





**034IN - FONDAMENTI DI AUTOMATICA -  
FUNDAMENTALS OF AUTOMATIC  
CONTROL  
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Part III: Stability**

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$$\begin{cases} \dot{x} = f(x, u, t) \\ y = g(x, u, t) \end{cases} \quad u(t) = \bar{u}, t \geq 0$$

  $0 = f(x, \bar{u})$    $\bar{x}$

Consider a **perturbation of the initial state** set on the equilibrium state  $\bar{x}$  :

$$\begin{cases} u(t) = \bar{u}, t \geq 0 \\ x(0) = \bar{x} + \delta\bar{x} \end{cases} \quad \xrightarrow{\text{Green arrow}} \quad x(t) \neq \bar{x}, t \geq 0$$

**perturbed state trajectory**

- The equilibrium state  $\bar{x}$  is **stable** if  $\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$  such that:

$$\forall x(0) : \|\delta\bar{x}\| < \delta(\varepsilon) \quad \longrightarrow \quad \|x(t) - \bar{x}\| < \varepsilon, \forall t \geq 0$$

- The equilibrium state  $\bar{x}$  is **asymptotically stable** if:

- it is stable, that is, if  $\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$  such that:

$$\forall x(0) : \|\delta\bar{x}\| < \delta(\varepsilon) \quad \longrightarrow \quad \|x(t) - \bar{x}\| < \varepsilon, \forall t \geq 0$$

- and

$$\lim_{t \rightarrow \infty} \|x(t) - \bar{x}\| = 0$$

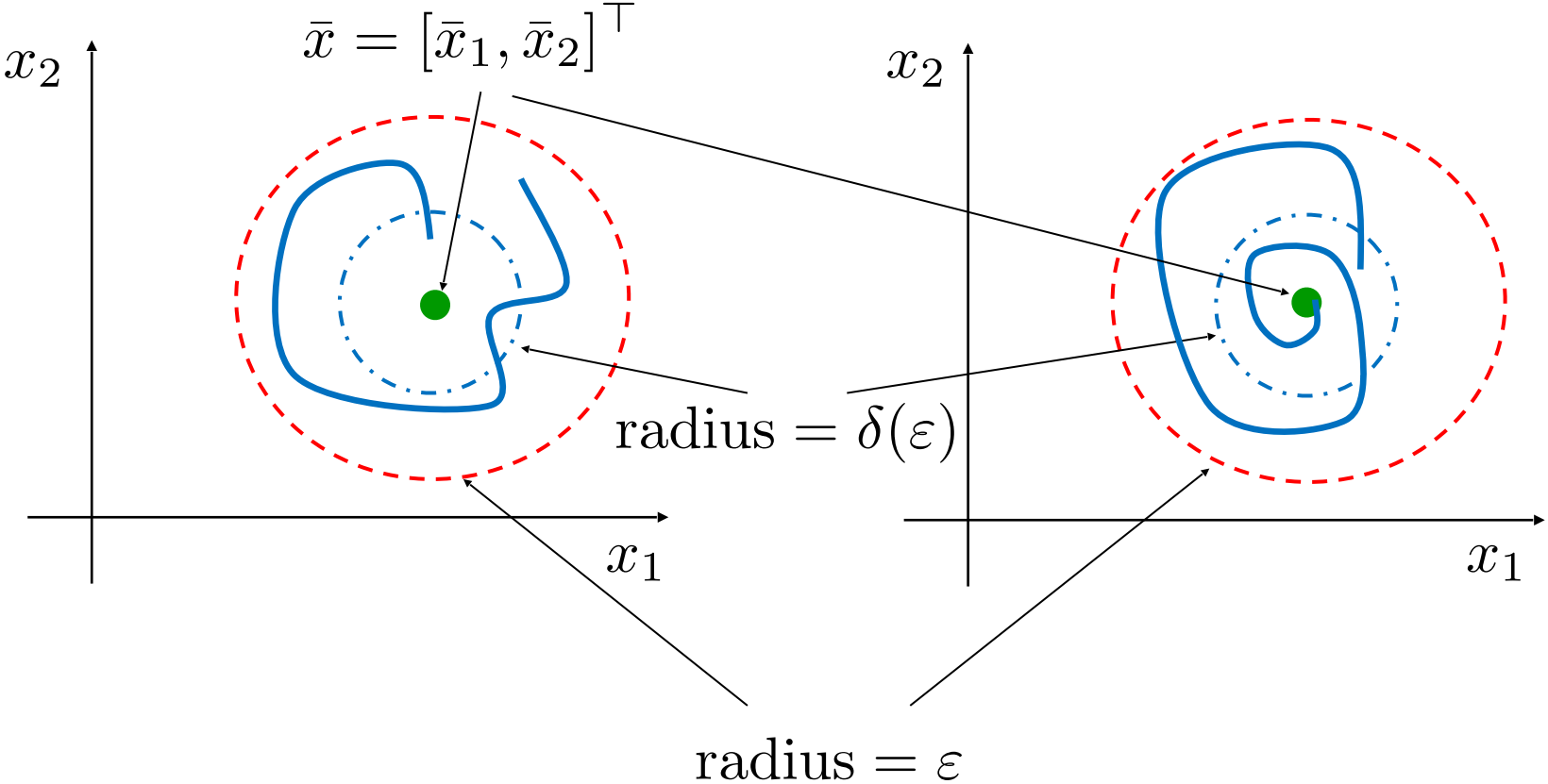
- The equilibrium state  $\bar{x}$  is **unstable** if it is not stable

# Stability of Equilibrium - Geometric Interpretation

$$f(\bar{x}, \bar{u}) = 0$$

**stability**


**asymptotic stability**



$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

- **a)** In equilibrium conditions:

$$\begin{cases} u(t) = \bar{u}, t \geq 0 \\ x(0) = \bar{x} \end{cases}$$


$$x(t) = e^{At}\bar{x} + \int_0^t e^{A(t-\tau)} B\bar{u}d\tau = \bar{x}, \forall t \geq 0$$

