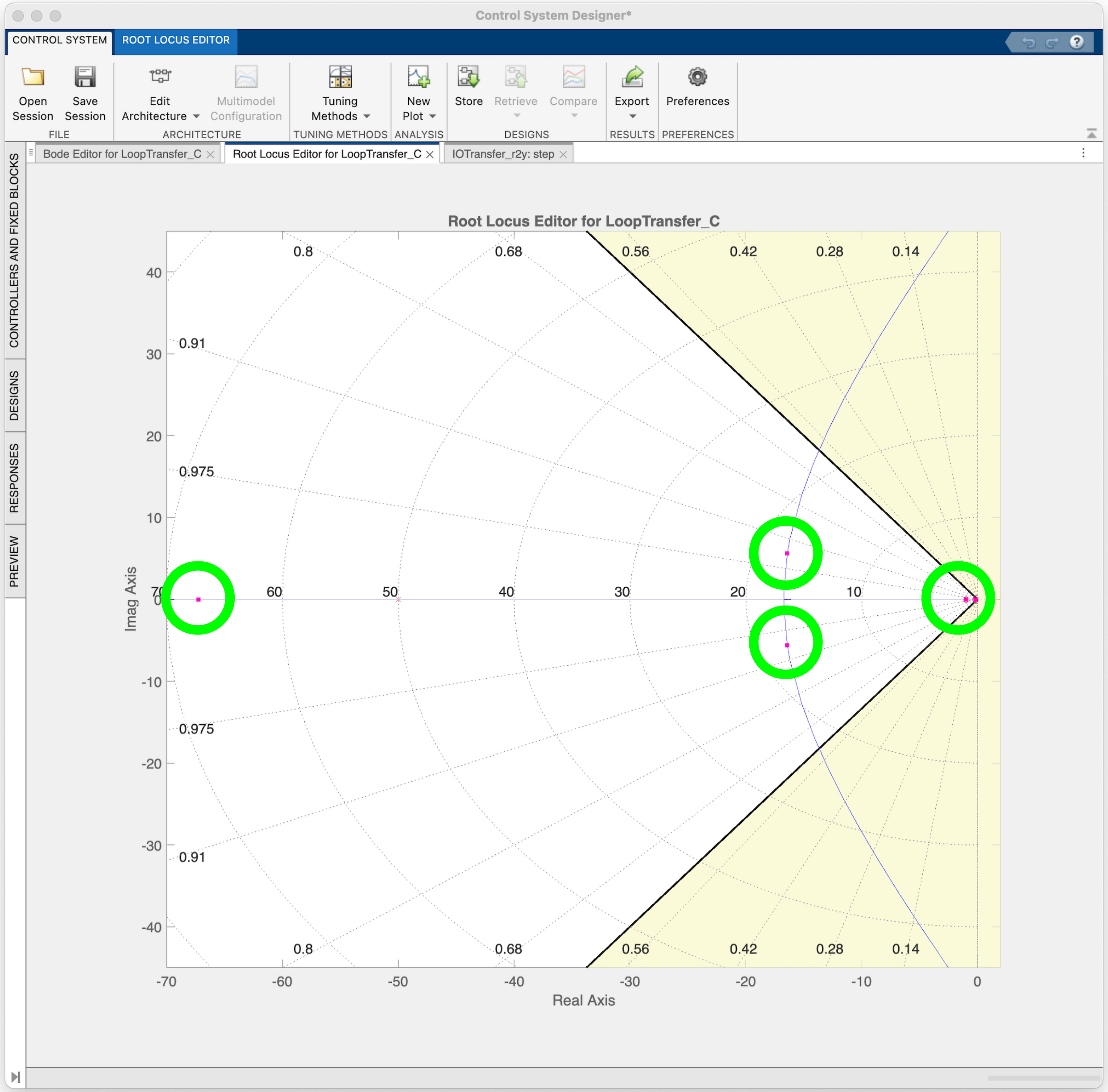
closed-loop pole

- All **closed-loop** poles are asymptotically stable.
- From the Requirement $\omega_c \geq 0.2$ we obtain the **RL graphical constraint**

$$\omega_n \simeq \omega_c \geq 0.2$$

- From the Requirement $\varphi_m \geq 60^\circ$ we obtain the **RL graphical constraint**

$$\xi \simeq \frac{\varphi_m}{100} \geq \frac{60^\circ}{100} = 0.6$$



Procedure followed in Option (B) :

- The static design is kept unchanged with $C_1(s) = 8$
- The “slower” pole in -0.1 is kept whereas the ”faster” poles in -0.2, -1 are cancelled by respective zeros introduced by the controller to ensure a slope $-20\mathrm{dB/dec}$ when crossing the point $(\omega_c, 0\mathrm{dB})$
- Two sufficiently fast (at frequency sufficiently larger than ω_c) poles are introduced by the controller
- The resulting asymptotic diagram of $|L''(j\omega)|_{\mathrm{dB}}$ gives:

$$L''(s) = \frac{80}{(1+10s)(1+0.02s)^2} \quad \Rightarrow \quad C_2(s) = \frac{L''(s)}{L'(s)} = \frac{(1+5s)(1+s)}{(1+0.02s)^2}$$

Static and Dynamic Design, Option (C):

Pick the static controller (thus we also modify the static design): $C_1(s) = \dfrac{\mu_C}{s}$

$$L(s) = \frac{G(s)}{s}\,\mu_C\, C_2(s) = L_1(s)\,\mu_C\, C_2(s)$$

$$\text{with} \quad L_1(s) = \frac{10}{s(1+10s)(1+5s)(1+s)}$$

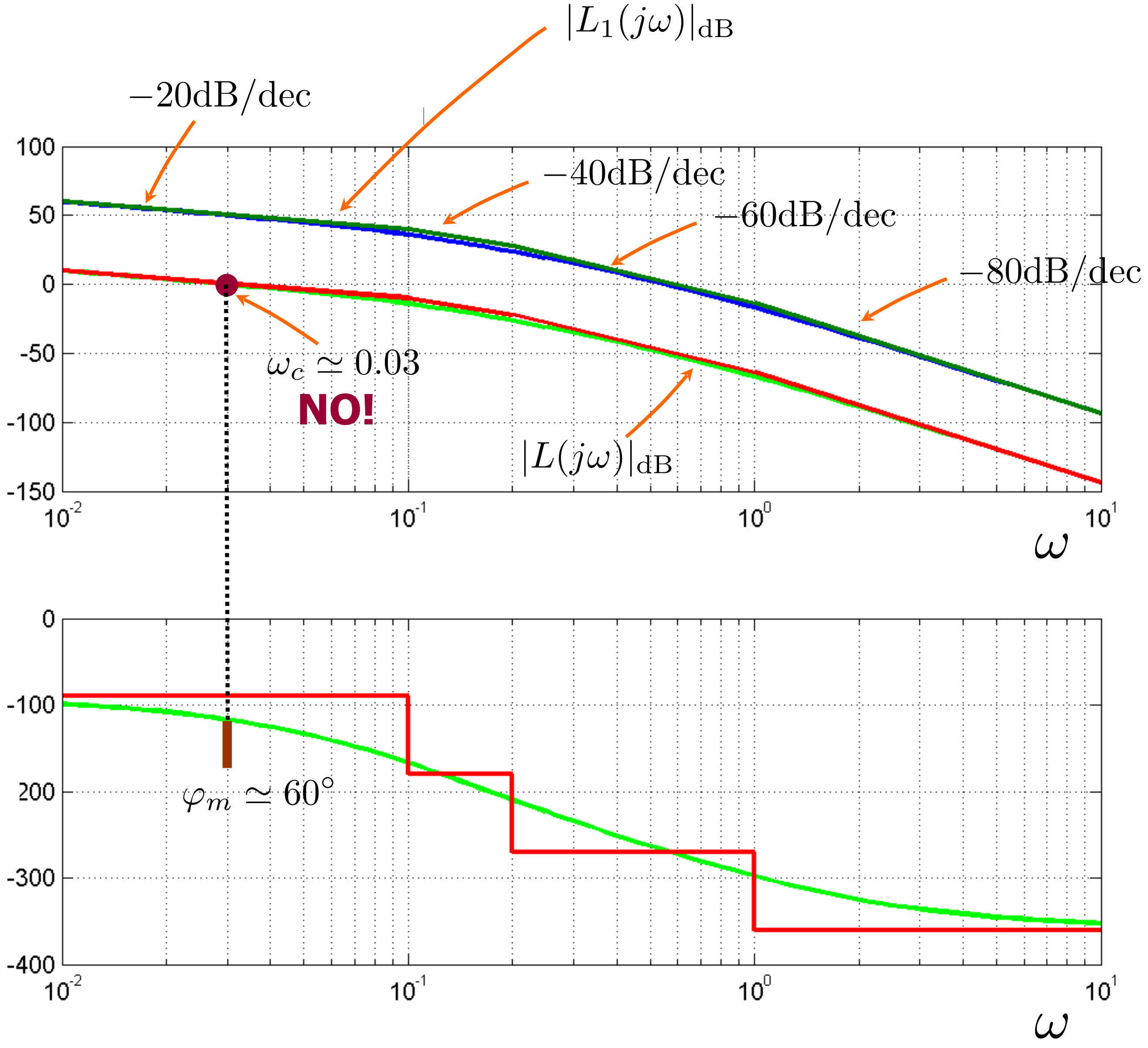
Again, we proceed by a **sequence of attempts** with increasing complexity of the structure of controller $C_2(s)$ by following the **iterative procedure** outlined in Slide 8:

Attempt 1:

$$C_2(s) = 1$$

$$L(s) = \mu_C L_1(s)$$

$$\mu_C = 10^{-5/2}$$



UNIVERSITÀ DEGLI STUDI DI TRIESTE

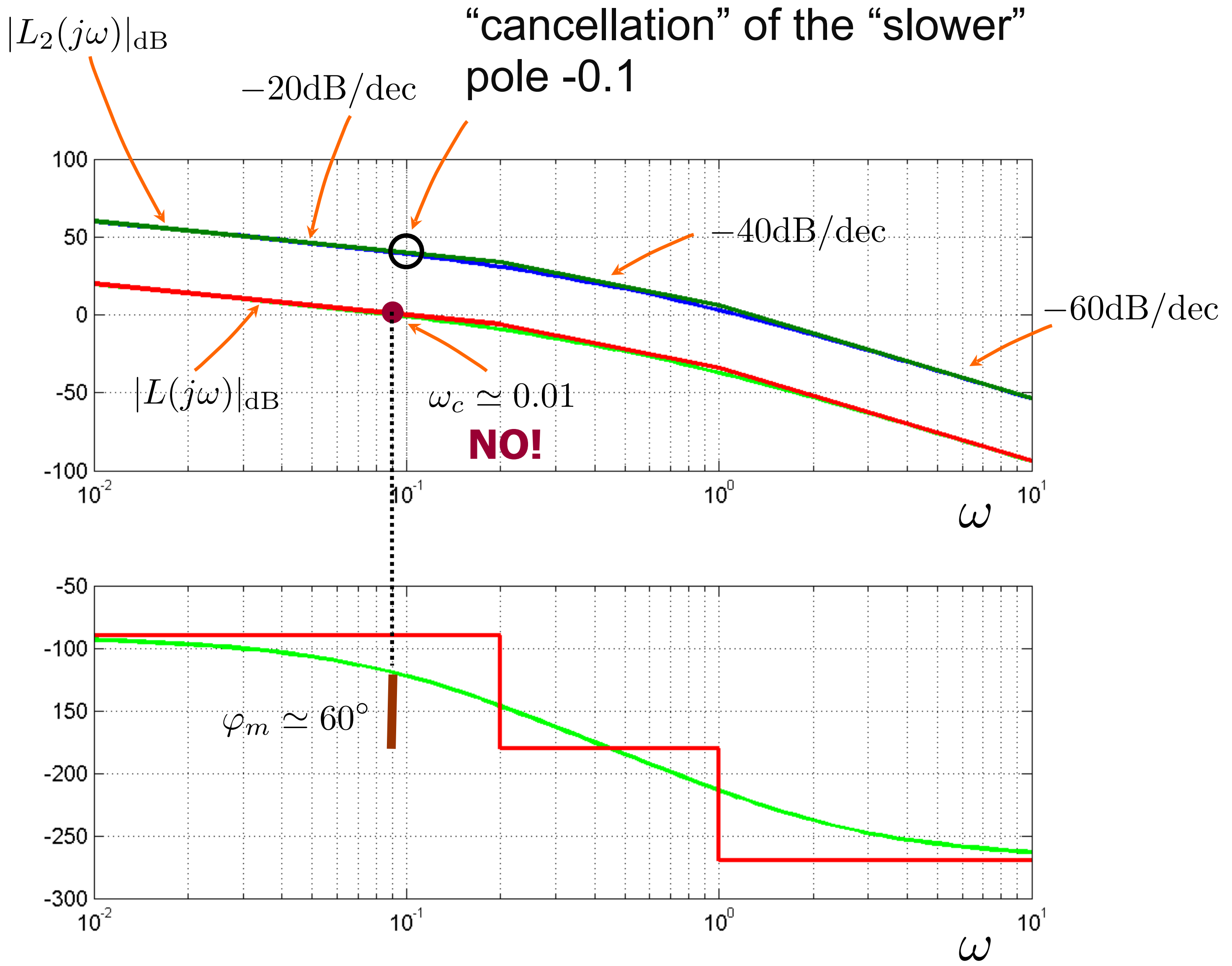
Attempt 2:

$$C_2(s) = (1 + 10s)$$

$$\Rightarrow L_2(s) = \frac{10}{s(1+5s)(1+s)}$$

$$\Rightarrow L(s) = \mu_C L_2(s)$$

$$\Rightarrow \mu_C \simeq 0.01$$



UNIVERSITÀ DEGLI STUDI DI TRIESTE

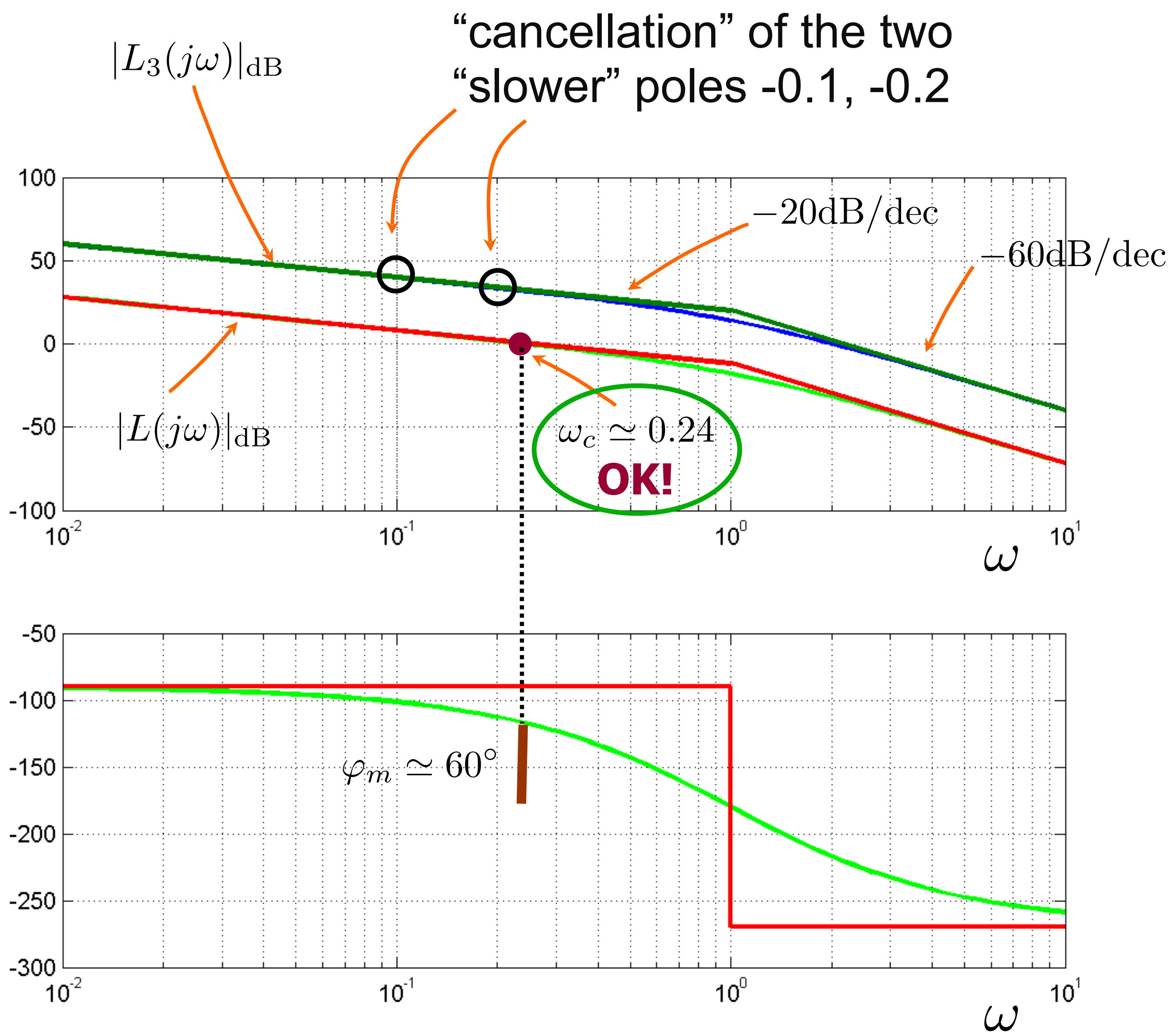
Attempt 3:

$$C_2(s) = \frac{(1+10s)(1+5s)}{1+s}$$

$$L_3(s) = \frac{10}{s(1+s)^2}$$

$$L(s) = \mu_C L_3(s)$$

$$\mu_C \simeq 0.0254$$



Loop-Shaping Iterative Controller Design: Example 1 (contd.)