- **b)** After a perturbation of the equilibrium state:

$$\begin{cases} u(t) = \bar{u}, t \geq 0 \\ x(0) = \bar{x} + \delta\bar{x} \end{cases}$$



$$x(t) \neq \bar{x}, t \geq 0$$

**perturbed state trajectory**



$$\begin{aligned} x(t) &= e^{At}(\bar{x} + \delta\bar{x}) + \int_0^t e^{A(t-\tau)} B\bar{u}d\tau \\ &= \bar{x} + e^{At}\delta\bar{x} \end{aligned}$$



$$x(t) - \bar{x} = e^{At}\delta\bar{x}$$

$$\begin{cases} u(t) = \bar{u}, t \geq 0 \\ x(0) = \bar{x} + \delta\bar{x} \end{cases}$$



$$x(t) - \bar{x} = e^{At} \delta\bar{x}$$

deviation of the perturbed  
state trajectory from the  
equilibrium state

- stability properties do not depend on the specific value taken on by  $\bar{x}$



stability is not a property of the equilibrium state (as in the general case) but it is a structural property of the system as a whole

- stability properties depend on the time-behaviour of  $e^{At}$  :

- stability



$e^{At}$  bounded  $\forall t \geq 0$

- asymptotic stability



$\lim_{t \rightarrow \infty} e^{At} = 0$

- instability

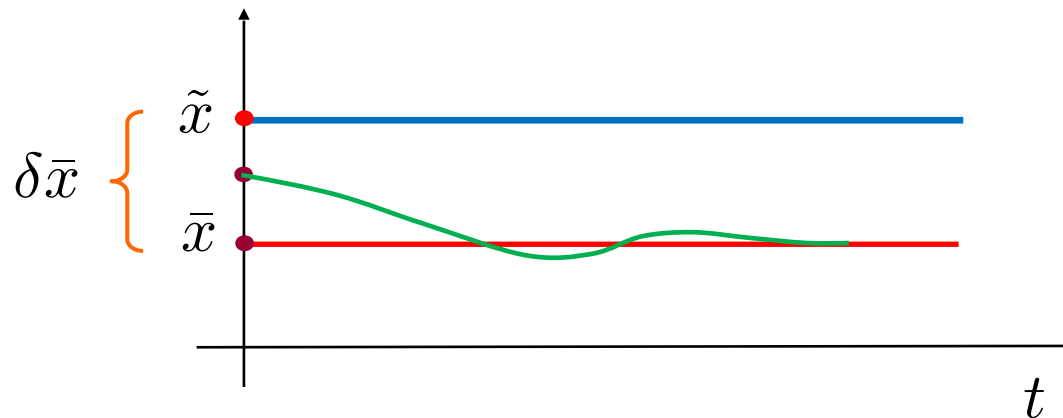


$e^{At}$  unbounded

- The system state moved from equilibrium "tends" getting back to it
- Given a specific **constant input**  $u(t) = \bar{u}, t \geq 0$  the corresponding equilibrium state  $\bar{x}$  is **unique**:

asymptotic stability  $\longrightarrow$   $\bar{x}$  unique  $\longleftrightarrow$   $\det(A) \neq 0$

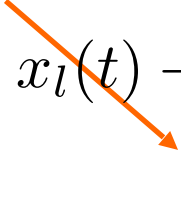
In fact, suppose that, for a given  $\bar{u}$ , two different equilibrium states  $\bar{x}, \tilde{x}$  would exist:



contradiction with the  
asymptotic stability  
assumption!!!

- The perturbed state trajectory asymptotically only depends on the input trajectory  $u(t)$  :

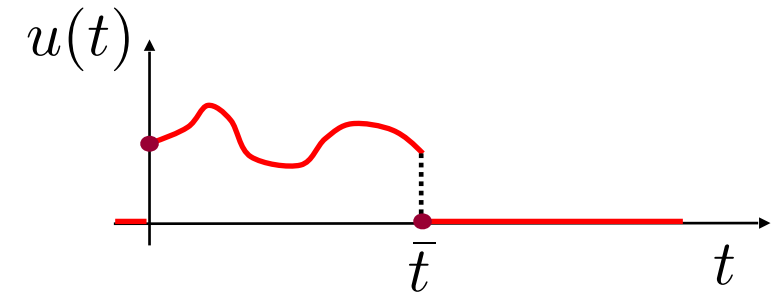
$$x(t) = x_l(t) + x_f(t)$$

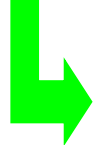
  
0

For example:

- $u(t) = 0, t \geq 0$    $\lim_{t \rightarrow \infty} x(t) = 0; \lim_{t \rightarrow \infty} y(t) = 0, \forall x_0$

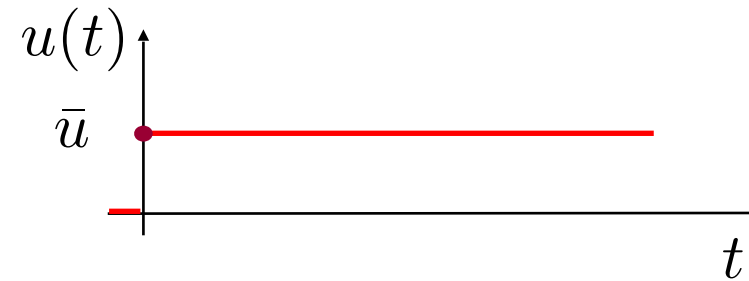
- $u(t) = \begin{cases} \text{any function} & 0 \leq t < \bar{t} \\ 0 & t \geq \bar{t} \end{cases}$



-   $\lim_{t \rightarrow \infty} x(t) = 0; \lim_{t \rightarrow \infty} y(t) = 0, \forall x_0$

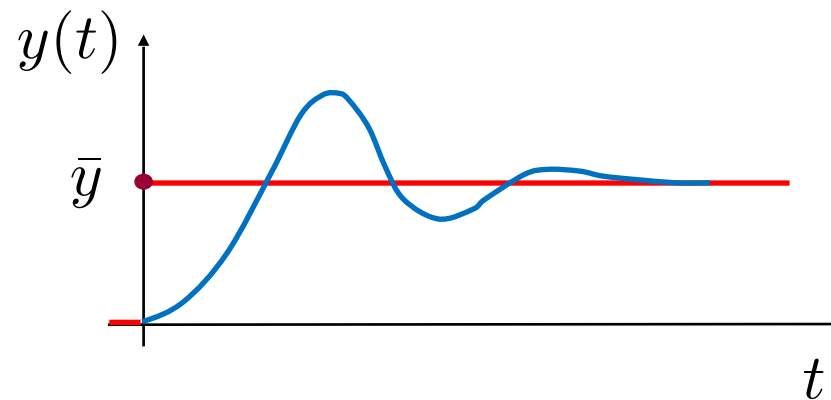
- If:

$$u(t) = \begin{cases} 0 & t < 0 \\ \bar{u} & t \geq 0 \end{cases}$$



$$\lim_{t \rightarrow \infty} y(t) = \bar{y}, \quad \forall x_0 \quad \text{where} \quad \bar{y} = \mu \bar{u} = (-CA^{-1}B + D) \bar{u}$$

static gain

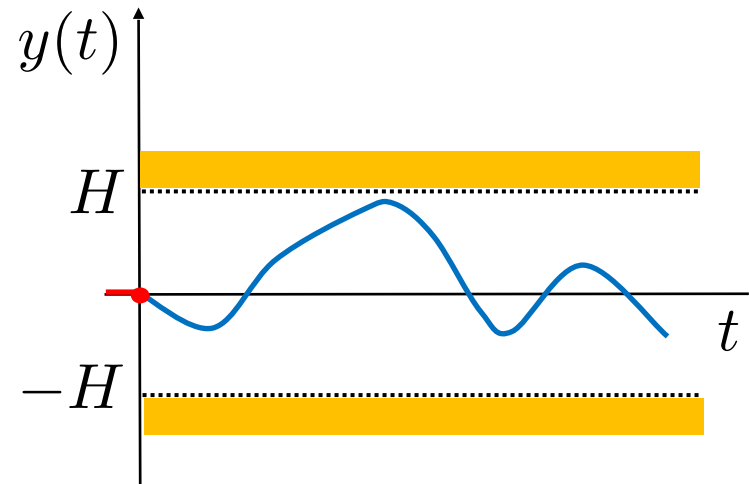
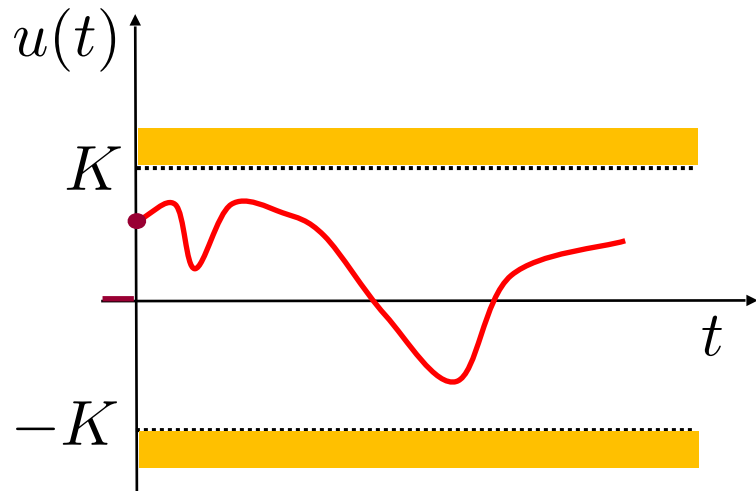


- It holds that:

$$u(t) \text{ bounded} \quad \longrightarrow \quad y(t) \text{ bounded}$$

that is:




$$\|u(t)\| < K, \forall t \geq 0 \quad \longrightarrow \quad \exists H : \|y(t)\| < H, \forall t \geq 0$$



- This property is named **Bounded Input Bounded Output (BIBO) Stability**
- Asymptotic Stability  $\longrightarrow$  BIBO Stability but **NOT** viceversa

Stability properties depend on the asymptotic time-behaviour of the free state trajectories or, equivalently, of the matrix exponential:

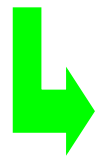
$$x_l(t) = e^{At}x(0)$$

- stability   $e^{At}$  **bounded**  $\forall t \geq 0$
- asymptotic stability   $\lim_{t \rightarrow \infty} e^{At} = 0$
- instability   $e^{At}$  **unbounded**

## Example: Scalar State Case

$$x_l(t) = e^{at}x(0), \quad a \in \mathbb{R}$$

- stability  $\longleftrightarrow a \leq 0$
- asymptotic stability  $\longleftrightarrow a < 0$
- instability  $\longleftrightarrow a > 0$



The stability property just depends on the **sign** of  $a \in \mathbb{R}$

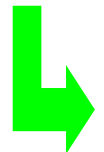
Consider the following cases:

1. Matrix  $A$  is **diagonal**
2. Matrix  $A$  has **real and distinct eigenvalues**
3. Matrix  $A$  has **complex and distinct eigenvalues**
4. Matrix  $A$  has eigenvalues with multiplicity larger than 1
  1. Matrix  $A$  **can** be transformed into a diagonal matrix
  2. Matrix  $A$  **cannot** be transformed into a diagonal matrix