UNIVERSITÀ DEGLI STUDI DI TRIESTE

Dynamic Design, Option (C):

$$C(s) = 0.0254 \frac{(1+10s)(1+5s)}{s(1+s)}$$

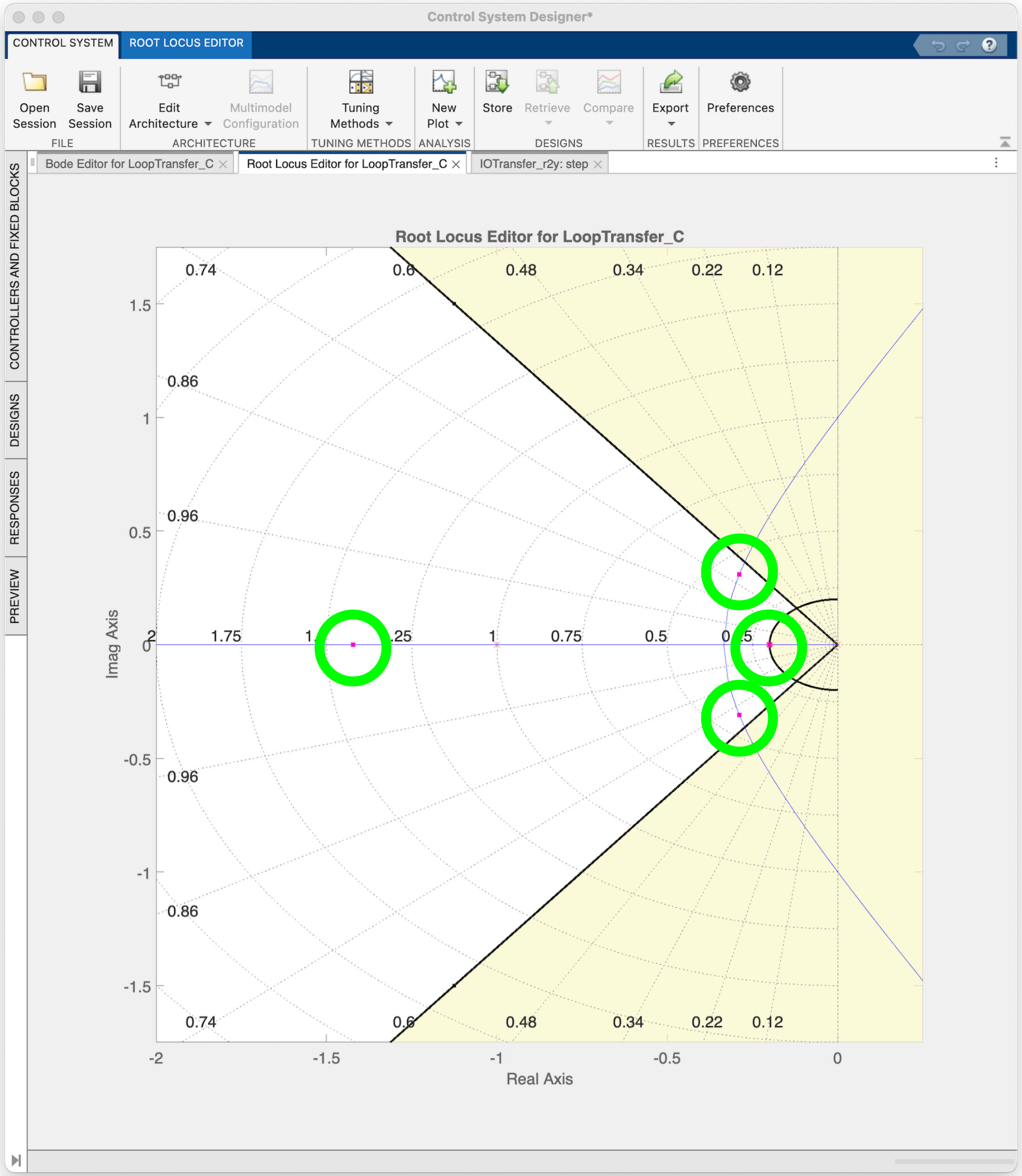
○ **closed-loop** pole

- All **closed-loop** poles are asymptotically stable.
- From the Requirement $\omega_c \geq 0.2$ we obtain the **RL graphical constraint**

$$\omega_n \simeq \omega_c \geq 0.2$$

- From the Requirement $\varphi_m \geq 60^\circ$ we obtain the **RL graphical constraint**

$$\xi \simeq \frac{\varphi_m}{100} \geq \frac{60^\circ}{100} = 0.6$$



Example 1: Comparison Between the Designed Controllers

UNIVERSITÀ DEGLI STUDI DI TRIESTE

$$G(s) = \frac{10}{(1+10s)(1+5s)(1+s)}$$

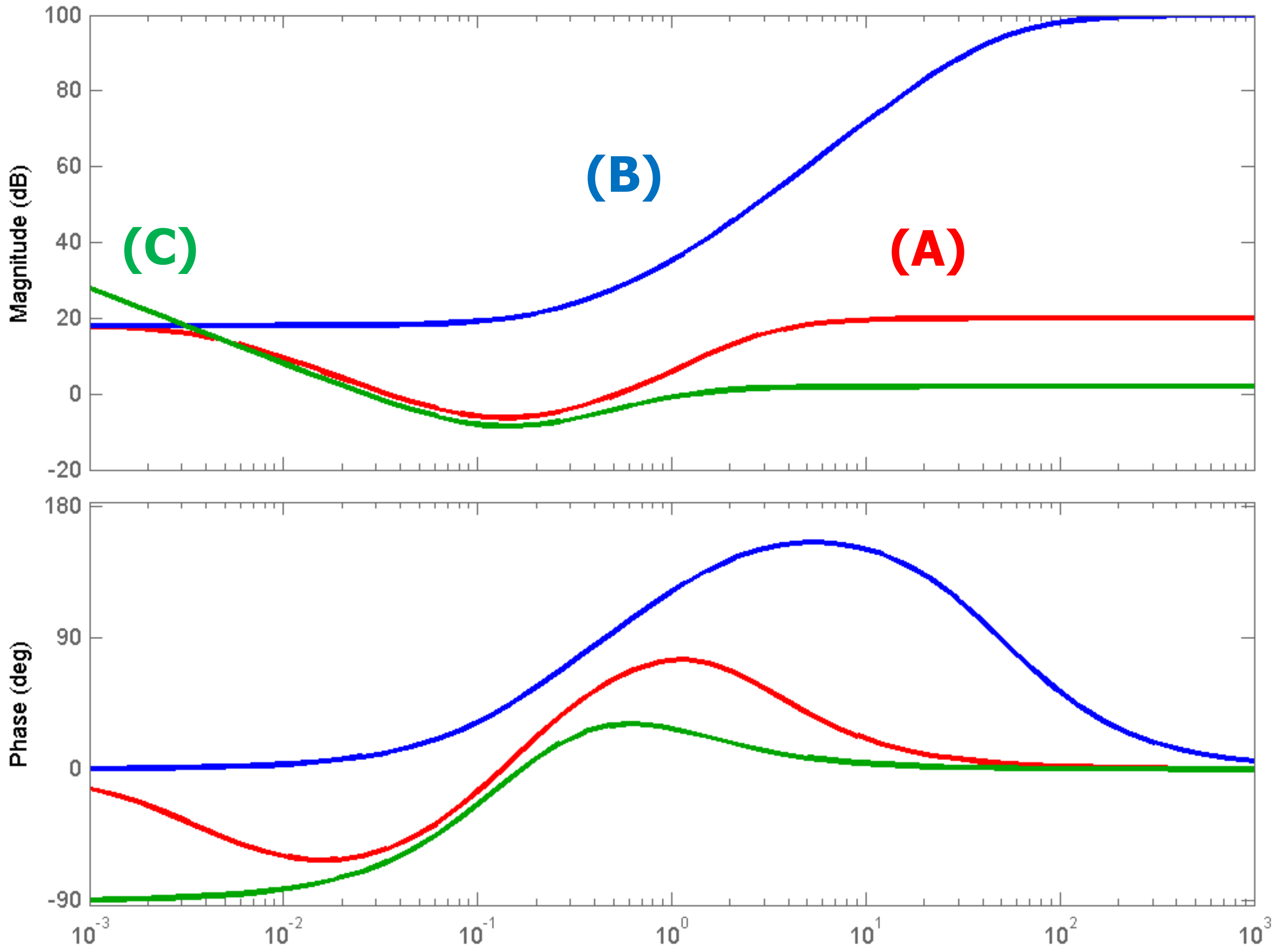
- **Design specifications**:
  - $|e(\infty)| \leq 0.1 \quad \text{with} \begin{cases} w(t) = A \cdot 1(t), \ |A| \leq 1 \\ d(t) = B \cdot 1(t), \ |B| \leq 5 \end{cases}$
  - $\omega_c \geq 0.2$
  - $\varphi_m \geq 60°$

(A): $C(s) = 8\dfrac{(1+10s)(1+5s)(1+s)}{(1+250s)(1+0.4s)^2}$ $\quad \begin{cases} \omega_c \simeq 0.3 \\ \varphi_m \simeq 76° \end{cases}$

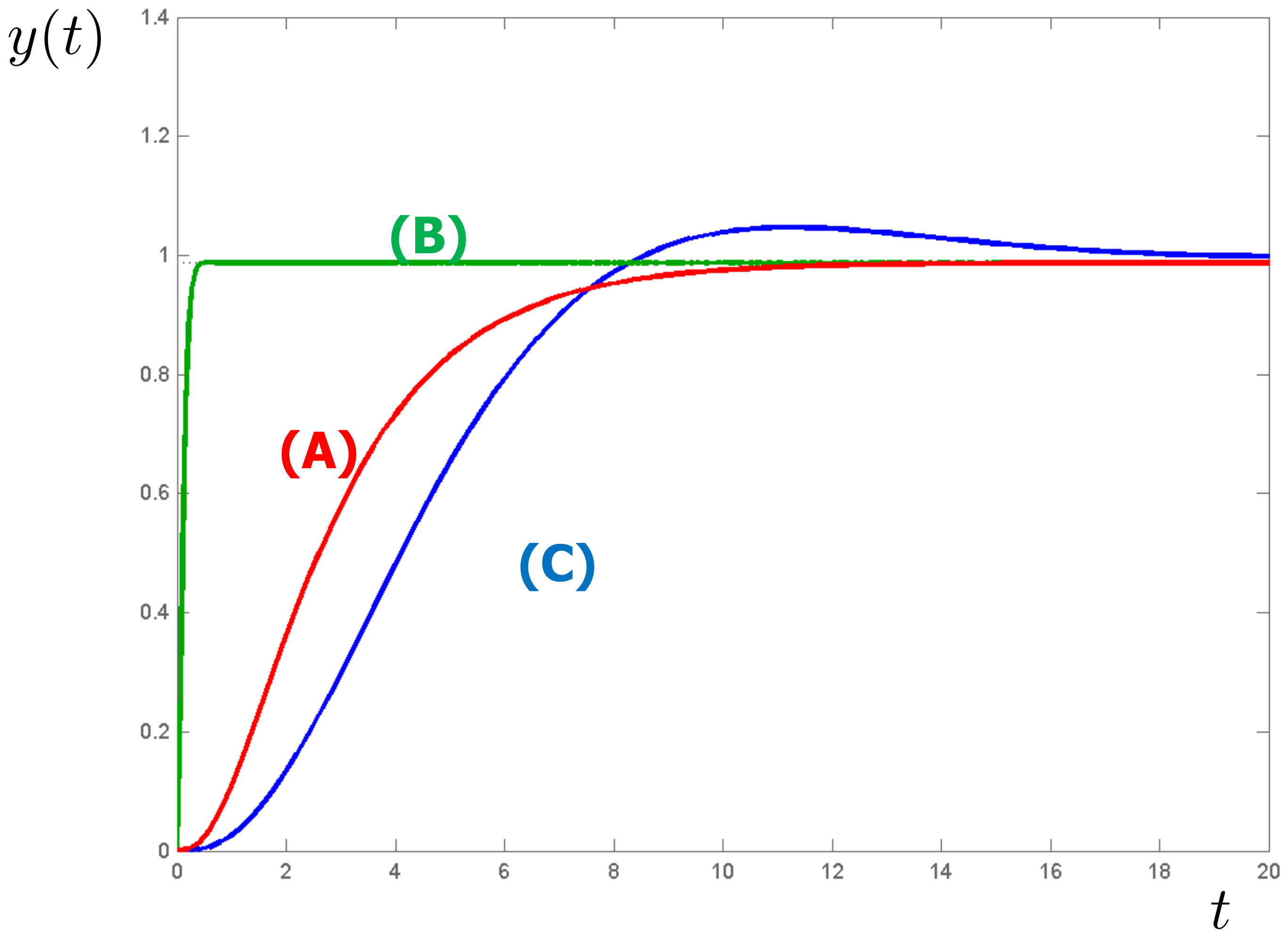
(B): $C(s) = 8\dfrac{(1+5s)(1+s)}{(1+0.02s)^2}$ $\quad \begin{cases} \omega_c \simeq 7.8 \\ \varphi_m \simeq 73° \end{cases}$

(C): $C(s) = 0.0254\dfrac{(1+10s)(1+5s)}{s(1+s)}$ $\quad \begin{cases} \omega_c \simeq 0.24 \\ \varphi_m \simeq 63° \end{cases}$ and $\quad e(\infty) = 0$

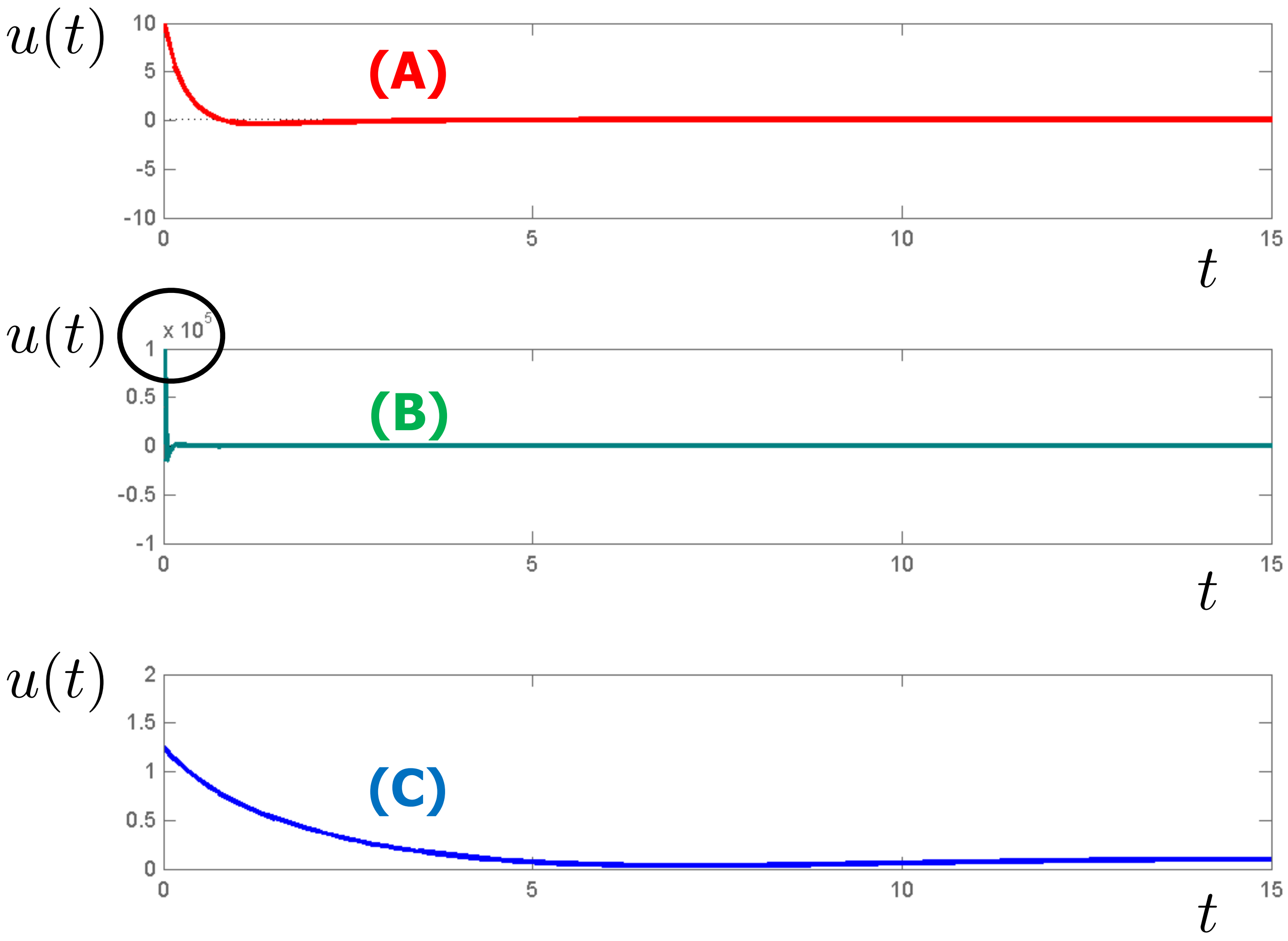
Bode diagrams of the controllers



Closed-loop step response



Closed-loop step response

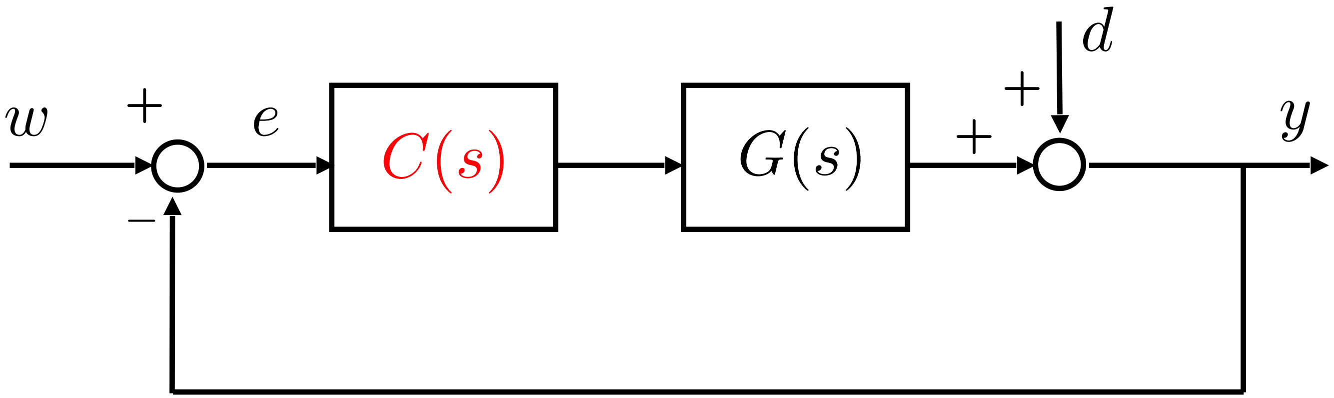


Control Design for Non-Minimum Phase Zeros/Unstable Systems

In scenarios where the open-loop system to be controlled has zeros and/or poles in the right-half complex plane **care must be exercised**:

- the empirical criterion presented in Part 9, slide 32 cannot be used
- the **Bode criterion** cannot be used for open-loop unstable systems
- the “**cancellation**” of zeros and poles with non-negative real part is **not allowed** not to generate unstable hidden dynamics
- the presence of zeros and/or poles with non-negative real part introduces **inherent limitations on the achievable performance**

Example 2 - System with a Non-Minimum Phase Zero



$$G(s) = \frac{10(1-2s)}{s(1+10s)(1+0.1s)}$$

Design specifications: $\varphi_m \geq 40^\circ$

The term $\dfrac{(1-2s)}{s}$:

- cannot be "canceled"
- it gives a **negative contribution** to the phase: $-90^\circ - \text{arcgtg}\,(2\omega)$

↳ for $\omega = 0.5$ the negative contribution to the phase is -135°

↳ the critical frequency ω_c cannot exceed "too much" $\omega_c = 0.5$

↳ strong limitation to the achievable speed of response

UNIVERSITÀ DEGLI STUDI DI TRIESTE

Note that:

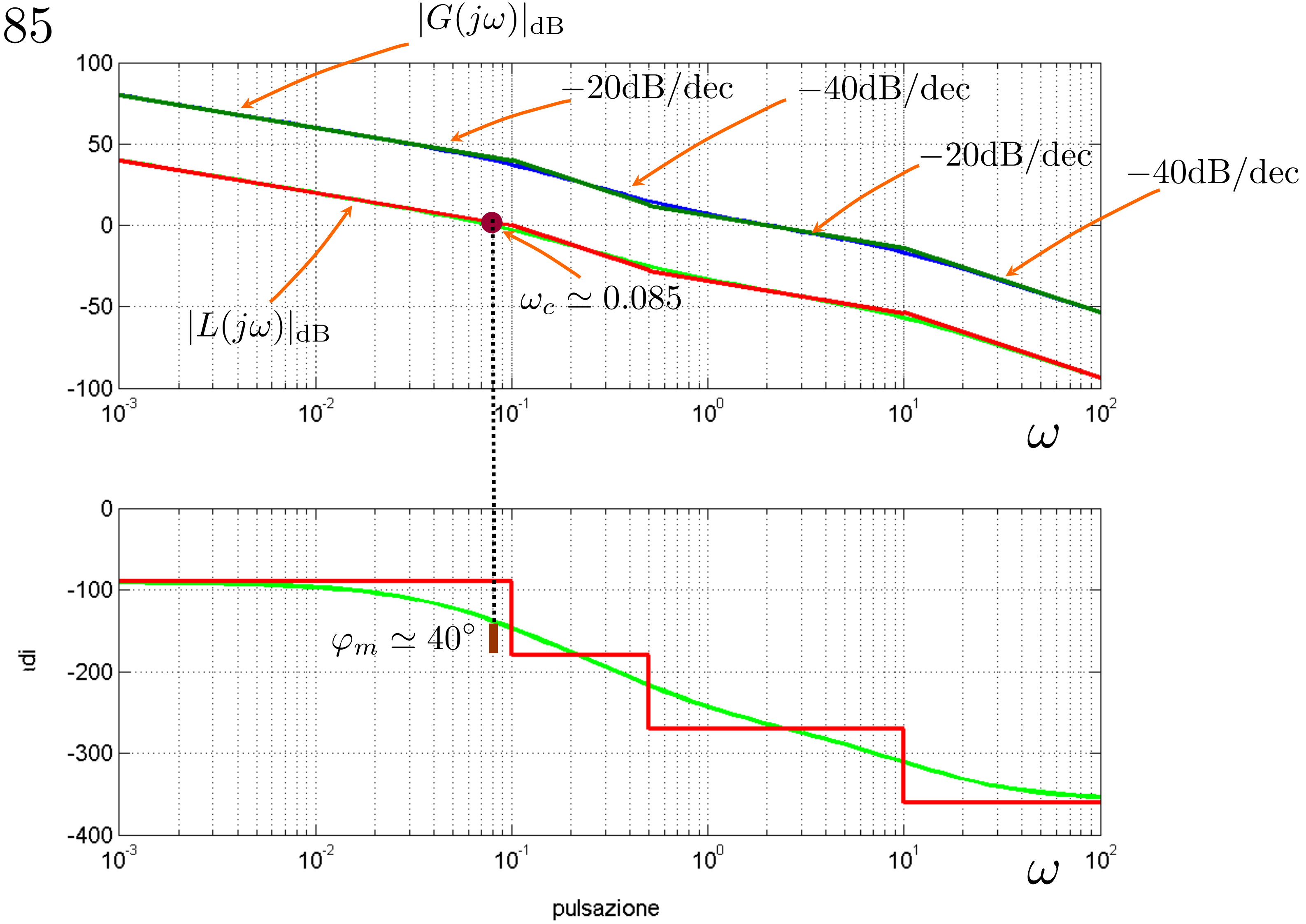
$$\arg G(j\omega) = -140^\circ \ \text{ for } \omega \simeq 0.085$$

Attempt 1:

$$C(s) = \mu_C \,;\ L(s) = \mu_C G(s)$$

$$\omega_{c\max} = 0.085$$

$$\mu_C \simeq 0.01$$



Attempt 2:

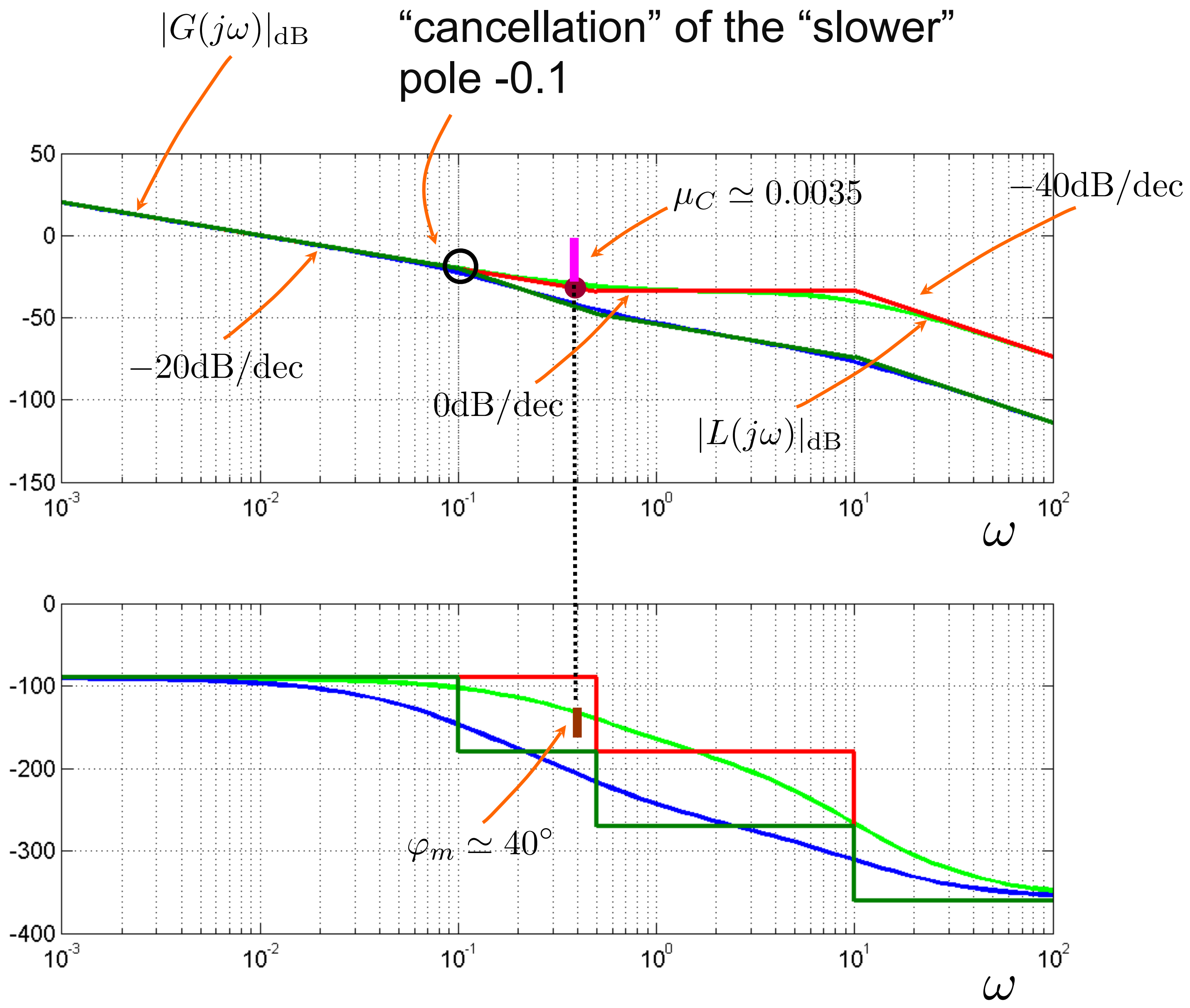
$$C(s) = \mu_C \frac{1+10s}{1+0.1s}$$

$$L(s) = \mu_C \frac{10(1-2s)}{s(1+0.1s)^2}$$

Note that:

$$\arg L(j\omega) = -140^\circ \text{ for } \omega \simeq 0.5$$

$$\mu_C \simeq 0.0035$$



UNIVERSITÀ DEGLI STUDI DI TRIESTE

Dynamic Design, Attempt 2:

$$C(s) = 0.0035\,\frac{1+10s}{1+0.1s}$$

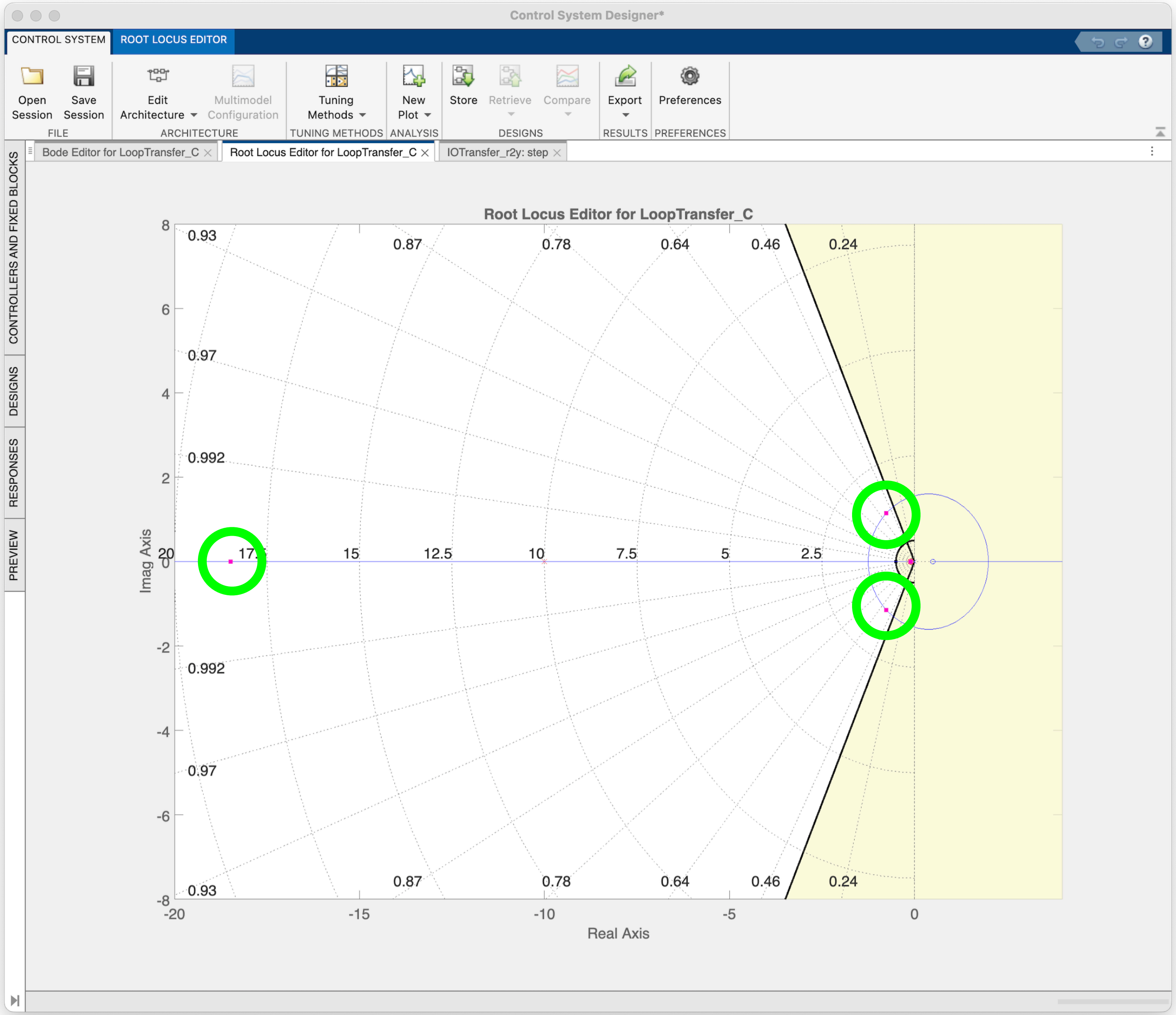
closed-loop pole

- All **closed-loop** poles are asymptotically stable.
- From the Requirement $\varphi_m \geq 40^\circ$ we obtain the **RL graphical constraint**

$$\xi \simeq \frac{\varphi_m}{100} \geq \frac{40^\circ}{100} = 0.4$$

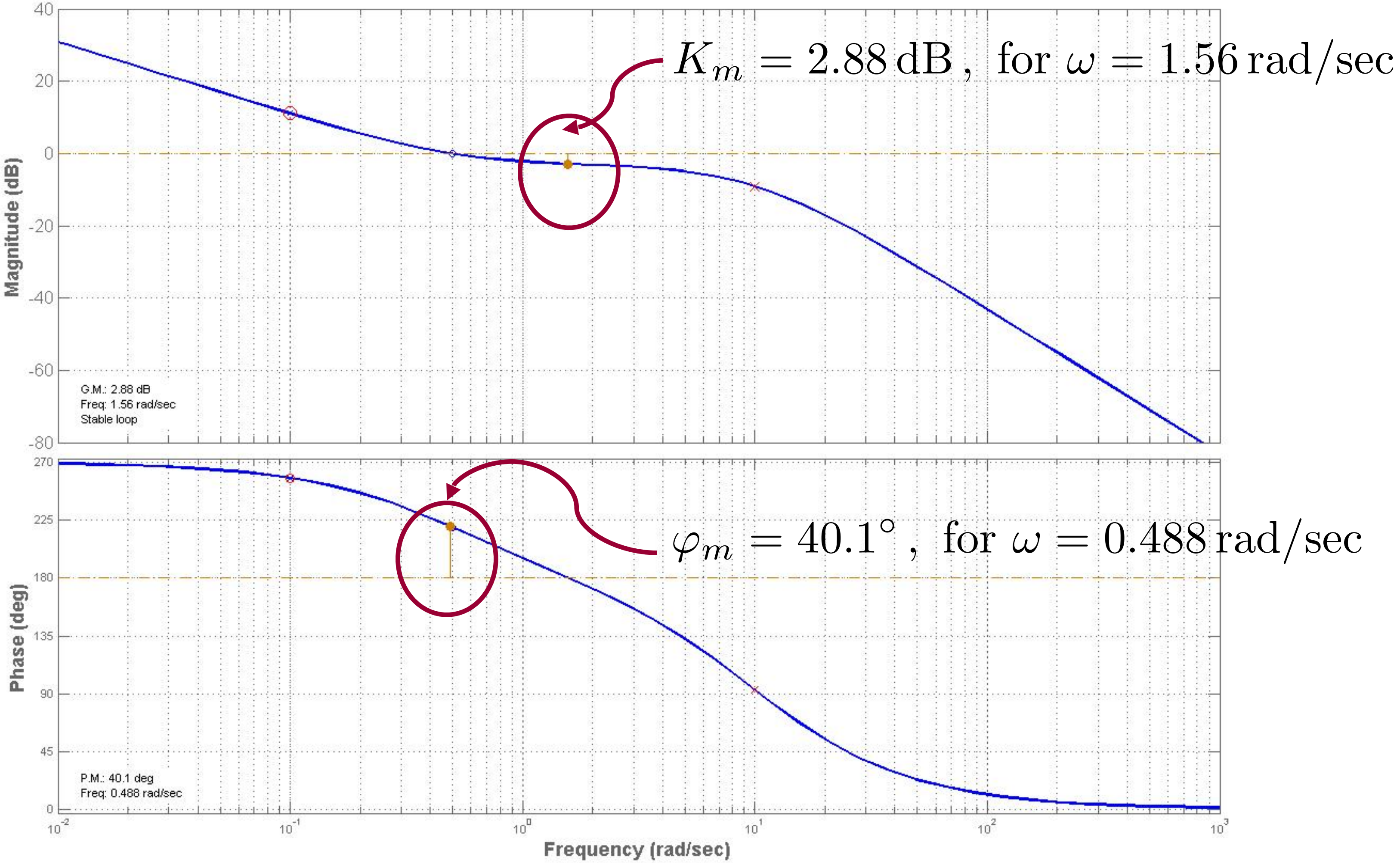
- Even if it is not a requirement, to get a controller that "pushes" the performance to the limits, we impose $\omega_c \geq 0.5$, thus we obtain the **RL graphical constraint**

$$\omega_n \simeq \omega_c \geq 0.5$$



Actual performance with controller obtained by Attempt 2:

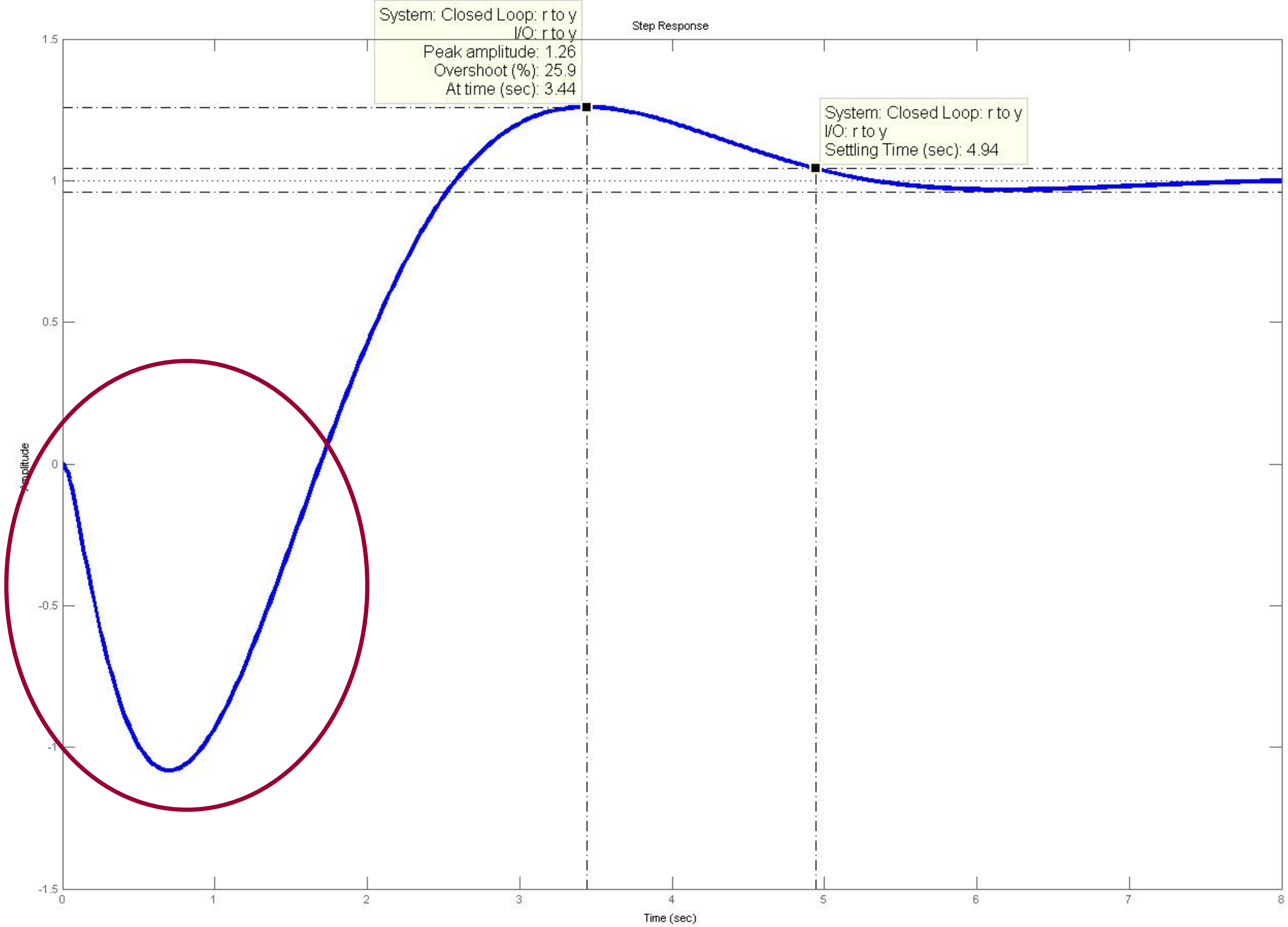
Note that the **gain margin** K_m is rather small



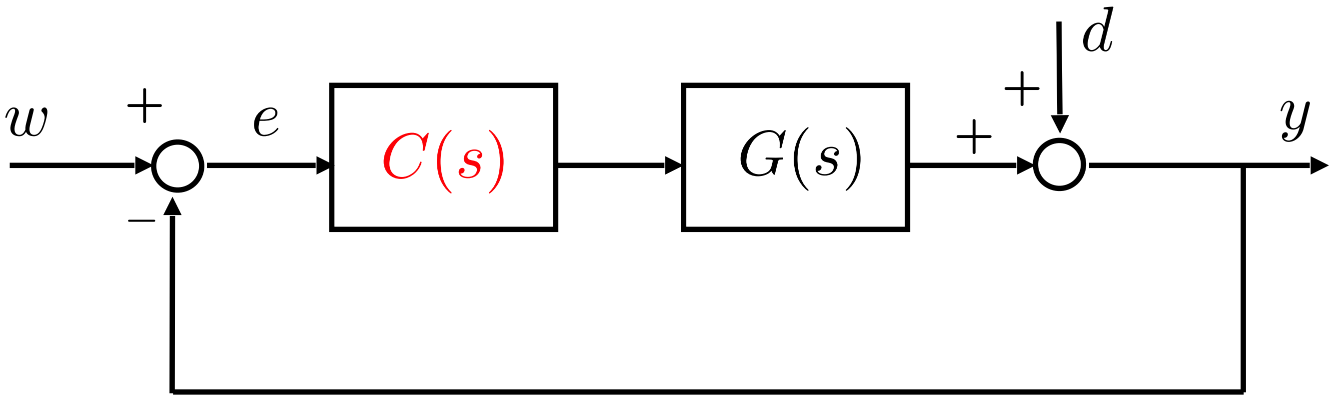
Example 2 - System with a Non-Minimum Phase Zero (contd.)