# Case 1: Matrix $A$ is Diagonal




$$A = \begin{bmatrix} s_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & s_n \end{bmatrix}$$

$s_1, s_2, \dots, s_n \in \mathbb{R}$   
eigenvalues of  $A$


$$e^{At} = \begin{bmatrix} e^{s_1 t} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & e^{s_n t} \end{bmatrix}$$

The matrix  $e^{At}$  is a square  $n \times n$  matrix whose elements  $e^{s_i t}$  are functions of time

Hence:

- stability   $s_i \leq 0, i = 1, \dots, n$
- asymptotic stability   $s_i < 0, i = 1, \dots, n$
- instability   $\exists i \in \{1, \dots, n\}$  such that  $s_i > 0$

## Case 2: Matrix $A$ has real and distinct eigenvalues

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad s_1, s_2, \dots, s_n \in \mathbb{R}, s_1 \neq s_2 \neq \cdots \neq s_n$$

eigenvalues of  $A$

Find matrix  $T \in \mathbb{R}^{n \times n}$ ,  $\det(T) \neq 0$  :  $A = T\tilde{A}T^{-1}$ ,  $\tilde{A} = \begin{bmatrix} s_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & s_n \end{bmatrix}$



$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \dots$$

$$= I + T\tilde{A}T^{-1}t + \frac{1}{2}(T\tilde{A}T^{-1}t)(T\tilde{A}T^{-1}t) + \dots$$

$$= T \left( I + \tilde{A}t + \frac{1}{2}(\tilde{A}t)^2 + \dots \right) T^{-1}$$

$$= Te^{\tilde{A}t}T^{-1}$$



$$e^{At} = T \begin{bmatrix} e^{s_1t} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & e^{s_nt} \end{bmatrix} T^{-1}$$

Hence:

- stability  $\longleftrightarrow s_i \leq 0, i = 1, \dots, n$
- asymptotic stability  $\longleftrightarrow s_i < 0, i = 1, \dots, n$
- instability  $\longleftrightarrow \exists i \in \{1, \dots, n\}$  such that  $s_i > 0$

$$\begin{cases} \dot{x}_1 = -2x_1 + 6x_2 \\ \dot{x}_2 = -2x_1 + 5x_2 \end{cases} \quad A = \begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix}$$

- Eigenvalues:

$$\begin{aligned} p_A(s) &= \det(sI - A) = \det \begin{bmatrix} s + 2 & -6 \\ 2 & s - 5 \end{bmatrix} = (s + 2)(s - 5) + 12 \\ &= s^2 - 3s + 2 = (s - 2)(s - 1) \end{aligned}$$



$$s_1 = 1; s_2 = 2$$

**Unstable** because both eigenvalues are positive

- Eigenvectors:

$$Av = s_1 v \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{matrix} \text{L} \\ \text{R} \end{matrix} \begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 1 \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{cases} -2v_1 + 6v_2 = v_1 \\ -2v_1 + 5v_2 = v_2 \end{cases}$$

$$Av = s_2 v$$

$$\begin{matrix} \text{L} \\ \text{R} \end{matrix} \begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2 \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{matrix} \text{L} \\ \text{R} \end{matrix} v_1 = 2v_2 \quad \text{for example:} \quad v^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{cases} -2v_1 + 6v_2 = 2v_1 \\ -2v_1 + 5v_2 = 2v_2 \end{cases}$$

$$\begin{matrix} \text{L} \\ \text{R} \end{matrix} v_1 = \frac{3}{2}v_2 \quad \text{for example:} \quad v^{(2)} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

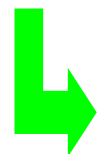
- Transformation into diagonal form:

$$T = \left[ v^{(1)} \mid v^{(2)} \right] = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \quad \longrightarrow \quad T^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\downarrow \quad \tilde{A} = T^{-1}AT = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

- Calculation of the matrix exponential  $e^{At}$  :

$$\begin{aligned} e^{At} &= T e^{\tilde{A}t} T^{-1} = T \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} T^{-1} \\ &= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 4e^t - 3e^{2t} & -6e^t + 6e^{2t} \\ 2e^t - 2e^{2t} & -3e^t + 4e^{2t} \end{bmatrix} \end{aligned}$$



The matrix exponential contains elements (all, in this specific example) that are **asymptotically unbounded** which is consistent with the previous statement about **instability** based on the **positive sign of the eigenvalues**

## Case 3: Matrix $A$ has complex and distinct eigenvalues

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$s_1, s_2, \dots, s_n \in \mathbb{C}, s_1 \neq s_2 \neq \dots \neq s_n$   
eigenvalues of  $A$

For simplicity, consider the case:  $n = 2; \quad s_1 = \sigma + j\omega; \quad s_2 = \sigma - j\omega$



Hence, the matrix exponential  $e^{At}$  contains terms such as:

$$\gamma e^{(\sigma + j\omega)t} + \bar{\gamma} e^{(\sigma - j\omega)t} \quad \text{where} \quad \gamma = \alpha + j\beta; \quad \bar{\gamma} = \alpha - j\beta$$

### Case 3: Matrix $A$ has complex and distinct eigenvalues (contd.)

Then:

$$\begin{aligned} & \gamma e^{(\sigma+j\omega)t} + \bar{\gamma} e^{(\sigma-j\omega)t} \\ &= \gamma e^{\sigma t} [\cos(\omega t) + j \sin(\omega t)] + \bar{\gamma} e^{\sigma t} [\cos(\omega t) - j \sin(\omega t)] \\ &= e^{\sigma t} [(\gamma + \bar{\gamma}) \cos(\omega t) + j(\gamma - \bar{\gamma}) \sin(\omega t)] \\ &= e^{\sigma t} [2\alpha \cos(\omega t) + j(j2\beta) \sin(\omega t)] \\ &= 2e^{\sigma t} [\alpha \cos(\omega t) - \beta \sin(\omega t)] \end{aligned}$$

bounded

this is the term responsible for boundedness/convergence/divergence over time


Hence, generalising:

- stability  $\longleftrightarrow \operatorname{Re}(s_i) \leq 0, i = 1, \dots, n$
- asymptotic stability  $\longleftrightarrow \operatorname{Re}(s_i) < 0, i = 1, \dots, n$
- instability  $\longleftrightarrow \exists i \in \{1, \dots, n\}$  such that  $\operatorname{Re}(s_i) > 0$

## Case 4: Matrix $A$ has multiple eigenvalues

- **Example 1:**

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \quad s_1 = s_2 = \alpha$$


$$e^{At} = \begin{bmatrix} e^{\alpha t} & 0 \\ 0 & e^{\alpha t} \end{bmatrix}$$

Matrix  $A$  is already in diagonal form, hence no need for resorting to an equivalent state equation. This case is equivalent to Case 1

- **Example 2:**

$$A = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} \quad s_1 = s_2 = \alpha$$

In this case, there **does not exist** an equivalent state space transformation bringing matrix  $A$  into a diagonal form


$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \dots$$

$$A^2 = \begin{bmatrix} \alpha^2 & 2\alpha \\ 0 & \alpha^2 \end{bmatrix} \quad A^3 = \begin{bmatrix} \alpha^3 & 3\alpha^2 \\ 0 & \alpha^3 \end{bmatrix} \quad \dots \quad A^k = \begin{bmatrix} \alpha^k & k\alpha^{k-1} \\ 0 & \alpha^k \end{bmatrix} \quad \dots$$

## Case 4: Matrix $A$ has multiple eigenvalues (contd.)

Therefore:

$$A = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} \quad s_1 = s_2 = \alpha$$


$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} t + \begin{bmatrix} \alpha^2 & 2\alpha \\ 0 & \alpha^2 \end{bmatrix} \frac{t^2}{2!} + \dots$$

$$+ \dots + \begin{bmatrix} \alpha^k & k\alpha^{k-1} \\ 0 & \alpha^k \end{bmatrix} \frac{t^k}{k!} + \dots$$

$$= \begin{bmatrix} e^{\alpha t} & t + \alpha t^2 + \dots + \alpha^{k-1} \frac{t^k}{(k-1)!} + \dots \\ 0 & e^{\alpha t} \end{bmatrix} = \begin{bmatrix} e^{\alpha t} & te^{\alpha t} \\ 0 & e^{\alpha t} \end{bmatrix}$$

## Case 4: Matrix $A$ has multiple eigenvalues (contd.)

Hence:

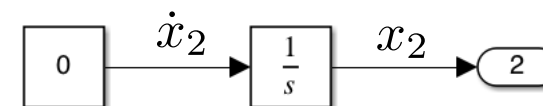
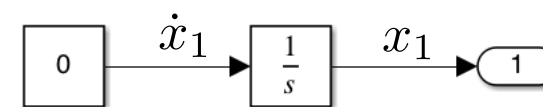
### - Concerning **Example 1**:

- stability  $\longleftrightarrow \alpha = 0$
- asymptotic stability  $\longleftrightarrow \alpha < 0$
- instability  $\longleftrightarrow \alpha > 0$

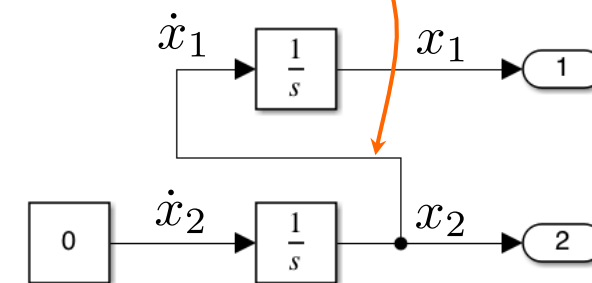
### - Concerning **Example 2**:

- asymptotic stability  $\longleftrightarrow \alpha < 0$
- instability  $\longleftrightarrow \alpha \geq 0$