Actual performance with controller obtained by Attempt 2:

Also, note the rather significant **under-shoot** (typical for non-minimum phase systems)



Example 3 - Unstable System



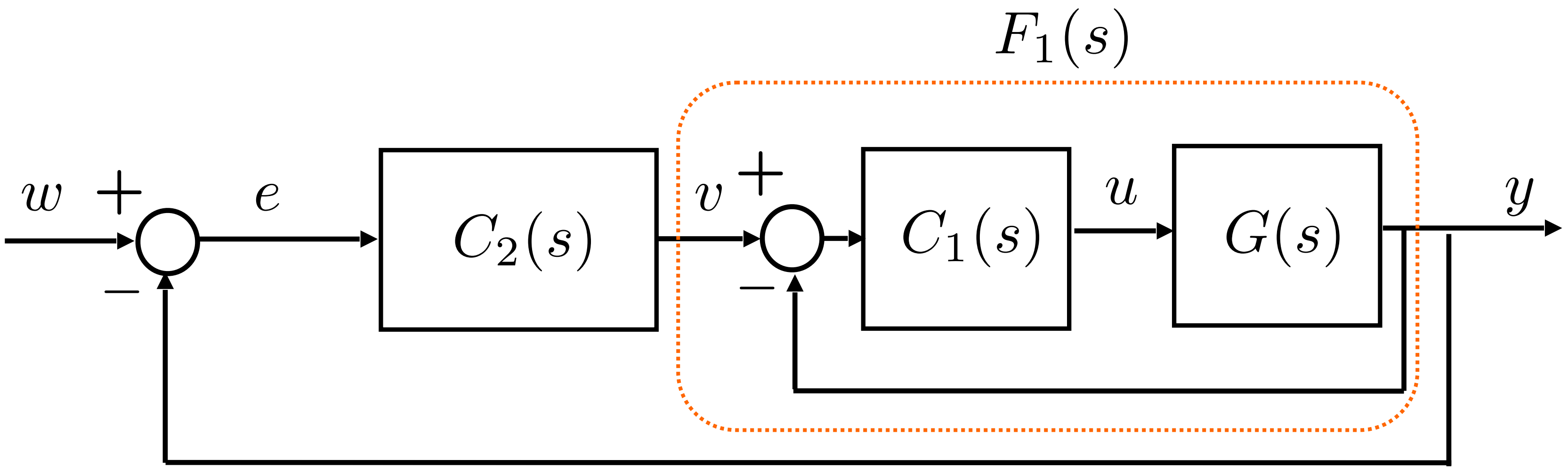
$$G(s) = \frac{1}{s-1}$$

Design specifications:

- $|e(\infty)| = 0 \quad \text{with} \begin{cases} w(t) = A \cdot 1(t), \ \forall A \\ d(t) = 0\,, \ \forall t \end{cases}$
- $\omega_c \geq 0.5$
- $\varphi_m \geq 45°$

Example 3 - Unstable System (contd.)

The simplest and frequently used approach is to consider a **dual-loop controller architecture**



- Controller $C_1(s)$ is designed to **stabilise** the system described by the inner closed-loop transfer function $F_1(s)$
- Controller $C_2(s)$ is designed to **meet the specifications** for the whole feedback control system

The design of the inner controller $C_1(s)$ can be carried out as:

$$C_1(s) = \mu_1 \Rightarrow L_1(s) = C_1(s) \cdot G(s) = \frac{\mu_1}{s-1}$$

$$\Rightarrow F_1(s) = \frac{L_1(s)}{1+L_1(s)} = \frac{\mu_1}{s-1+\mu_1}$$

By choosing (for example): $\mu_1 = 11$

$$\Rightarrow F_1(s) = \frac{1.1}{1+0.1s}$$ **asymptotically stable**

Example 3 - Unstable System (contd.)

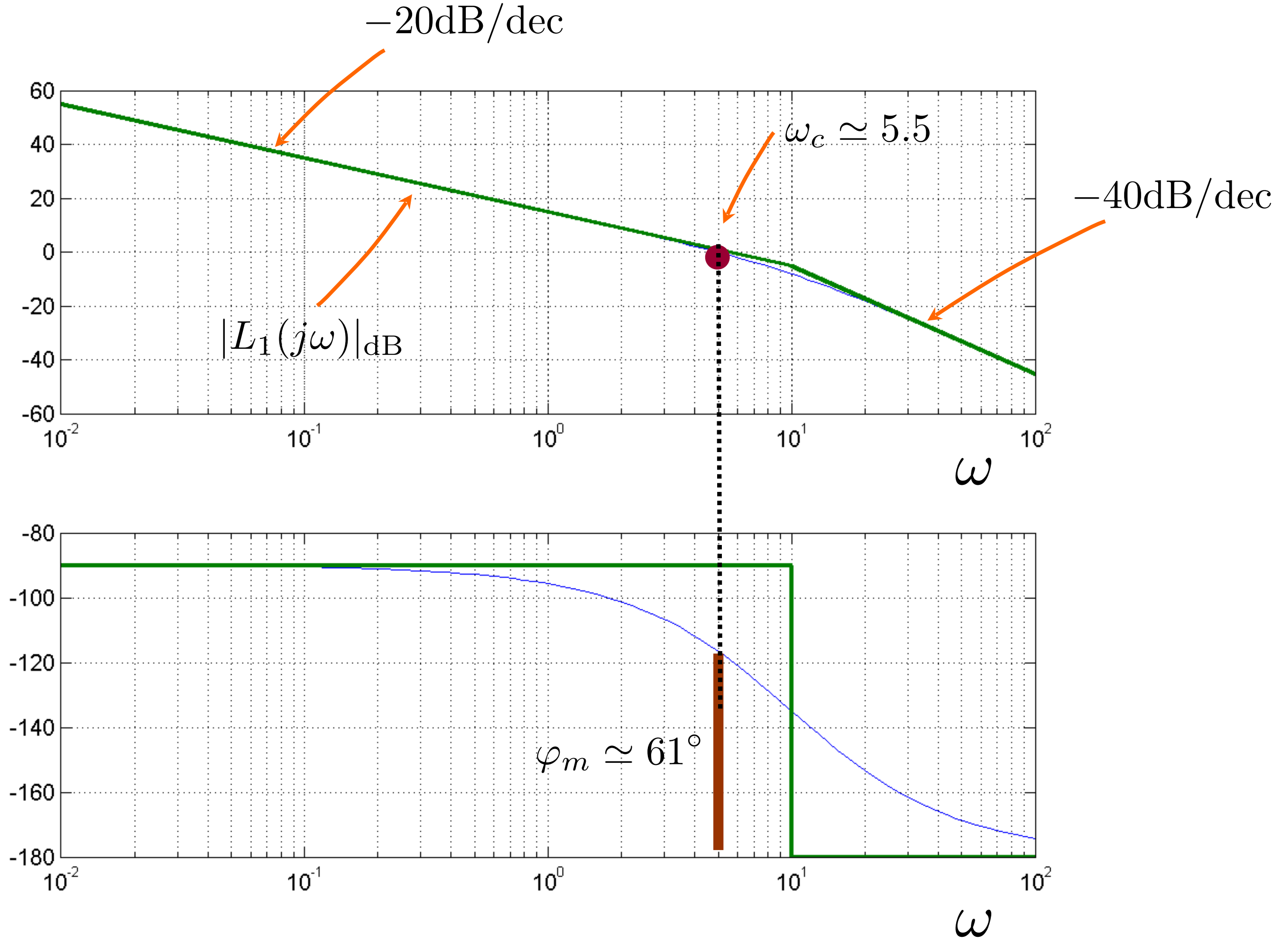
- Now, the design of the outer controller $C_2(s)$ is very simple:
- To meet the specification on the static error, an **integrator** on the direct path has to be introduced.
- Hence, by choosing:

$$C_2(s) = \frac{5}{s}$$

$$L_1(s) = C_2(s) \cdot F_1(s)$$

$$= \frac{5.5}{s(1+0.1s)}$$

All specifications are met

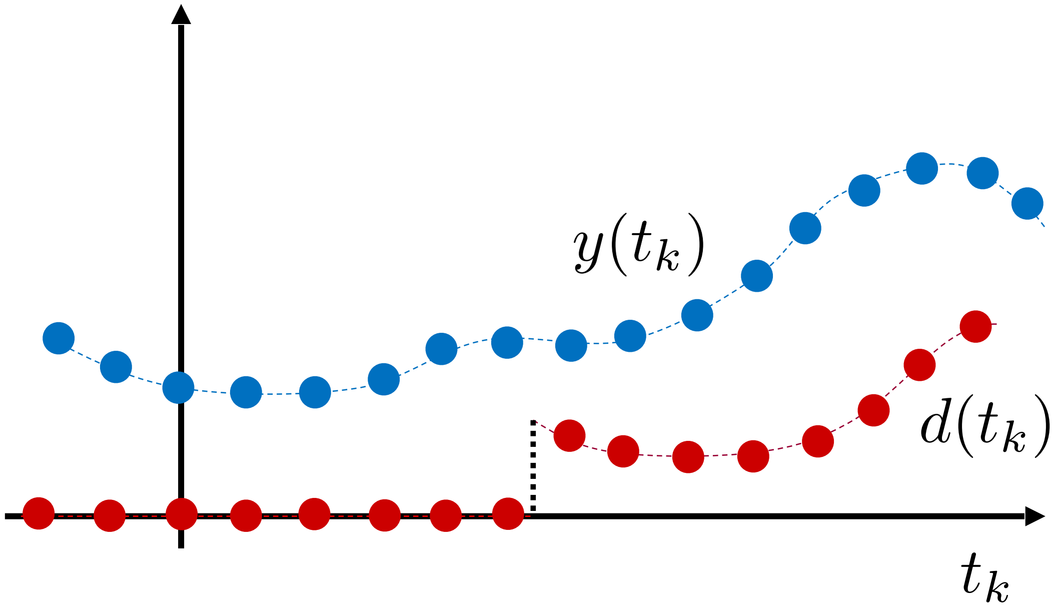


Digital Implementation of Continuous-Time Feedback Control Systems

Assumptions on Discrete-time/Sampled-time Variables

Discrete-time variables

$$t_k,\ k \in \mathbb{Z}$$
$$y(t_k) \in \mathbb{R}$$
$$d(t_k) \in \mathbb{R}$$
$$\vdots$$

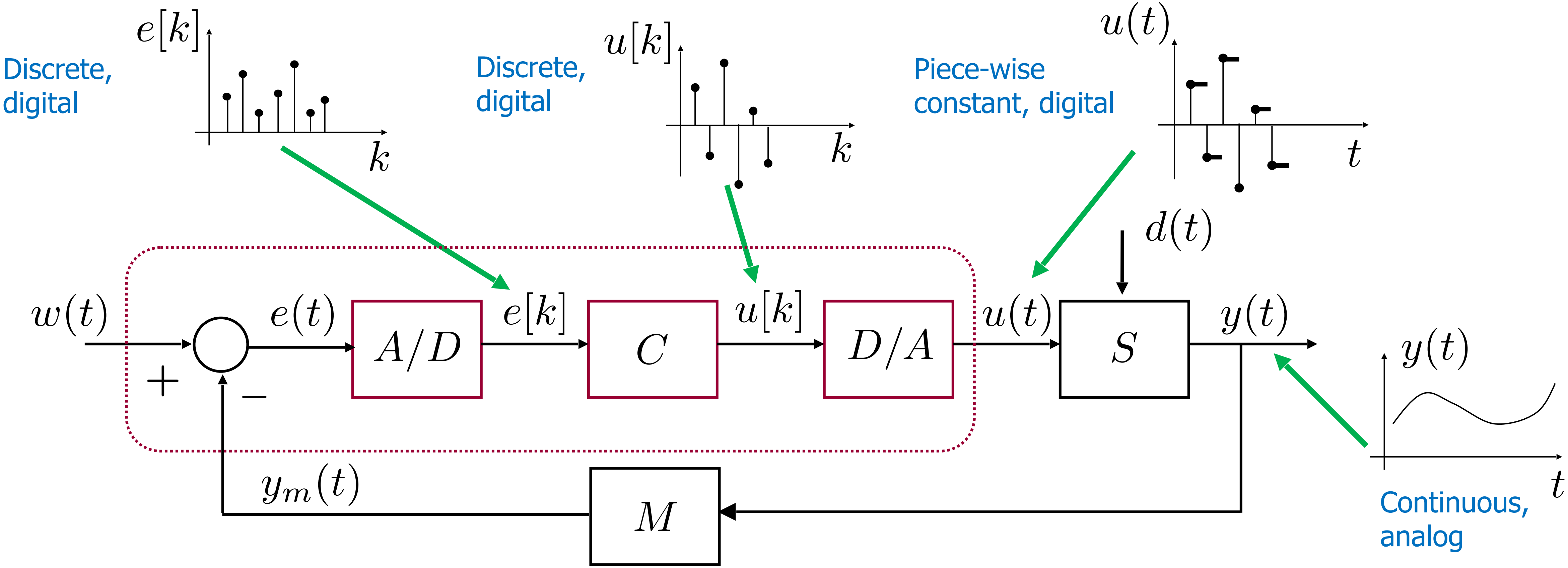


In case of regular sampling: $y(t_k)\,:\ t_k = kT_s,\ k \in \mathbb{Z}$

$$y[k], d[k], \ldots\ k \in \mathbb{Z}$$

Digital Control: Typical Structure

UNIVERSITÀ DEGLI STUDI DI TRIESTE



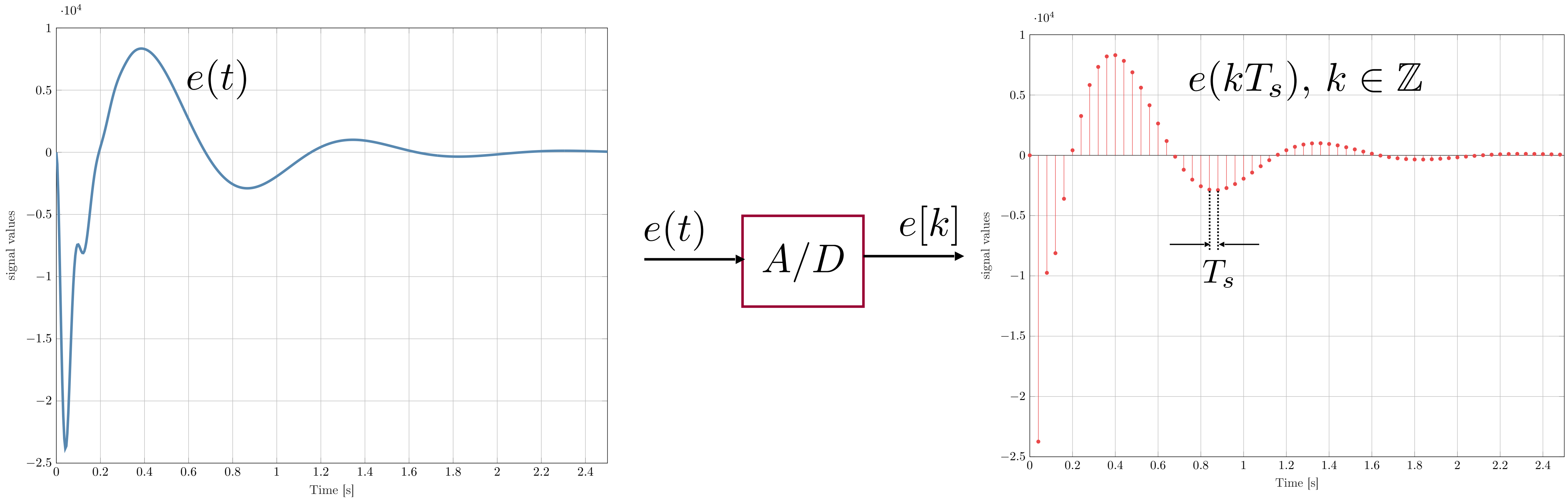
From a mathematical point of view, digital control systems are **hybrid systems** since continuous-time and discrete-time variables are present at the same time

- **Analog** variables: their values change with continuity
- **Digital** variables: their values are quantised

Analog Controller vs. Digital Controller

- The analog controller receives analog inputs in continuous-time and yields analog outputs in continuous-time
- The digital controller is implemented on a digital computing device (micro-controller, DSP card, PLC, etc.)
- Digital computing devices can only elaborate digital sampled-time variables, and suitable conversion devices are needed at the interface with the controlled system: Analog/Digital Converters (**A/D**) Digital/Analog Converters (**D/A**)
- A/D and D/A converters need to be synchronised via a clock signal with period T_s (**sampling time**)
- The control unit gets the input variables from the A/D converter and yields the output variable to the D/A converter **only on the clock time-instants**
- Such variables are denoted as **discrete-time** variables
- When there is no risk of ambiguity, we drop the sampling-time notation and only keep the discrete-time index: $t_k = kT_s \quad \Longrightarrow \quad k$

A/D Conversion



Ideally, the A/D conversion implements the **sampling mechanism**

$$\{e(t) :\ t = kT_s,\ k \in \mathbb{Z}\} \Longrightarrow e[k]$$

with **sampling frequency**

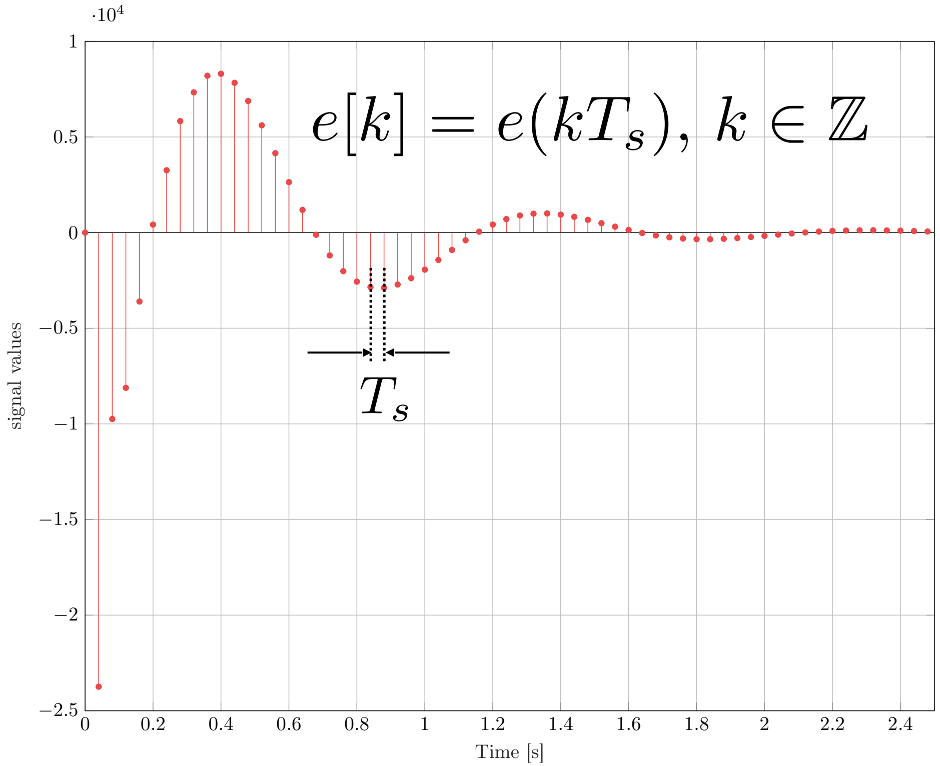
$$f_s = \frac{1}{T_s}$$

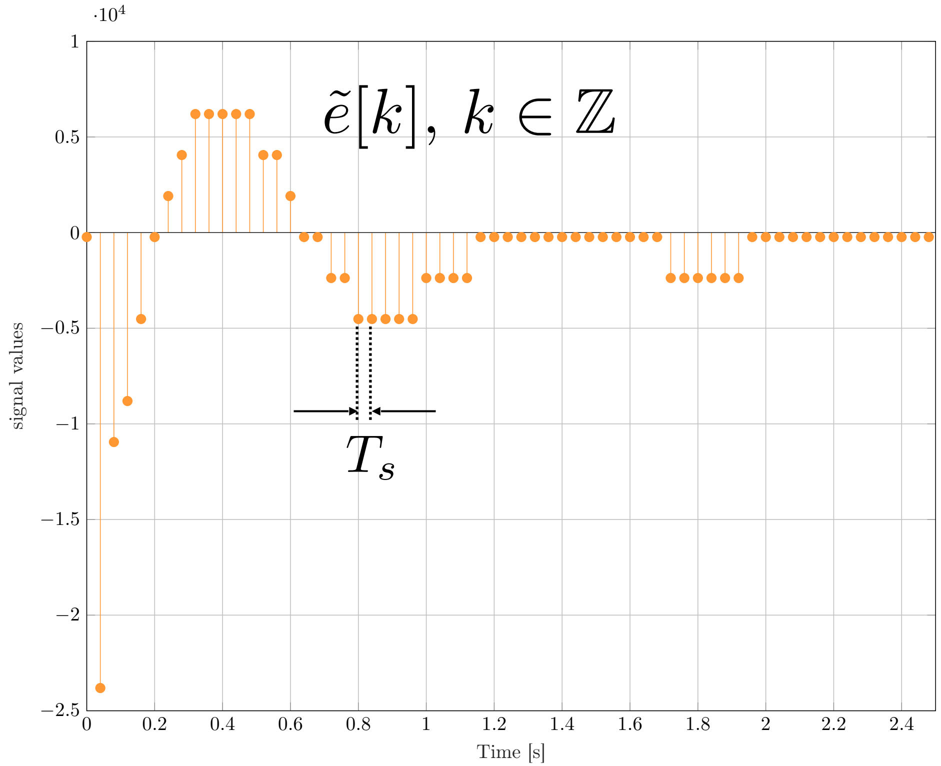
A/D Conversion (contd.)

The A/D conversion implies:

- loss of information (continuous-time/discrete-time)
- quantisation noise and distortion (analog to digital)

$e(t)$ → $e[k],\ k \in \mathbb{Z}$ → $\tilde{e}[k] \neq e[k],\ k \in \mathbb{Z}$ 0 1 10 10 ...





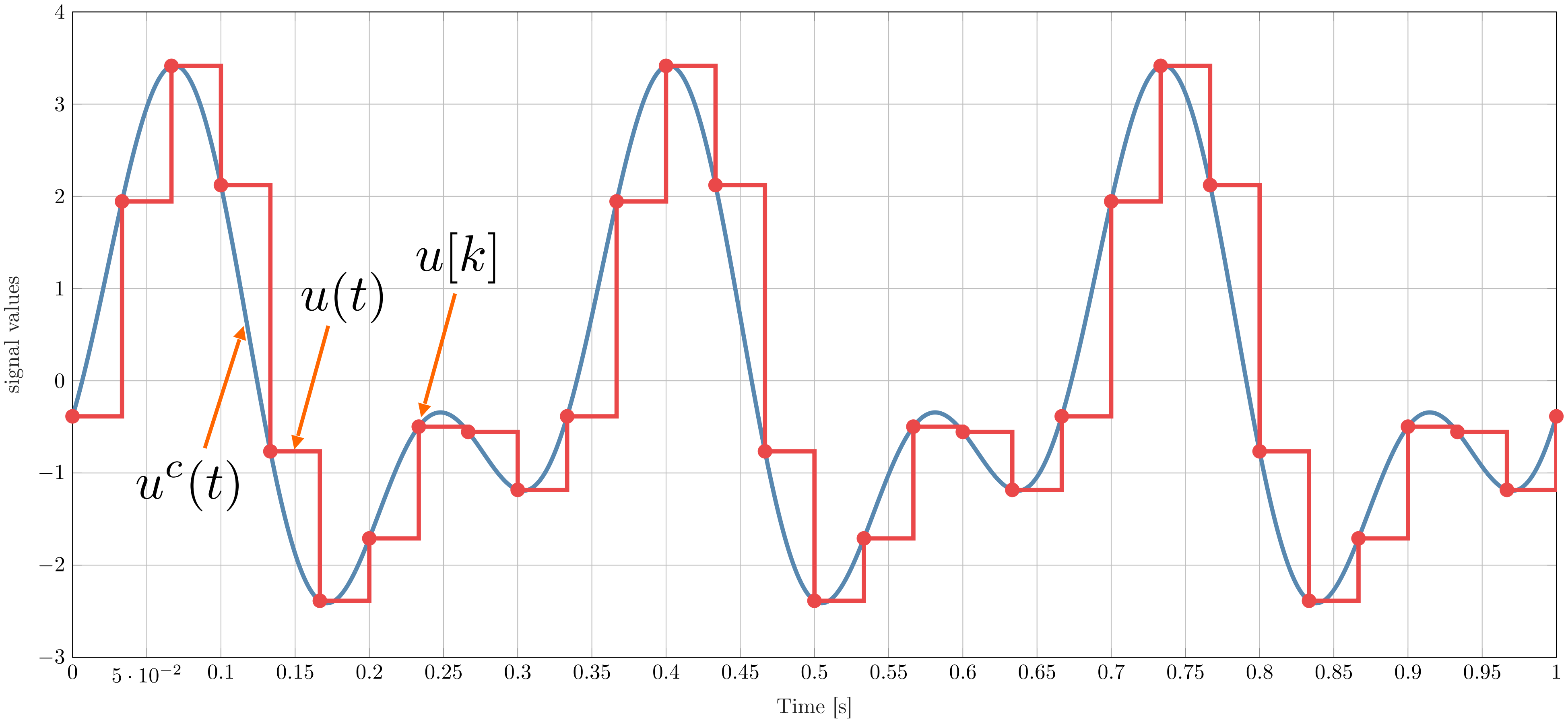
[In this course, we suppose $\tilde{e}[k] = e[k],\ k \in \mathbb{Z}$]

D/A Conversion

$$u[k] \rightarrow \boxed{D/A} \rightarrow u(t)$$

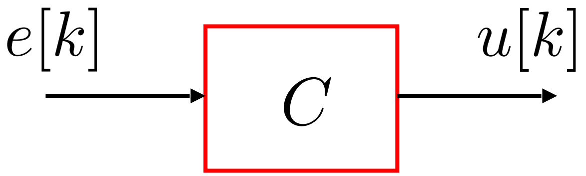
Zero-order hold (ZOH): $u(t) = u[k]\,,\ kT_s \leq t < (k+1)T_s,\ k \in \mathbb{Z}$

The D/A conversion using a ZOH is a stair-wise **delayed** approximation of the unknown underlying continuous time function $u^c(t)$



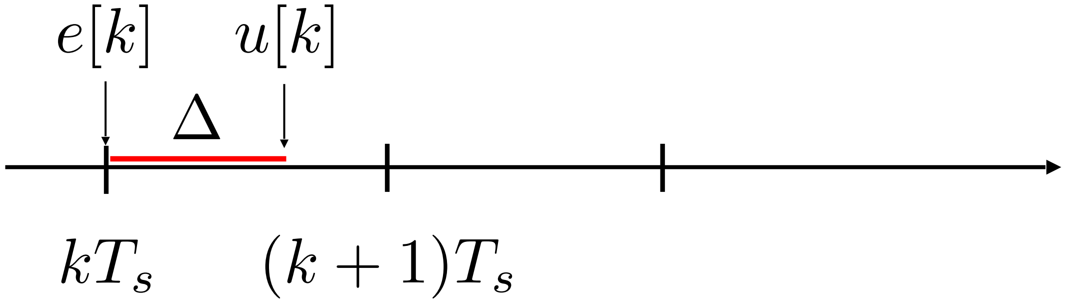
Digital Discrete-Time Controller

The controller is a discrete-time system, that is: a **computational algorithm**!



$$u[k] = f(u[k-1], u[k-2], \ldots, e[k], e[k-1], \ldots)$$

Temporisation:



The controller computation time should satisfy: $\Delta < T_s$

A Glimpse on the Z Transform

- The $\mathcal{Z}$ transform of a sequence $\{x[k],\, k = 0, 1, \ldots\}$ with $x[k] = 0,\, \forall\, k < 0$ is defined as:

$$X(z) = \mathcal{Z}\{x[k]\} = \sum_{k=0}^{\infty} x[k] z^{-k}$$

- The $\mathcal{Z}$ transform, analogously to the Laplace transform, maps discrete-time sequences into functions of the complex variable z
- The $\mathcal{Z}$ transform is very useful to represent in a compact form difference equations like the ones representing a **control algorithm** of the form:

$$u[k] = f(u[k-1], u[k-2], \ldots, e[k], e[k-1], \ldots) \qquad (\star)$$

A Glimpse on the Z Transform

- In our introductory course, the $\mathcal{Z}$ transform will only be used:
  - **(a)** to obtain in compact form the digital implementation of a continuous-time controller and afterwards
  - **(b)** to obtain the **control algorithm** of the form $(\star)$

- The expression of a $\mathcal{Z}$ transform can be **equivalently** represented in **negative** or **positive** powers of z as in the following example (the expression in negative powers of z can be immediately obtained by dividing numerator and denominator by the highest power of z):

$$X(z) = \frac{1 - 2z^{-1}}{4 + 6z^{-1} + 8z^{-2}} \iff X(z) = \frac{z^2 - 2z}{4z^2 + 6z + 8}$$

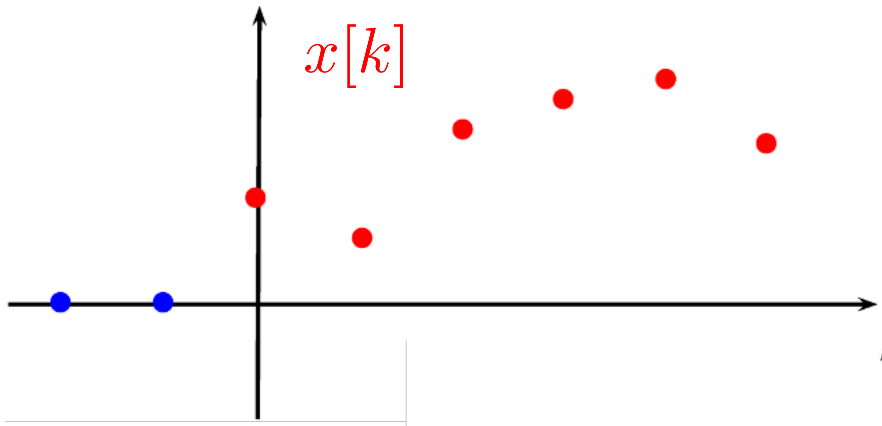
- To obtain the control algorithm, the expression in negative powers of z will be used.

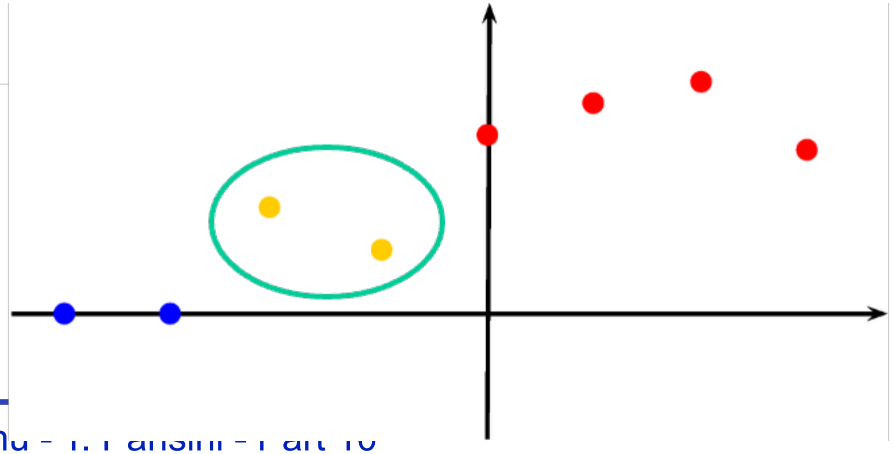
Main properties of the Z Transform

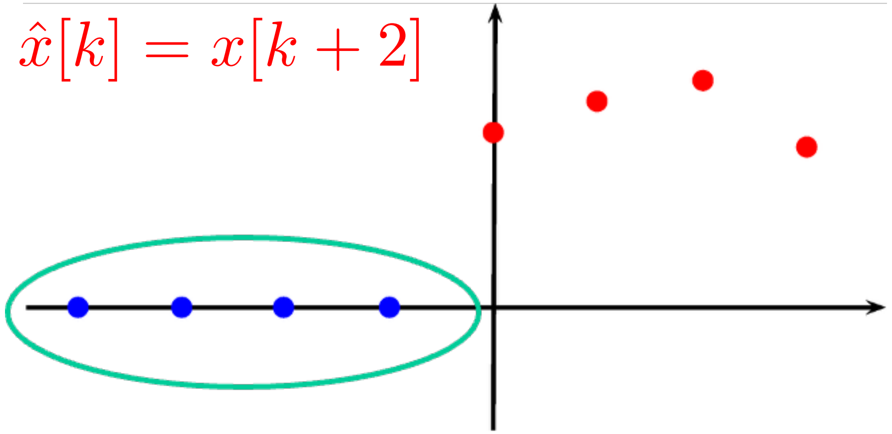
Linearity: $\mathcal{Z}\{c_1 f[k] + c_2 g[k]\} = c_1 \mathcal{Z}\{f[k]\} + c_2 \mathcal{Z}\{g[k]\}$

One-step forward time-translation: $\mathcal{Z}\{x[k+1]\} = z\{X(z) - x[0]\}$

m-steps forward time-translation: $\mathcal{Z}\{x[k+m]\} = z^m \left\{ X(z) - \sum_{k=0}^{m-1} x[k] z^{-k} \right\}$



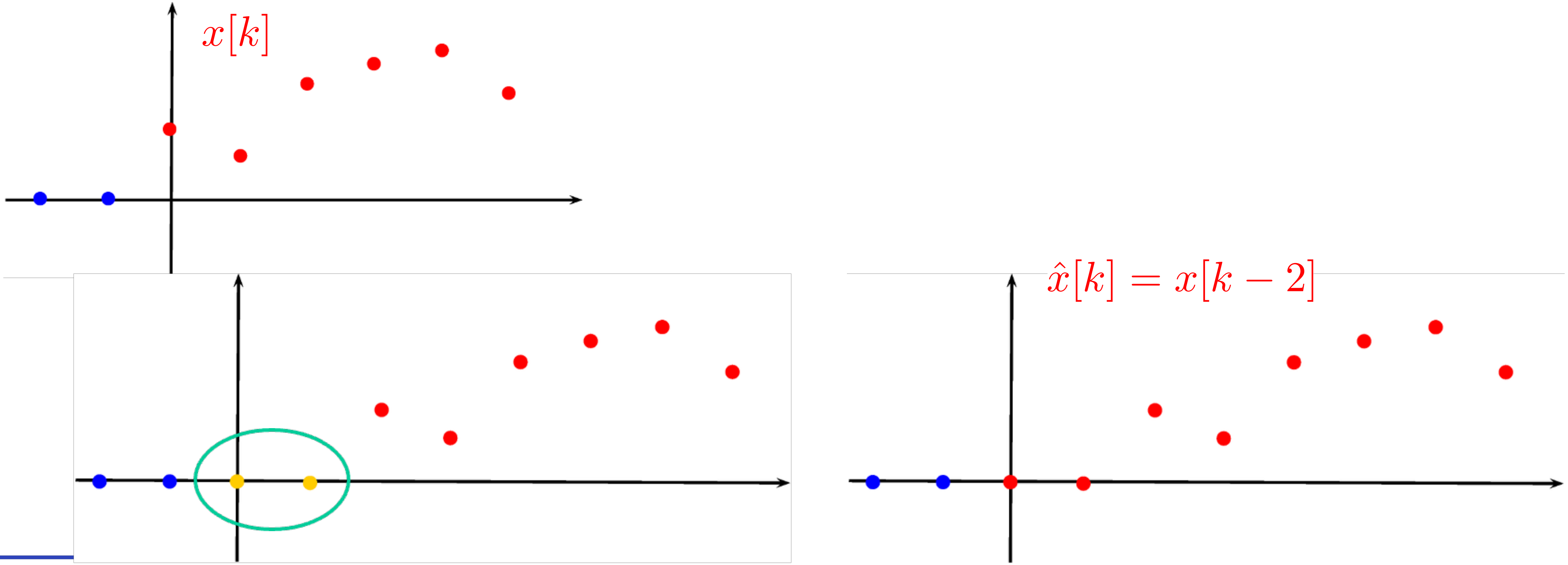




Main properties of the Z Transform

One-step backward time-translation (delay): $\mathcal{Z}\{x[k-1]\} = z^{-1}X(z)$

m-steps backward time-translation: $\mathcal{Z}\{x[k-m]\} = z^{-m}X(z)$



Sampling and Anti-Aliasing Filtering

From basic sampling theory:

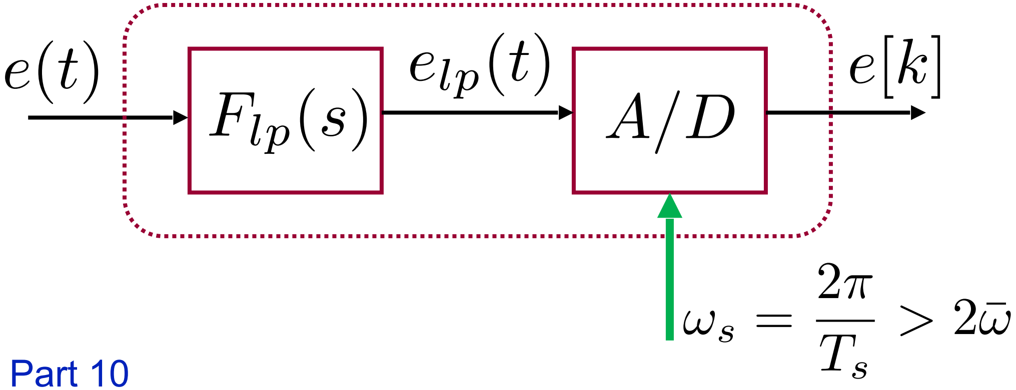
- If the continuous-time signal $e(t)$ is **band-limited** with bandwidth $B = [0, \bar{\omega}]$, the sampling angular frequency has to satisfy:

$$\omega_s = \frac{2\pi}{T_s} > 2\bar{\omega}$$

according to the *Nyquist Sampling Criterion*

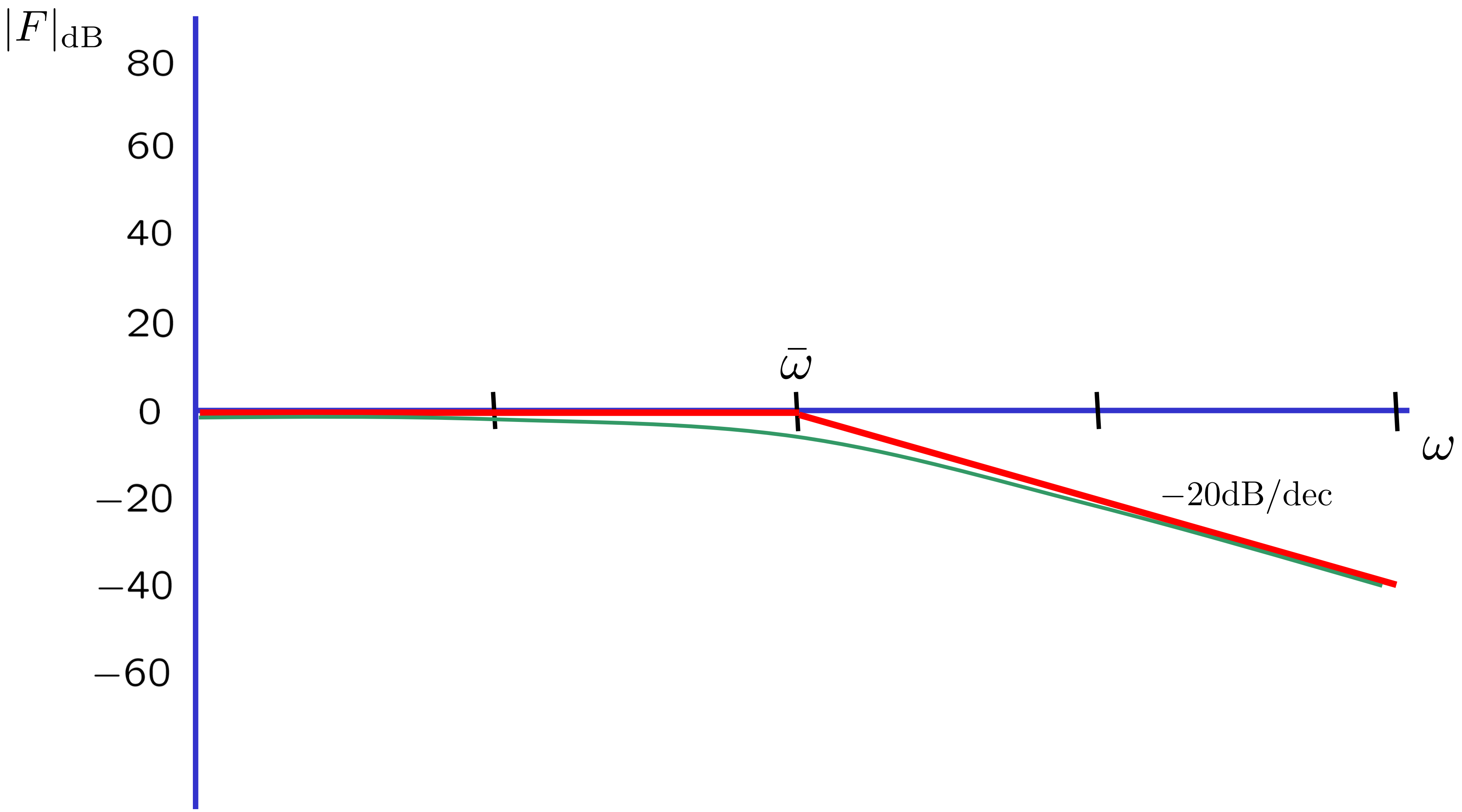
Livescripts in MS Teams: see Part 10: aliasing_demo

- Since, in general, $e(t)$ is **not band-limited**, to mitigate the *aliasing phenomenon* an **anti-aliasing lowpass filter** $F_{lp}(s)$ is designed and used:



For example, a simple first-order anti-aliasing filter could be used:

$$F(s) = \frac{1}{1 + s/\bar{\omega}}$$



UNIVERSITÀ DEGLI STUDI DI TRIESTE

Recall the frequency response behaviour of the complementary sensitivity function from Part 9, slide 40:

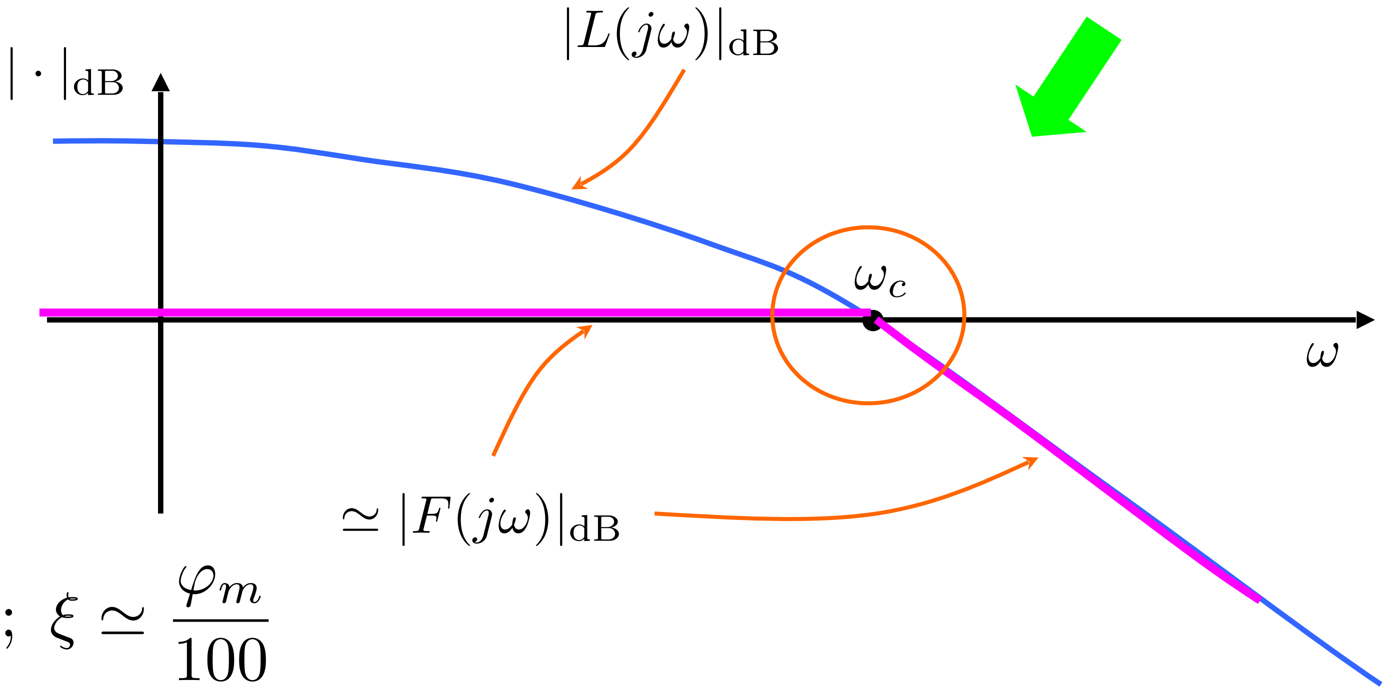
$$|F(j\omega)| = \frac{|L(j\omega)|}{|1+L(j\omega)|} \simeq \begin{cases} 1 & \text{if } |L(j\omega)| \gg 1 \\ |L(j\omega)| & \text{if } |L(j\omega)| \ll 1 \end{cases}$$

- $F(s)$ **low-pass** filter with bandwidth $B \simeq [0, \omega_c]$
- Gain:

$$\mu_F = \begin{cases} 1 & \text{if } g > 0 \\ \dfrac{\mu}{1+\mu} & \text{if } g = 0 \end{cases}$$

- Dominant poles:
  - if real: $\tau \simeq \dfrac{1}{\omega_c}$
  - if complex: $\omega_n \simeq \omega_c\,;\ \xi \simeq \dfrac{\varphi_m}{100}$

a typical scenario



(a) Choice of ω_s based on the **closed-loop bandwidth:**

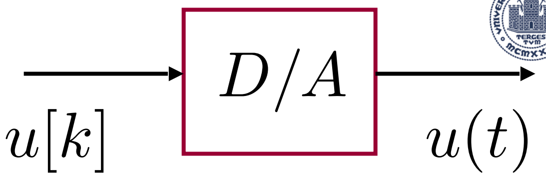
- The **closed-loop bandwidth** approximately is $B_{cl} = [0, \omega_c]$
- **Empirical rule**:

$$\omega_s \ : \ \alpha\omega_c < \omega_s < 10\alpha\omega_c \text{ with } \alpha \in [5, 10] \Longrightarrow \frac{2\pi}{10\alpha\omega_c} < T_s < \frac{2\pi}{\alpha\omega_c}$$

(b) Choice of T_s based on **number of samples in the transient:**

- The closed-loop system has **one** (with $\tau \simeq \dfrac{1}{\omega_c}, \ t_{s,0.01} \simeq 4.6\tau$) or **two dominant poles** (with $\omega_n \simeq \omega_c \, ; \ \xi \simeq \dfrac{\varphi_m}{100}, \ t_{s,0.01} \simeq \dfrac{4.6}{\xi\omega_c}$)
- **Empirical rule**:

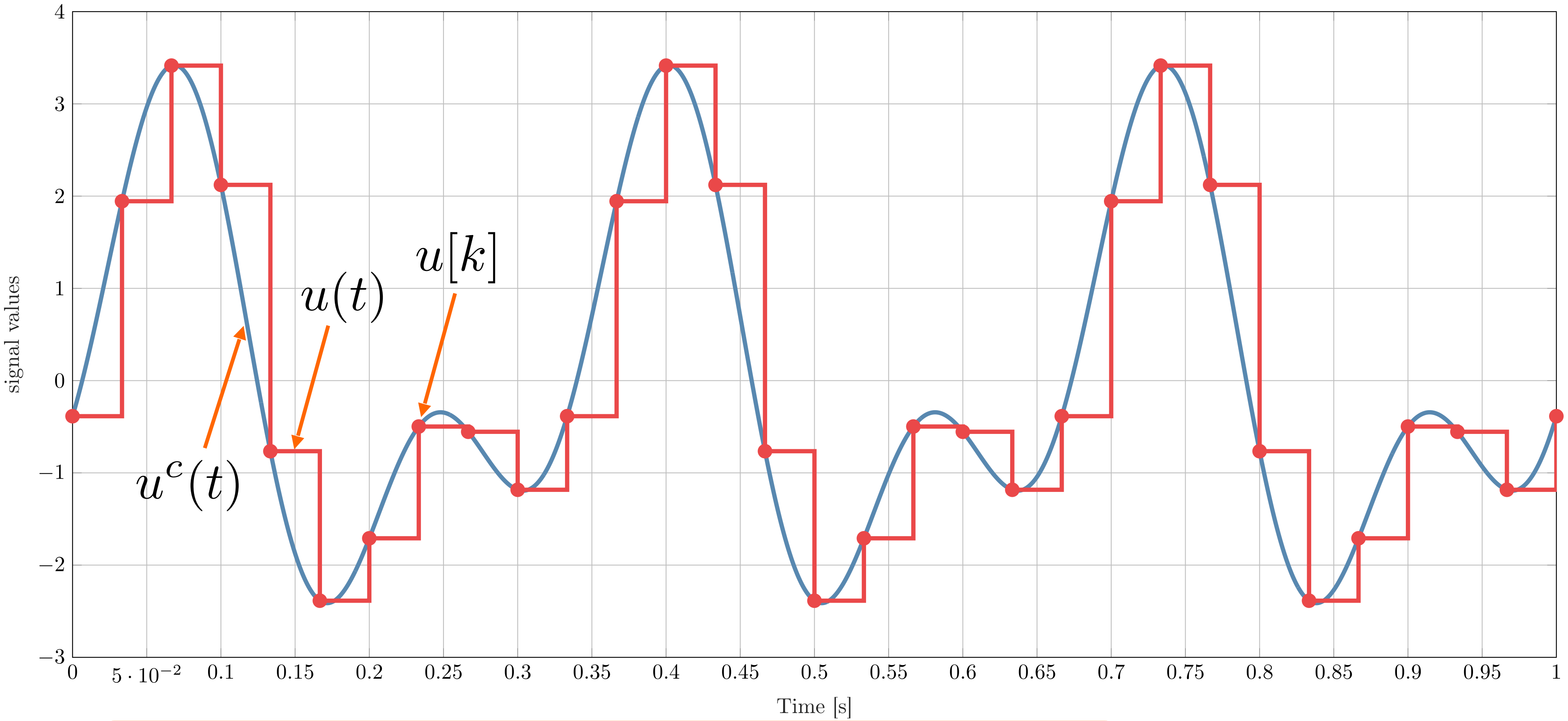
$$T_s \ : \ \frac{t_{s,0.01}}{10\alpha} < T_s < \frac{t_{s,0.01}}{\alpha} \text{ with } \alpha \in [5, 10]$$



Recall the **D/A conversion** from slide 51:

Zero-order hold (ZOH): $u(t) = u[k]\,,\; kT_s \le t < (k+1)T_s,\; k \in \mathbb{Z}$

The D/A conversion using a ZOH is a stair-wise **delayed** approximation of the unknown underlying continuous time function $u^c(t)$



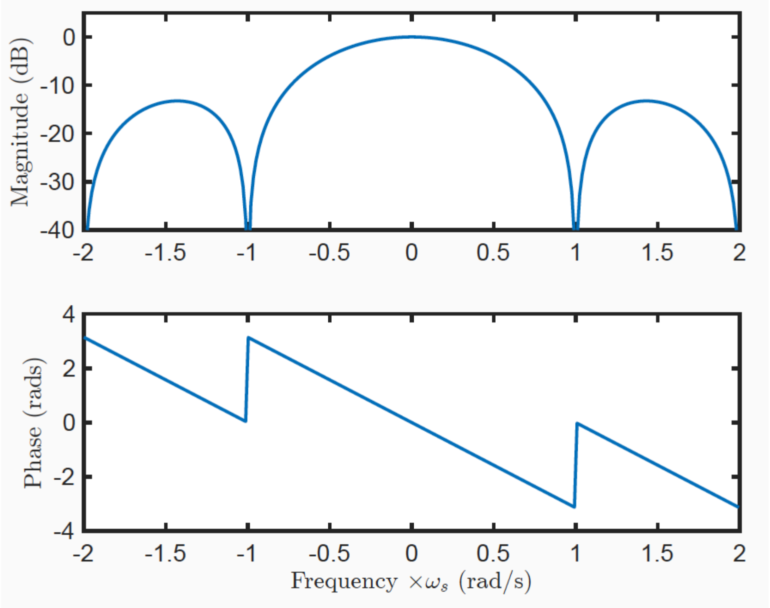
The ZOH block introduces a time-delay

Digital Feedback Control Systems: (contd.)

- The ZOH has a frequency response:

$$H_0(j\omega) = 2e^{-j\omega T_s/2}\frac{\sin(\omega T_s/2)}{\omega}$$

The diagram of $H_0(j\omega)$ is shown on the left for $T_s = 1\mathrm{sec}$



- The slope of the phase is $-T_s/2$ which corresponds to a **delay** of $T_s/2$
- Moreover:

$$|H_0(j\omega)| \simeq 0\mathrm{dB}\,,\ \text{for } \omega \in [0, \bar{\omega}] \text{ with } \bar{\omega} \ll \frac{\omega_s}{2}$$

- From theory of Laplace and Z transforms under ideal sampling it follows that

$$G(z) \xrightarrow{z = e^{sT_s}} G(s) \qquad\qquad G(s) \xrightarrow{s = \frac{1}{T_s}\ln z} G(z)$$

- If $G(s)$ and $G(z)$ are **transfer functions**, these relations **should not be directly used** (for example, direct substitution would not lead to ratios of polynomials!)