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \quad \alpha = 0$$



$$A = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} \quad \alpha = 0$$



## Case 4: Matrix $A$ has multiple eigenvalues (contd.)



[Livescripts in MS Teams](#): see Part 3:  
STABILITY\_STATE\_SPACE



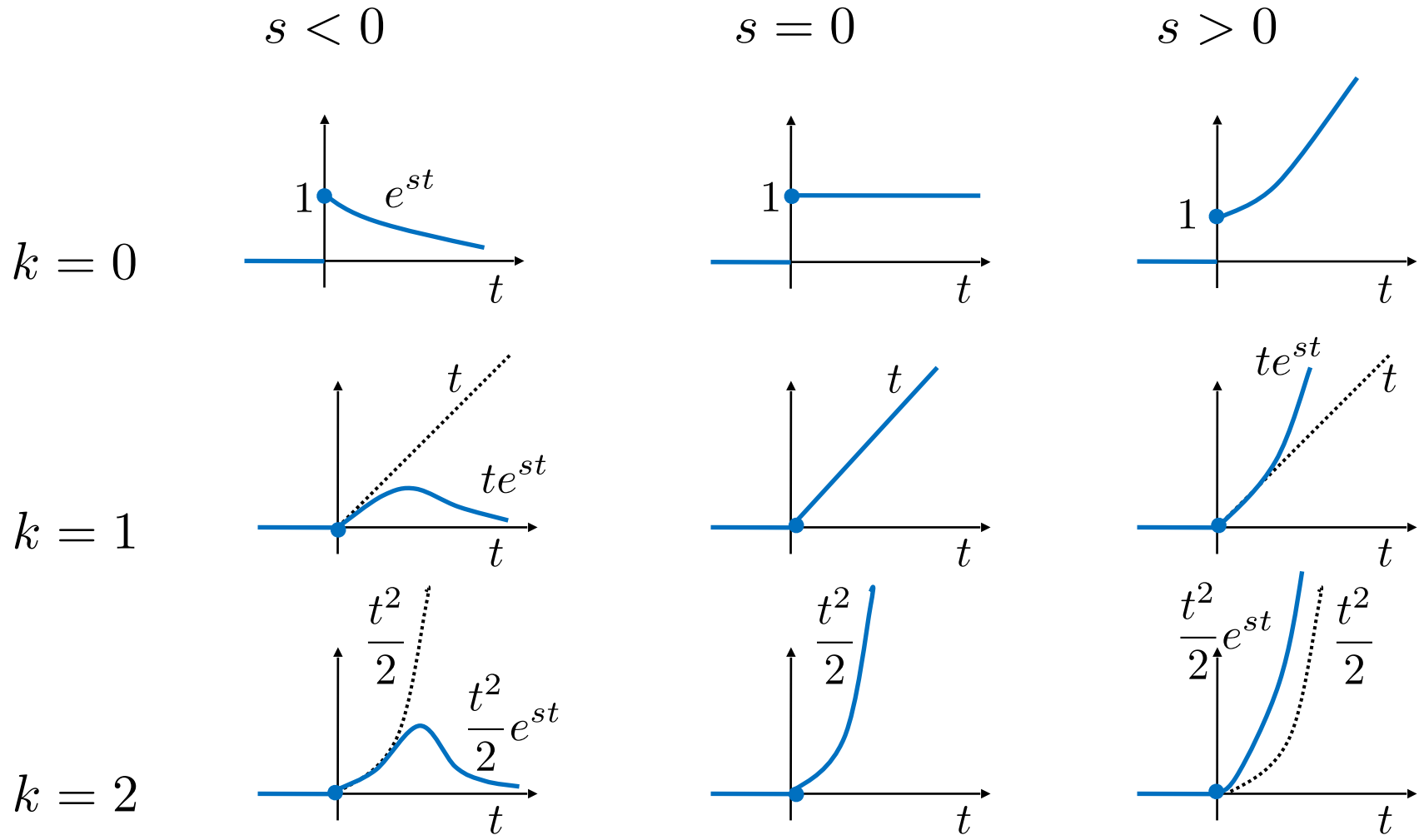
**In general**, consider a matrix  $A$  such that:

- $A$  has eigenvalues with multiplicity  $\nu > 1$
- matrix  $A$  **cannot** be transformed into equivalent diagonal form

  $e^{At}$  contains terms of the form  $t^k e^{s_i t}$ ,  $k = 1, 2, \dots, \nu - 1$

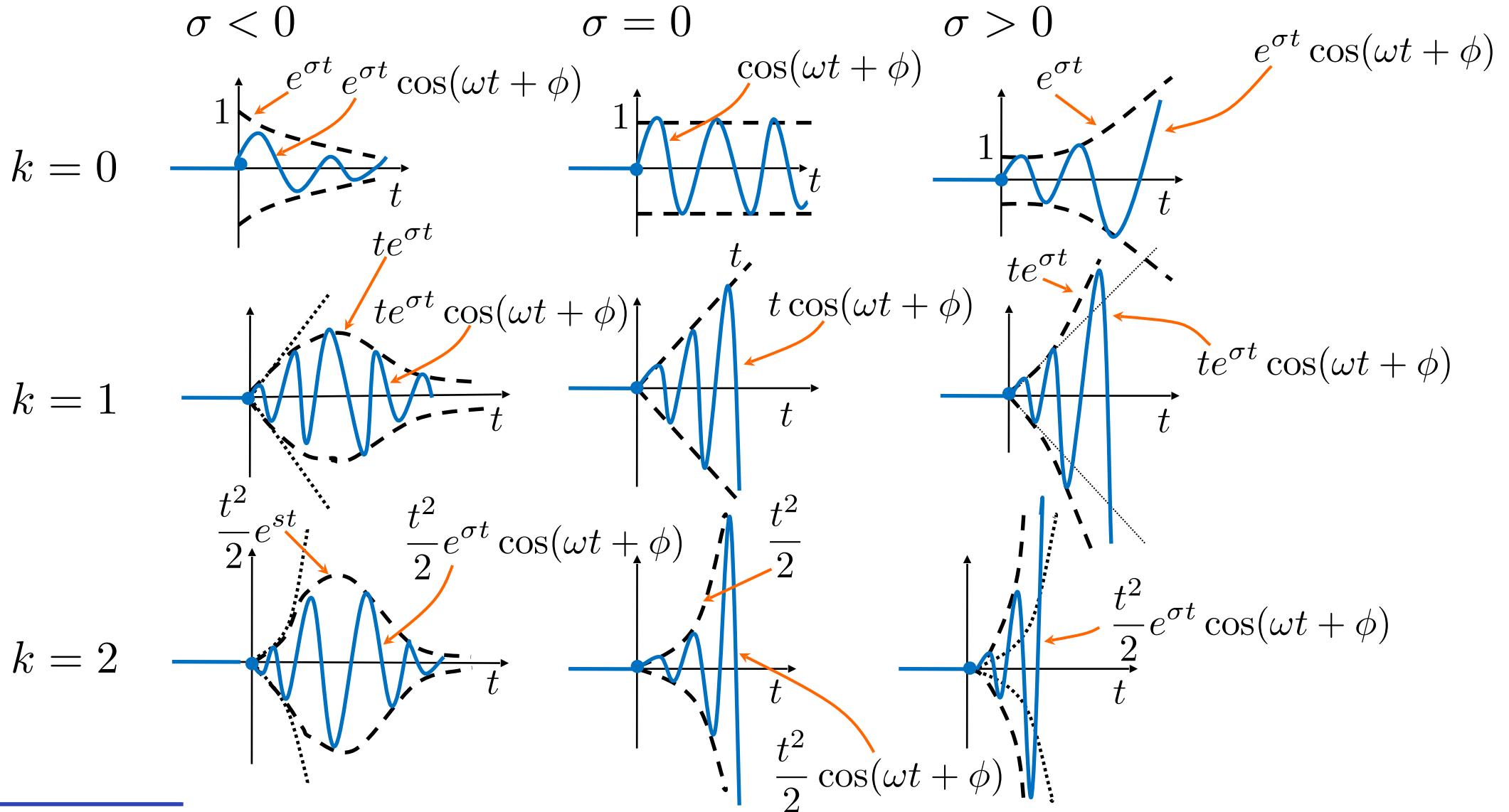
# Qualitative Analysis of the generic term $t^k e^{st}$ , $s \in \mathbb{C}$ , $k \in \mathbb{Z}^+$