- Moreover, it is of key importance to ensure that the approximate transformations from s to z and viceversa **preserve the stability properties**

- In the context of constructing a digital implementation of a continuous-time feedback control system, we are interested in the approximate transformation from s to z

- In the course, we consider the **Backward Euler** and the **Tustin** approximations

- The **Backward Euler** (BE) approximate transformation:

$$s = \frac{z-1}{T_s\, z}$$

$$G(s) \Longrightarrow \widehat{G}_{\mathrm{BE}}(z)$$

The BE approximation **preserves the stability properties** (asymptotically stable poles of $G(s)$ are mapped into asymptotically stable poles of $\widehat{G}_{\mathrm{BE}}(z)$)

- The **Tustin** (Tu) approximate transformation:

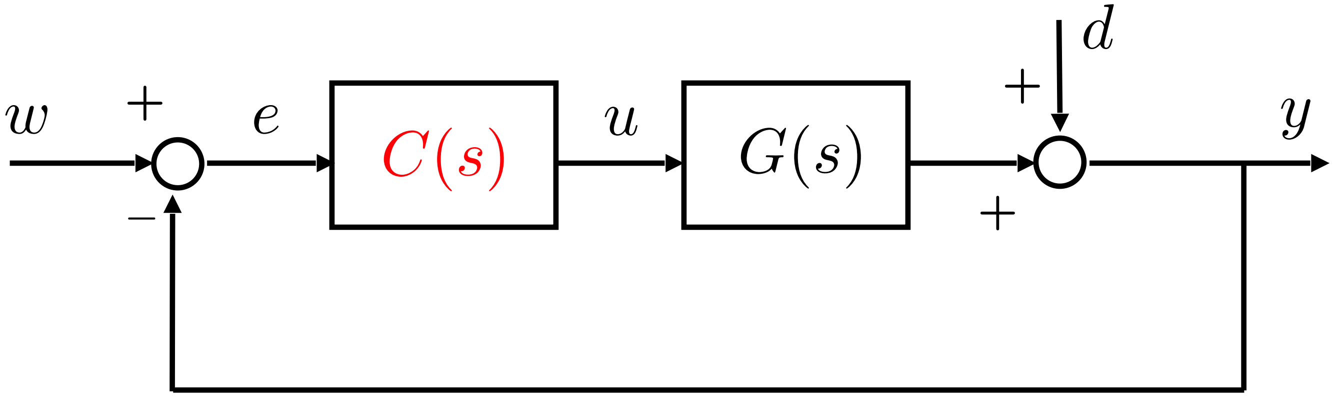
$$s = \frac{2}{T_s}\,\frac{z-1}{z+1}$$

$$G(s) \Longrightarrow \widehat{G}_{\mathrm{Tu}}(z)$$

The Tu approximation **preserves the stability properties** (asymptotically stable poles of $G(s)$ are mapped into asymptotically stable poles of $\widehat{G}_{\mathrm{Tu}}(z)$)

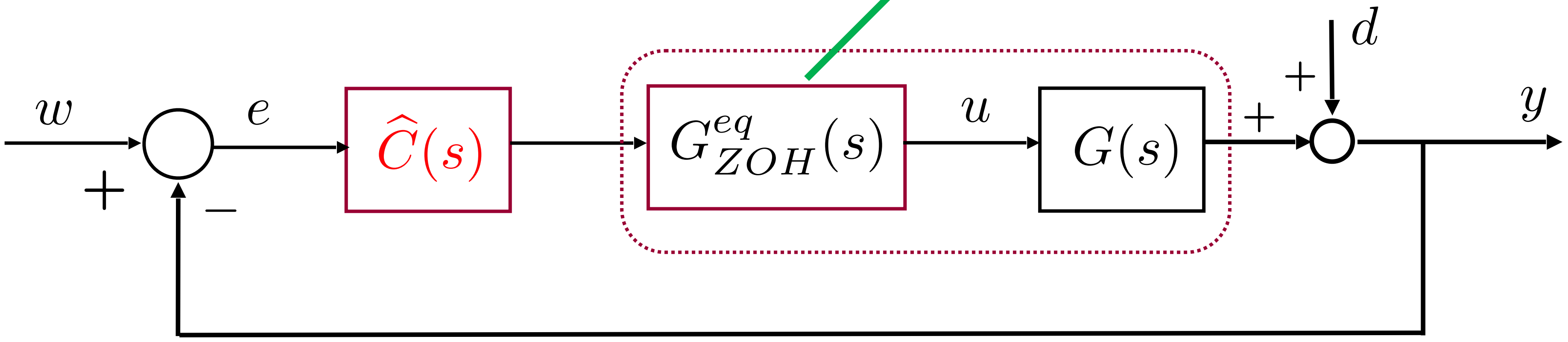
The following **operational procedure** is carried out:

(1) the original control system



is replaced by

$$G_{ZOH}^{eq}(s) = e^{-s\frac{T_s}{2}}$$



Remark: The addition of the block $G_{ZOH}^{eq}(s)$ takes into account the **delay introduced by the ZOH.** This additional block does not influence the requirements on closed-loop bandwidth and gain margin.

(2) the controller $\widehat{C}(s)$ is designed in continuous-time by loop-shaping techniques ensuring that the original specifications are met

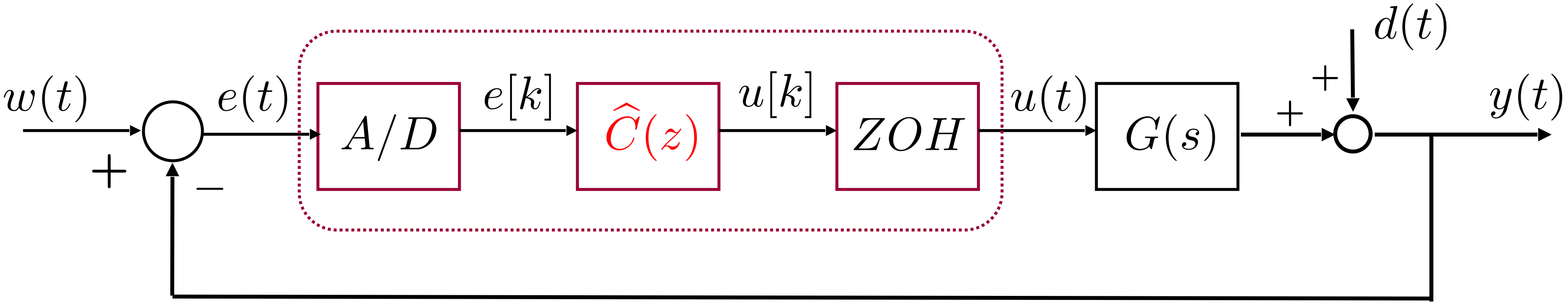
(3) the sampling-time T_s is selected according to the empirical rules in Slide 60

(4) the discrete-time approximation $\widehat{C}(z)$ of the continuous-time controller $\widehat{C}(s)$ is computed by the **BE** or by the **Tu** transformation:

$$s = \frac{z-1}{T_s\, z} \qquad\qquad s = \frac{2}{T_s}\frac{z-1}{z+1}$$

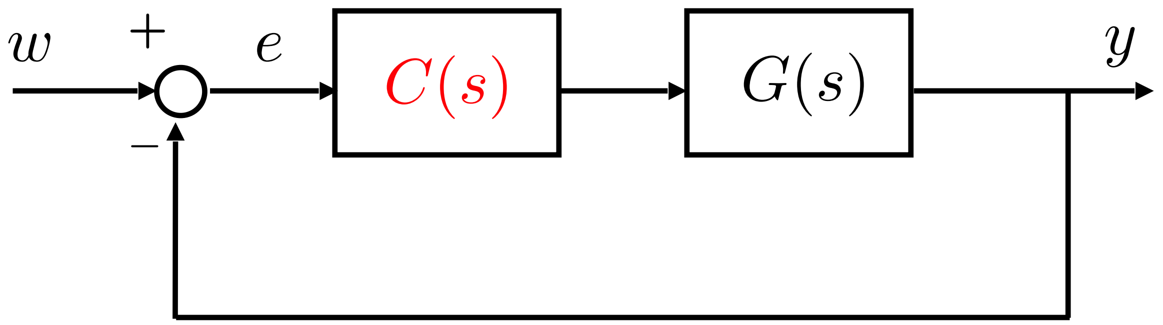
$$\widehat{C}(s) \Rightarrow \widehat{C}_{\mathrm{BE}}(z) \qquad\qquad \widehat{C}(s) \Rightarrow \widehat{C}_{\mathrm{Tu}}(z)$$

(5) the **digital control system** is implemented:



(6) the digital implementation is constructed via numerous **approximations**. Hence, the performance of the digitally-controlled system has to be verified *a posteriori* with the possible need for an improved re-design

Example - Digital Implementation of C.T. Control Systems



$$G(s) = 0.1 \cdot \frac{(1-2s)}{s(1+10s)(1+0.1s)}$$

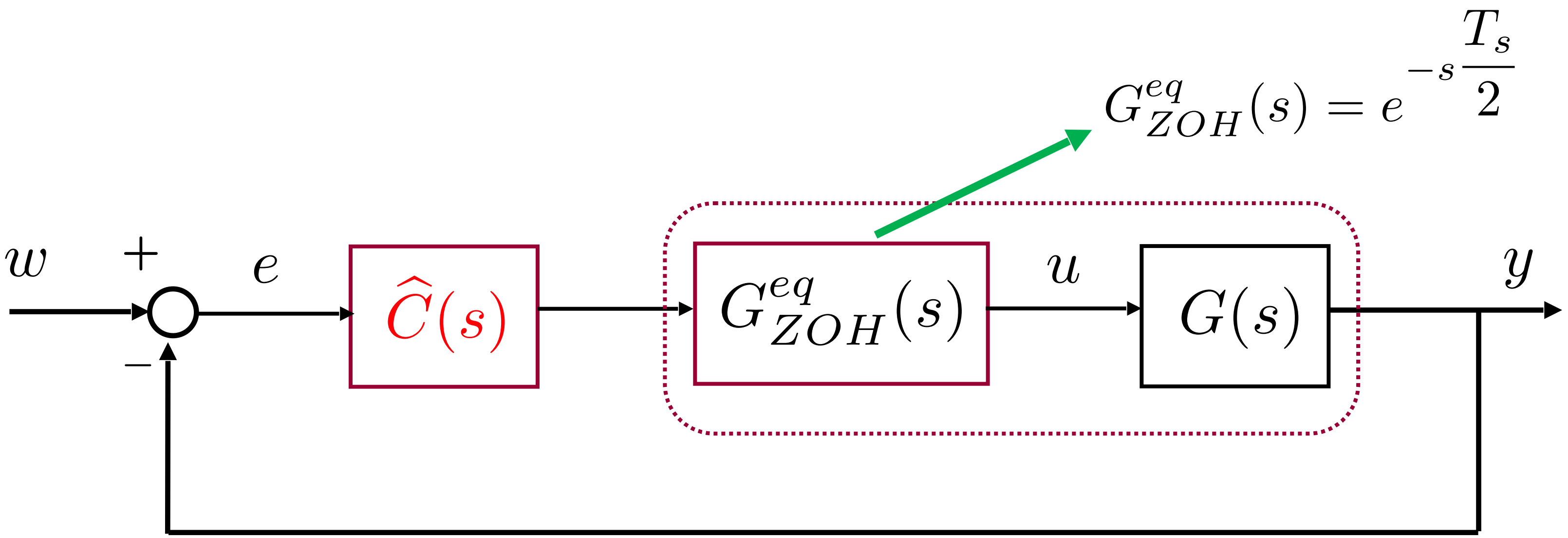
Design specifications (in continuous-time): $\varphi_m \geq 60^\circ$

- The controlled system has the same pole-zero configuration of the non-minimum phase system considered in Slide 35 and we choose the controller considered in Attempt 2:

$$C(s) = \mu_C \frac{1+10s}{1+0.1s} \quad \Longrightarrow \quad L(s) = \mu_C \frac{0.1 \cdot (1-2s)}{s(1+0.1s)^2}$$

Example - Digital Implementation of C.T. Control System (contd.)

- The **design** of the controller needs to take into account suitable **additional margins** so that the original specifications are met with the modified scheme below that takes into account the **digital implementation**:



- Hence, we design:

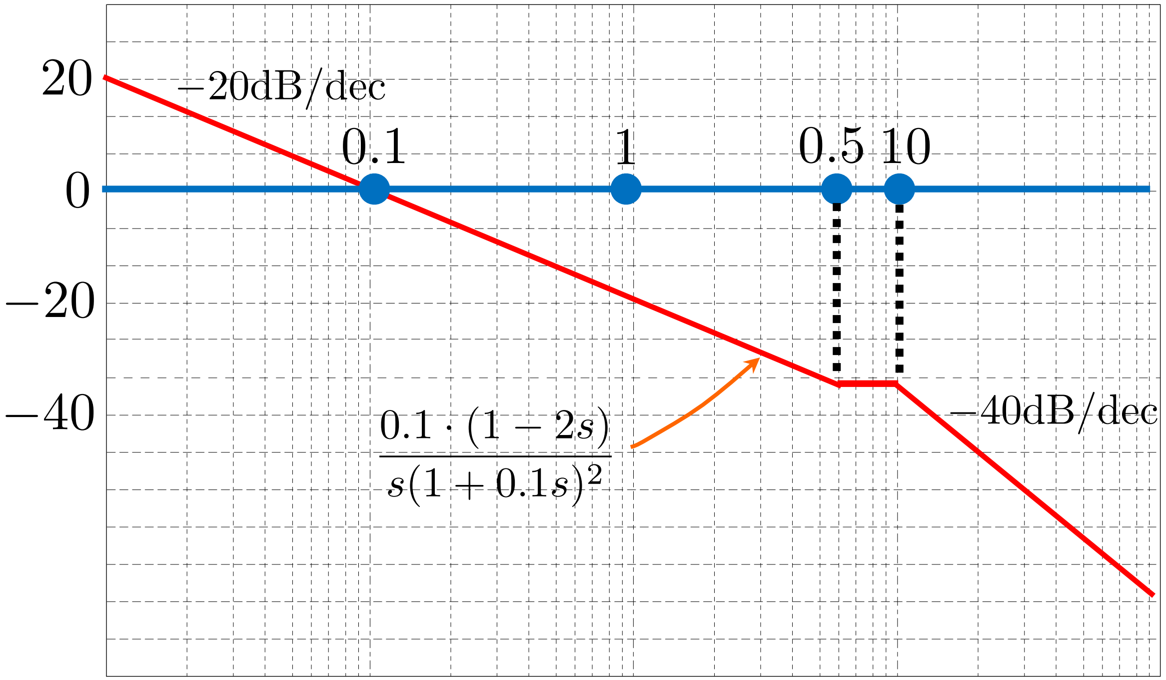
$$\widehat{C}(s) = \hat{\mu}_C \frac{1+10s}{1+0.1s} \quad \text{such that} \quad \varphi_m \geq 60^\circ + \delta_\varphi$$

where δ_φ is sufficiently large so that the ZOH-induced delay can be accommodated

Example - Digital Implementation of C.T. Control System (contd.)

- We set: $\delta_\varphi \geq 5^\circ$
- Consider:

$$\widehat{L}(s) = \hat{\mu}_C \frac{0.1 \cdot (1-2s)}{s(1+0.1s)^2}$$



- For $\hat{\mu}_C = 2$ let's compute

$$\omega_c \; : \; |\widehat{L}(j\omega_c)|_{\text{dB}} = 0$$

$$-20\log\omega + 20\log(2 \cdot 0.1) = 0 \Longrightarrow \omega_c \simeq 0.2\,\text{rad/sec}$$

$$\varphi_c = \arg L(j\omega_c) = -90^\circ - \arctan(2 \cdot 0.2) \cdot \frac{180^\circ}{\pi} + \arg \frac{1}{(1+0.1 \cdot 0.2j)^2}$$

$$\simeq -114.3^\circ \Longrightarrow \varphi_m \simeq 66^\circ \Longrightarrow \delta_\varphi \simeq 6^\circ$$

Example - Digital Implementation of C.T. Control System (contd.)

UNIVERSITÀ DEGLI STUDI DI TRIESTE

- Recall the empirical rule on Slide 60:

$$\omega_s \ : \ \alpha\omega_c < \omega_s < 10\alpha\omega_c \text{ with } \alpha \in [5, 10] \quad \Longrightarrow \quad \frac{2\pi}{10\alpha\omega_c} < T_s < \frac{2\pi}{\alpha\omega_c}$$

- Since $\omega_c \simeq 0.2\,\mathrm{rad/sec}$

$$\frac{\pi}{\alpha} < T_s < \frac{10\pi}{\alpha} \text{ with } \alpha \in [5, 10] \quad \Longrightarrow$$ a suitable choice is $T_s = 1\mathrm{sec}$

- Now: $\omega_s = \dfrac{2\pi}{T_s} = 2\pi \simeq 6.28\,\mathrm{rad/sec} \quad \Longrightarrow \quad \omega_c \simeq 0.2\,\mathrm{rad/sec} \ll \dfrac{\omega_s}{2} \simeq 3.14\,\mathrm{rad/sec}$

- Hence (recall Slide 65) the effect of the ZOH can be **approximated** by

$$G_{ZOH}^{eq}(s) = e^{-s\frac{T_s}{2}} = e^{-\frac{s}{2}}$$

which introduces a phase contribution of: $-\dfrac{\omega_c T_s}{2} \cdot \dfrac{180^\circ}{\pi} = -\dfrac{0.2}{2} \cdot \dfrac{180^\circ}{\pi} \simeq -5.7^\circ$

the phase margin becomes $\widehat{\varphi}_m = \varphi_m - 5.7^\circ \simeq 60^\circ$

Example - Digital Implementation of C.T. Control System (contd.)

- Recall the approximate transformations for the digital controllers on Slide 66:

$$s = \frac{z-1}{T_s\, z} \qquad\qquad s = \frac{2}{T_s}\frac{z-1}{z+1}$$

$$\widehat{C}(s) \Longrightarrow \widehat{C}_{\mathrm{BE}}(z) \qquad\qquad \widehat{C}(s) \Longrightarrow \widehat{C}_{\mathrm{Tu}}(z)$$

- We obtain (recall that $T_s = 1\mathrm{sec}$):

$$\widehat{C}_{\mathrm{BE}}(z) = 2\cdot \left.\frac{1+10s}{1+0.1s}\right|_{s=\frac{z-1}{z}} = \cdots = 20\cdot\frac{z-10/11}{z-1/11} = 20\cdot\frac{1-10/11z^{-1}}{1-1/11z^{-1}}$$

$$\Rightarrow u[k] = \frac{1}{11}u[k-1] + 20e[k] - \frac{200}{11}e[k-1]$$

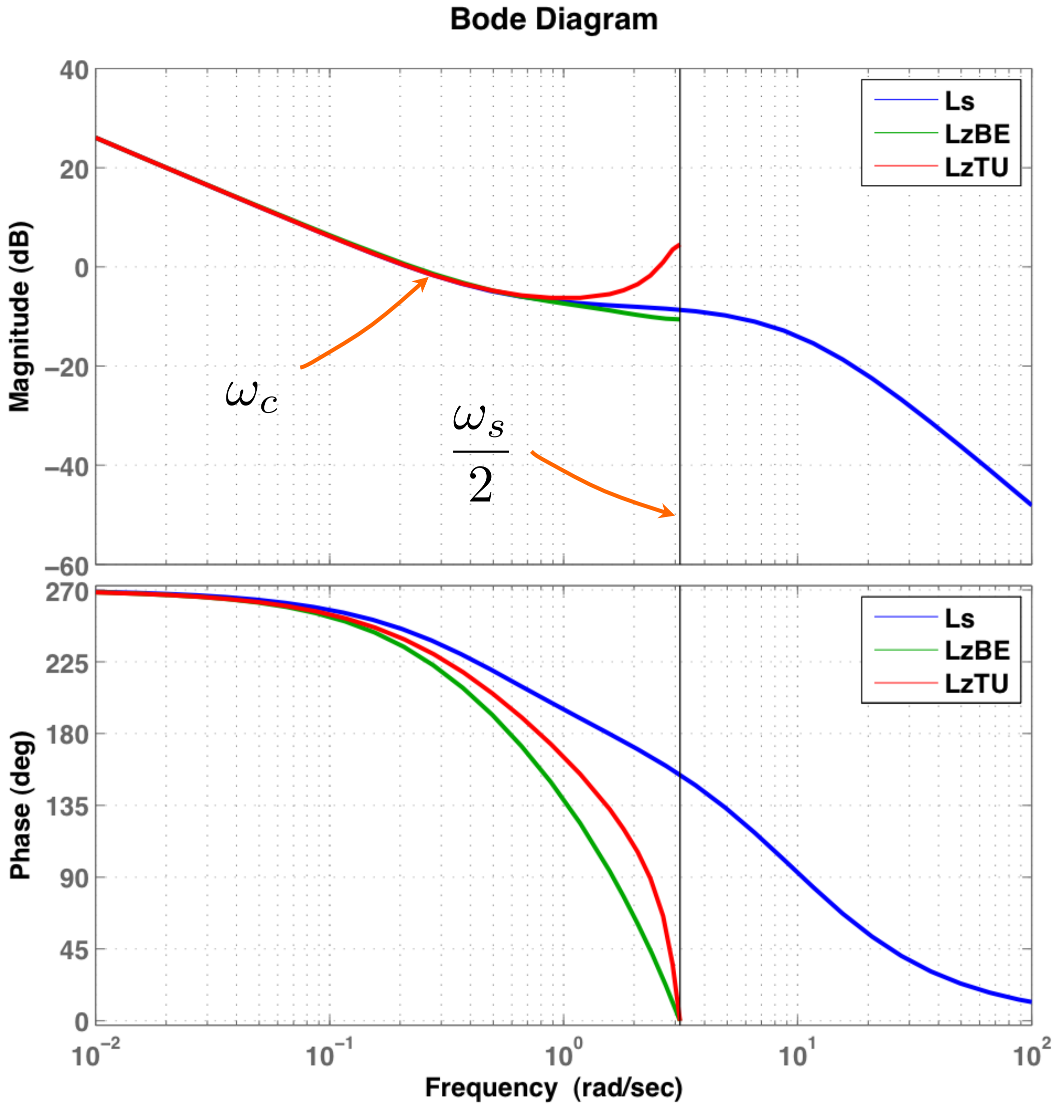
$$\widehat{C}_{\mathrm{Tu}}(z) = 2\cdot \left.\frac{1+10s}{1+0.1s}\right|_{s=2\frac{z-1}{z+1}} = \cdots = 35\cdot\frac{z-19/21}{z+2/3} = 35\cdot\frac{1-19/21z^{-1}}{1+2/3z^{-1}}$$

$$\Rightarrow u[k] = -\frac{2}{3}u[k-1] + 35e[k] - \frac{95}{3}e[k-1]$$

Example - Digital Implementation of C.T. Control System (contd.)