• **Case 1:**  $s \in \mathbb{R}$



# Qualitative Analysis of the generic term $t^k e^{st}$ , $s \in \mathbb{C}$ , $k \in \mathbb{Z}^+$

- **Case 2:**  $s \in \mathbb{C}$ ,  $s_1 = \sigma + j\omega$ ,  $s_2 = \sigma - j\omega$



$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad \begin{array}{l} s_1, s_2, \dots, s_n \in \mathbb{C} \\ \text{eigenvalues of } A \end{array}$$

## Result 1:

$$\operatorname{Re}(s_i) < 0, i = 1, \dots, n \quad \longleftrightarrow \quad \text{Asymptotic Stability}$$

## Result 2:

$$\exists i \in \{1, \dots, n\} \text{ such that } \operatorname{Re}(s_i) > 0 \quad \longrightarrow \quad \text{Instability}$$

## Result 3:

$$\begin{cases} \operatorname{Re}(s_i) \leq 0, i = 1, \dots, n \\ \exists \tilde{i} \in \{1, \dots, n\} \text{ such that } \operatorname{Re}(s_{\tilde{i}}) = 0 \end{cases} \quad \longrightarrow \quad \text{No Asymptotic Stability}$$

- If multiplicity of **all**  $s_{\tilde{i}}$  such that  $\operatorname{Re}(s_{\tilde{i}}) = 0$  is equal to 1

 Stability (Non-Asymptotic, also called Marginal)

- If  $\exists \tilde{i} \in \{1, \dots, n\}$  such that  $\operatorname{Re}(s_{\tilde{i}}) = 0$  with multiplicity  $> 1$


 Instability or Stability (non asymptotic anyway) 

- So far, the stability analysis has been carried out by **evaluating the eigenvalues** of matrix  $A$  and their **location** in the complex plane
- Other criteria can be devised **not requiring** the calculation of the eigenvalues of matrix  $A$  but based on the:
  - analysis of the **elements** of matrix  $A$
  - analysis of the **characteristic polynomial** of matrix  $A$  :

$$\varphi_A(s) = \det(sI - A) = \varphi_0 s^n + \varphi_1 s^{n-1} + \cdots + \varphi_{n-1} s + \varphi_n$$

- **Criterion 1:**

If matrix  $A$  is triangular:

  $a_{ii} < 0, i = 1, \dots, n$   Asymptotic Stability

- **Criterion 2:**

Letting  $\text{tr}(A) := \sum_{i=1}^n a_{ii}$  :

Asymptotic Stability   $\text{tr}(A) < 0$

  $\text{tr}(A) > 0$   Instability

- **Criterion 3:**

Asymptotic Stability   $\det(A) \neq 0$

- **Criterion 4 (valid for second-order systems  $n = 2$ ):**

Asymptotic Stability  $\longleftrightarrow$   $\text{Re}(s_i) < 0, i = 1, 2$   $\longleftrightarrow$   $\begin{cases} \{\varphi_0, \varphi_1, \varphi_2\} \\ \neq 0 \\ \text{same sign} \end{cases}$

- **Criterion 5:**

Asymptotic Stability  $\longleftrightarrow$   $\text{Re}(s_i) < 0, i = 1, 2$   $\longrightarrow$   $\begin{cases} \{\varphi_0, \dots, \varphi_n\} \\ \neq 0 \\ \text{same sign} \end{cases}$

$$\varphi(s) = s^3 + 3s^2 + 2s$$

Not Asymptotically Stable

$$\varphi(s) = s^3 + 2s + 5$$

Not Asymptotically Stable

$$\varphi(s) = s^3 + 5s^2 - 2s + 4$$

Not Asymptotically Stable

$$\varphi(s) = s^3 + 5s^2 + 2s + 4$$

???