UNIVERSITÀ DEGLI STUDI DI TRIESTE

A posteriori performance verification:



$$T_s = 1\text{sec}$$

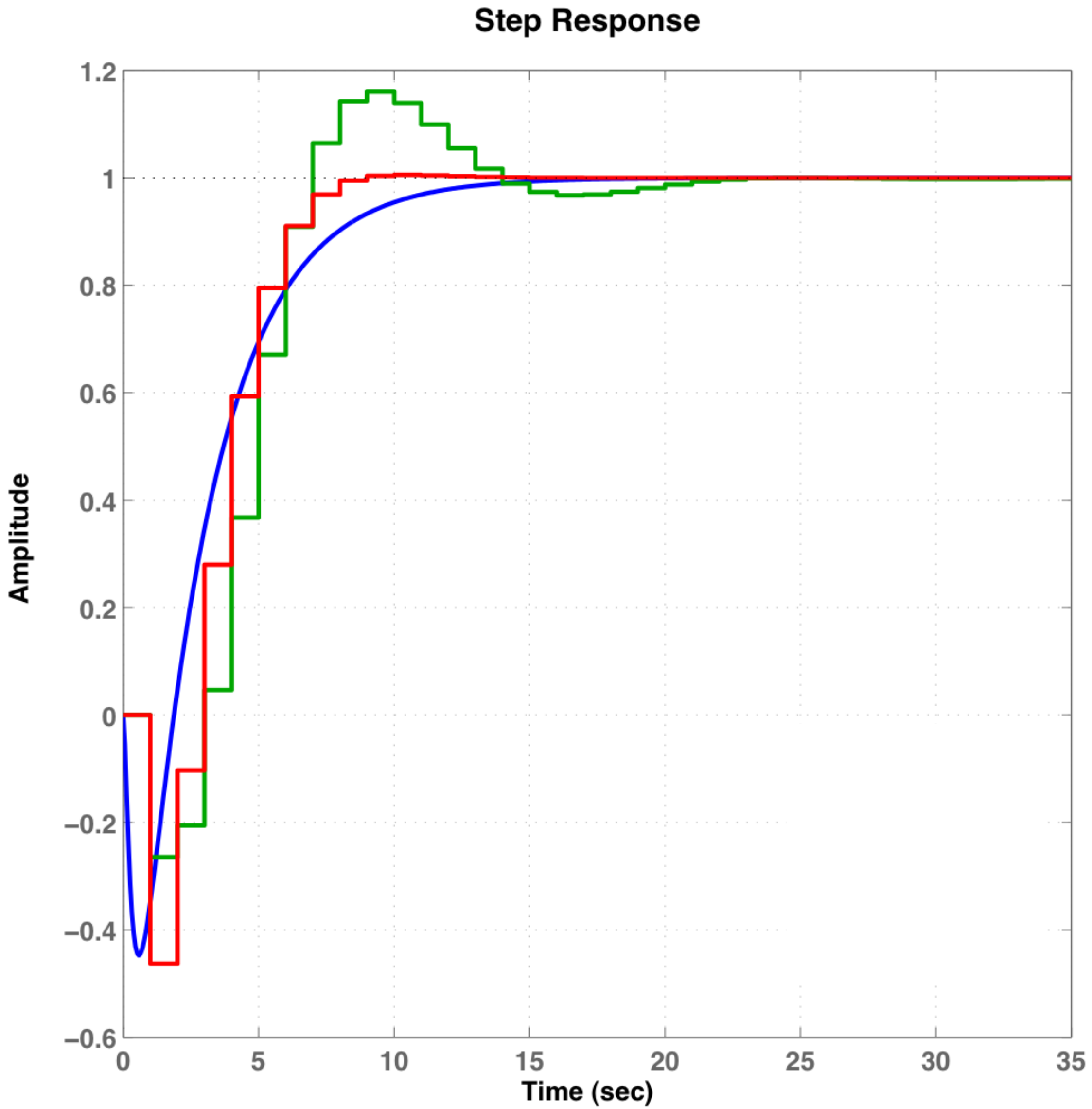
$\widehat{C}(s)\ :\ K_m \simeq 7.74\,\text{dB at } 1.56\,\text{rad/sec}\,;\ \varphi_m \simeq 63.9^\circ \text{ at } 0.218\,\text{rad/sec}$

$\widehat{C}_{\text{BE}}(z)\ :\ K_m \simeq 5.56\,\text{dB at } 0.592\,\text{rad/sec}\,;\ \varphi_m \simeq 51^\circ \text{ at } 0.227\,\text{rad/sec}$

$\widehat{C}_{\text{Tu}}(z)\ :\ K_m \simeq 6.13\,\text{dB at } 0.791\,\text{rad/sec}\,;\ \varphi_m \simeq 57.7^\circ \text{ at } 0.218\,\text{rad/sec}$

Remark: as expected, there is a performance deterioration. Depending on the requirements, a re-design might be needed

Step-response simulation verification:



$$T_s = 1\text{sec}$$

$\widehat{C}(s)\ :\ K_m \simeq 7.74\,\text{dB at }1.56\,\text{rad/sec}\,;\ \varphi_m \simeq 63.9^\circ\ \text{at }0.218\,\text{rad/sec}$

$\widehat{C}_{\text{BE}}(z)\ :\ K_m \simeq 5.56\,\text{dB at }0.592\,\text{rad/sec}\,;\ \varphi_m \simeq 51^\circ\ \text{at }0.227\,\text{rad/sec}$

$\widehat{C}_{\text{Tu}}(z)\ :\ K_m \simeq 6.13\,\text{dB at }0.791\,\text{rad/sec}\,;\ \varphi_m \simeq 57.7^\circ\ \text{at }0.218\,\text{rad/sec}$

- For comparison, let us now set a **smaller sampling time**: $T_s = 0.1\text{sec}$

- Now (recall Slide 65) the effect of the ZOH can be approximated by

$$G_{ZOH}^{eq}(s) = e^{-s\frac{T_s}{2}} = e^{-0.05s}$$

which introduces a phase contribution of: $-\frac{\omega_c T_s}{2} \cdot \frac{180°}{\pi} = -\frac{0.02}{2} \cdot \frac{180°}{\pi} \simeq -0.57°$

↳ the phase margin becomes $\widehat{\varphi}_m = \varphi_m - 0.57° \simeq 65.4°$

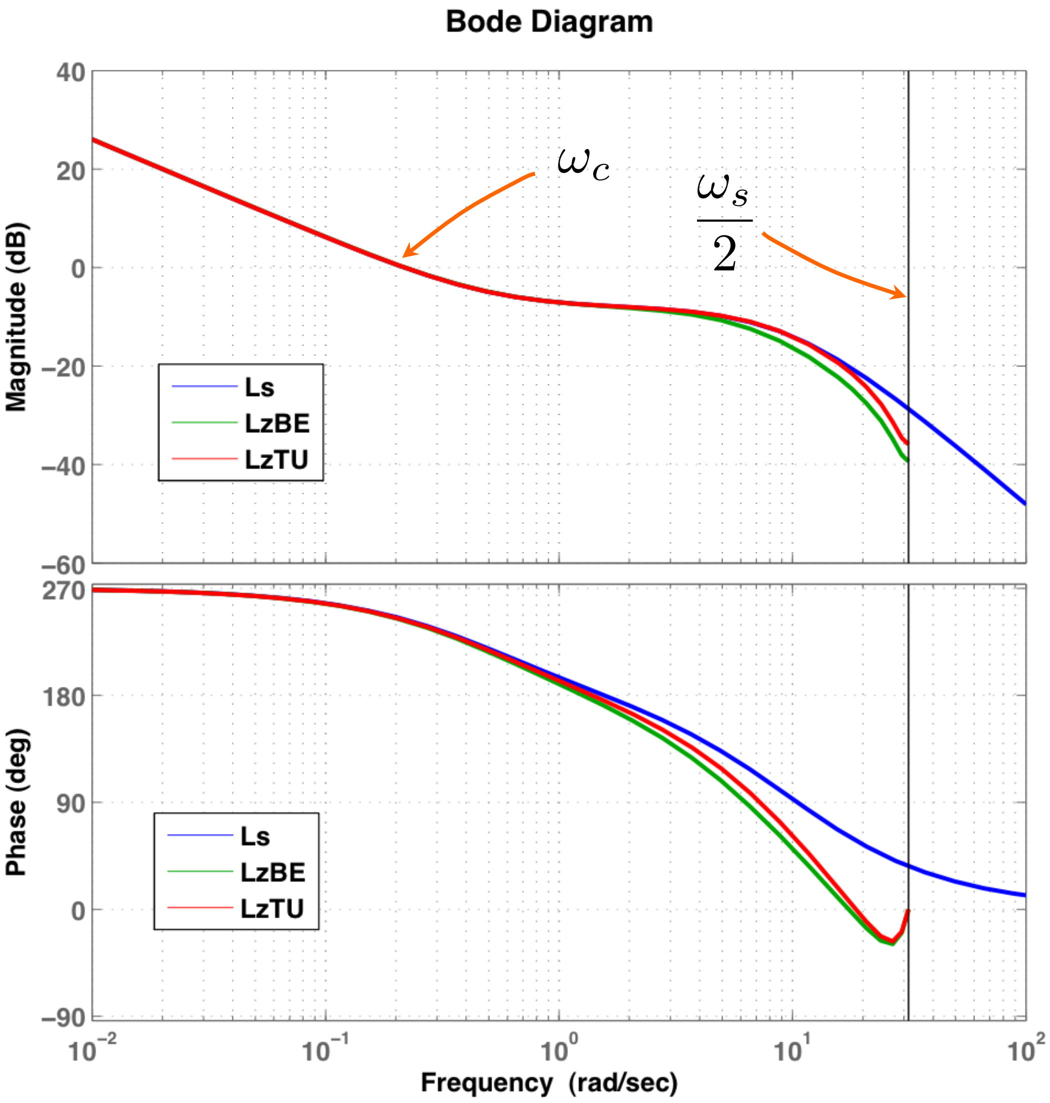
- We obtain:

$$\widehat{C}_{\text{BE}}(z) = 2 \cdot \left.\frac{1+10s}{1+0.1s}\right|_{s=\frac{z-1}{0.1z}} = \cdots = 101 \cdot \frac{z - 100/101}{z - 1/2} = 101 \cdot \frac{1 - 100/101 z^{-1}}{1 - 1/2 z^{-1}}$$

$$\widehat{C}_{\text{Tu}}(z) = 2 \cdot \left.\frac{1+10s}{1+0.1s}\right|_{s=\frac{2}{0.1}\frac{z-1}{z+1}} = \cdots = 134 \cdot \frac{z - 199/201}{z - 1/3} = 134 \cdot \frac{1 - 199/201 z^{-1}}{1 - 1/3 z^{-1}}$$

UNIVERSITÀ DEGLI STUDI DI TRIESTE

A posteriori performance verification:



$$T_s = 0.1\text{sec}$$

$\widehat{C}(s)\ :\ K_m \simeq 7.74\,\text{dB at } 1.56\,\text{rad/sec}\,;\ \varphi_m \simeq 63.9^\circ \text{ at } 0.218\,\text{rad/sec}$

$\widehat{C}_{\text{BE}}(z)\ :\ K_m \simeq 7.5\,\text{dB at } 1.27\,\text{rad/sec}\,;\ \varphi_m \simeq 62.7^\circ \text{ at } 0.219\,\text{rad/sec}$

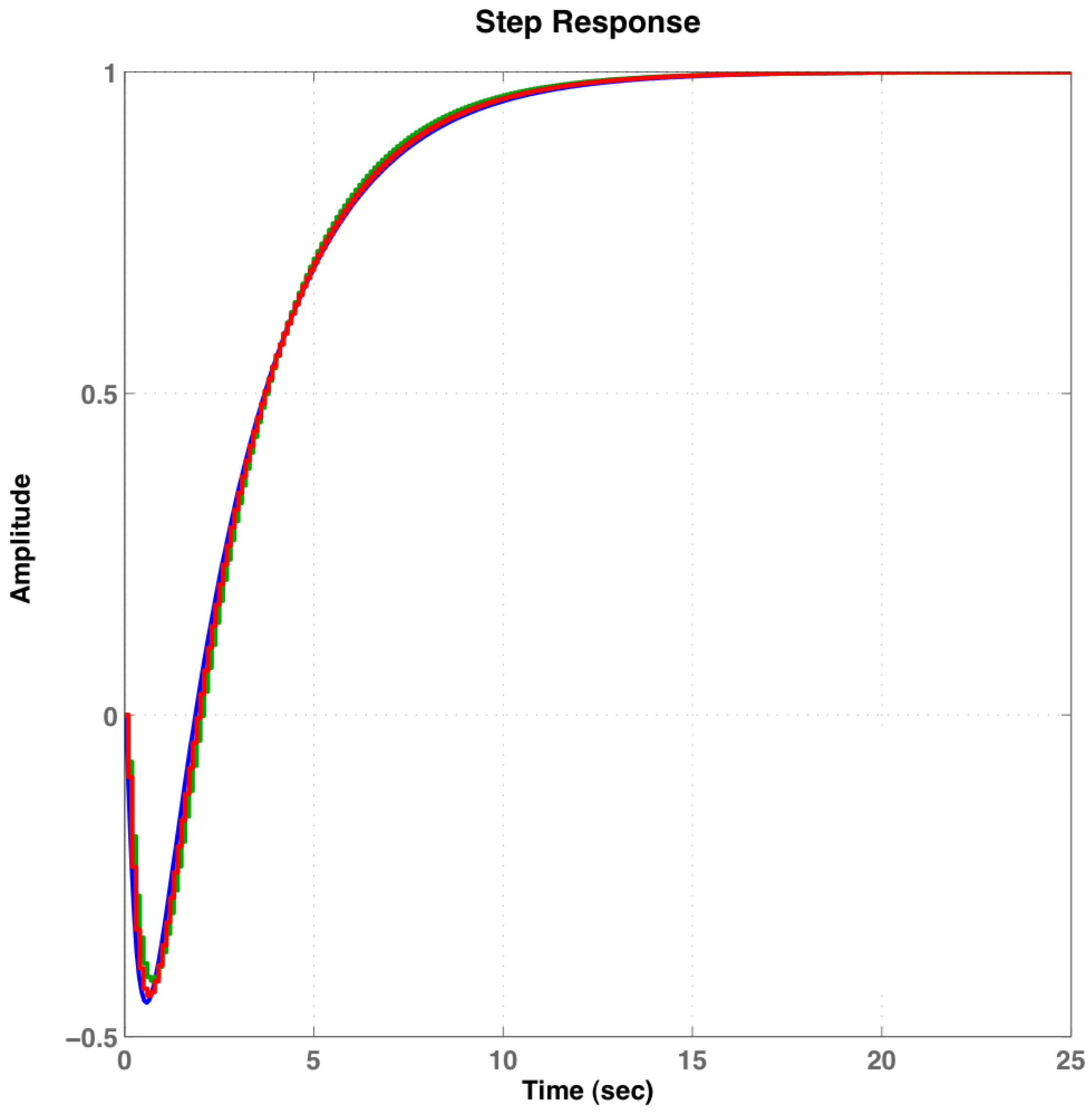
$\widehat{C}_{\text{Tu}}(z)\ :\ K_m \simeq 7.59\,\text{dB at } 1.39\,\text{rad/sec}\,;\ \varphi_m \simeq 63.3^\circ \text{ at } 0.218\,\text{rad/sec}$

Remark: as expected, there is **much less performance deterioration** using a smaller sampling-time

Example - Digital Implementation of C.T. Control System (contd.)

UNIVERSITÀ DEGLI STUDI DI TRIESTE

Step-response simulation verification:



$$T_s = 0.1\text{sec}$$

$\widehat{C}(s)\ :\ K_m \simeq 7.74\,\text{dB at } 1.56\,\text{rad/sec}\,;\ \varphi_m \simeq 63.9^\circ \text{ at } 0.218\,\text{rad/sec}$

$\widehat{C}_{\text{BE}}(z)\ :\ K_m \simeq 7.5\,\text{dB at } 1.27\,\text{rad/sec}\,;\ \varphi_m \simeq 62.7^\circ \text{ at } 0.219\,\text{rad/sec}$

$\widehat{C}_{\text{Tu}}(z)\ :\ K_m \simeq 7.59\,\text{dB at } 1.39\,\text{rad/sec}\,;\ \varphi_m \simeq 63.3^\circ \text{ at } 0.218\,\text{rad/sec}$

Fundamentals of Automatic Control

... The End ...

Main properties of the Z Transform

UNIVERSITÀ DEGLI STUDI DI TRIESTE

Initial Value Theorem:

Given a sequence $\{x[k]\}_{k=0}^{\infty}$ that admits the $\mathcal{Z}$ transform, if $\lim_{z\to\infty} X(z)$ **does exist** and is **finite**, then

$$\lim_{z\to\infty} X(z) = \lim_{k\to 0} x[k] = x[0]$$

Final Value Theorem:

Given a sequence $\{x[k]\}_{k=0}^{\infty}$ that admits the $\mathcal{Z}$ transform, if all roots of the denominator of $X(z)$ are located strictly inside the unit circle in the complex plane with the possible exception of a single root located on the unit circle of the complex plane, then $\lim_{k\to\infty} x[k]$ **does exist**, is **finite** and

$$\lim_{k\to\infty} x[k] = \lim_{z\to 1} \left[(1 - z^{-1})X(z)\right]$$