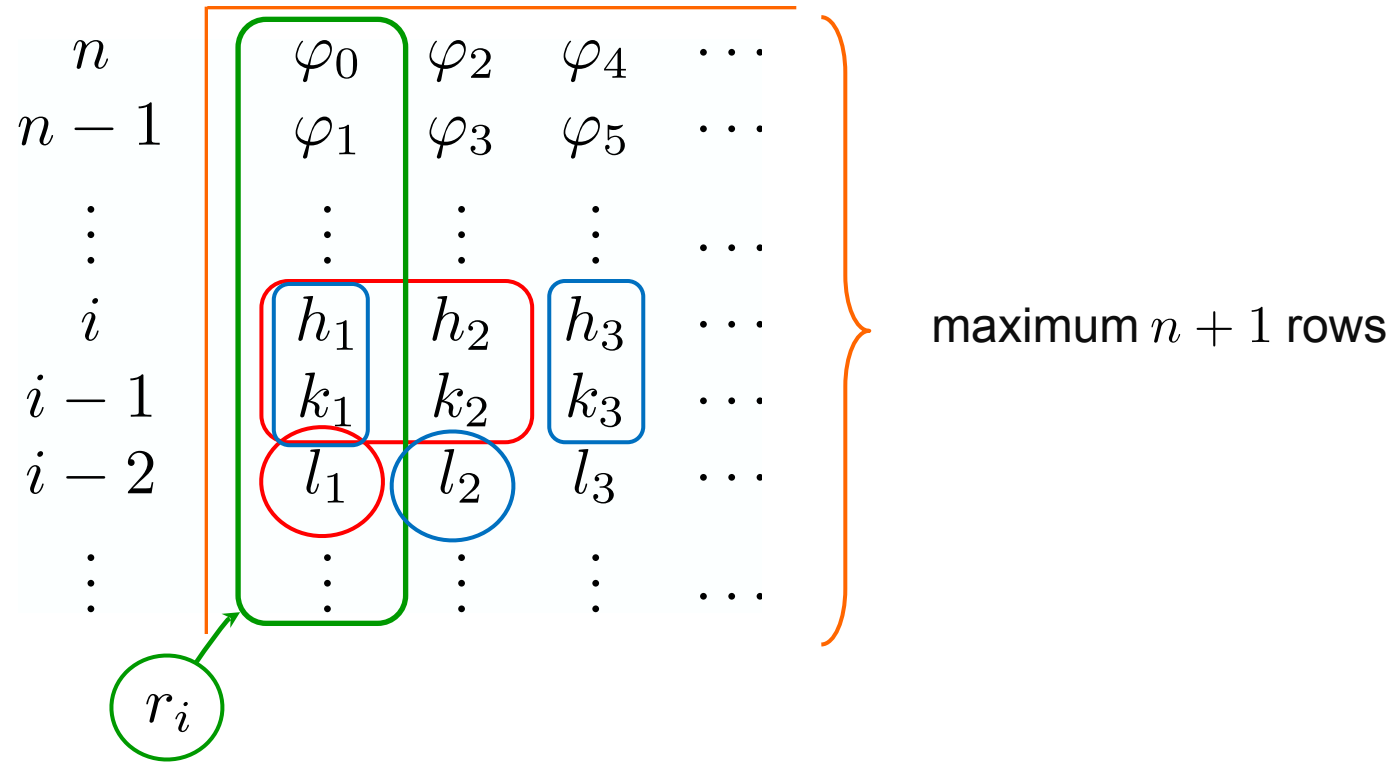


In the general case, a **necessary and sufficient** condition is needed

# Routh Table

For a given system matrix  $A$  the characteristic polynomial  $\varphi_A(s)$  is:




$$\varphi_A(s) = \varphi_0 s^n + \varphi_1 s^{n-1} + \dots + \varphi_{n-1} s + \varphi_n$$



$$l_1 = -\frac{1}{k_1} \det \begin{bmatrix} h_1 & h_2 \\ k_1 & k_2 \end{bmatrix} \quad l_2 = -\frac{1}{k_1} \det \begin{bmatrix} h_1 & h_3 \\ k_1 & k_3 \end{bmatrix} \quad \dots \quad l_j = -\frac{1}{k_1} \det \begin{bmatrix} h_1 & h_{j+1} \\ k_1 & k_{j+1} \end{bmatrix}$$

Consider a given system matrix  $A$  and its characteristic polynomial  $\varphi_A(s)$ :

$$\varphi_A(s) = \varphi_0 s^n + \varphi_1 s^{n-1} + \dots + \varphi_{n-1} s + \varphi_n$$

- If the Routh Table cannot be completed  no asymptotic stability
- If the Routh Table can be completed ( $n + 1$  rows)
  - The number of roots of  $\varphi_A(s)$  with **positive real part** is equal to the **number of sign-changes** in the first column
  -  • The number of roots of  $\varphi_A(s)$  with **negative real part** is equal to the **number of sign-permanencies** in the first column  $r_i$
  - **No sign-changes** in the first column  $r_i$   **asymptotic stability**

# Example 1

$$\varphi_A(s) = s^3 + 5s^2 + 2s + 4$$

3	1	2	0
2	5	4	0
1	$\alpha$	0	0
0	$\beta$	0	0

$r_0$

$$\alpha = -\frac{1}{5} \det \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} = \frac{6}{5}$$

$$\beta = -\frac{1}{\alpha} \det \begin{bmatrix} 5 & 4 \\ \alpha & 0 \end{bmatrix} = 4$$



No sign-changes in  $r_0$



asymptotic stability

## Example 2

$$\varphi_A(s) = s^4 + 2s^3 + 4s^2 + 9s + 6$$

4	1	4	6
3	2	9	0
2	$\alpha$	6	0
1	$\beta$	0	0
0	6	0	0

$r_0$

$$\alpha = -\frac{1}{2} \det \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix} = -\frac{1}{2}$$

$$\beta = -\frac{1}{\alpha} \det \begin{bmatrix} 2 & 9 \\ \alpha & 6 \end{bmatrix} = 33$$



Two sign-changes in  $r_0$



instability

## Example 3 (important)



$$\varphi_A(s) = s^4 + 6s^3 + 11s^2 + 6s + K$$

[Livescripts in MS Teams](#): see Part 3:  
STABILITY\_ROUTH\_HURWITZ\_APPLICATIONS

4	1	11	$K$
3	6	6	0
2	10	$K$	0
1	$\alpha$	0	0
0	$K$	0	0

$r_0$

$$\alpha = -\frac{1}{10} \det \begin{bmatrix} 6 & 6 \\ 10 & K \end{bmatrix} = \frac{3}{5}(10 - K)$$

If  $K > 0$  and  $K < 10$

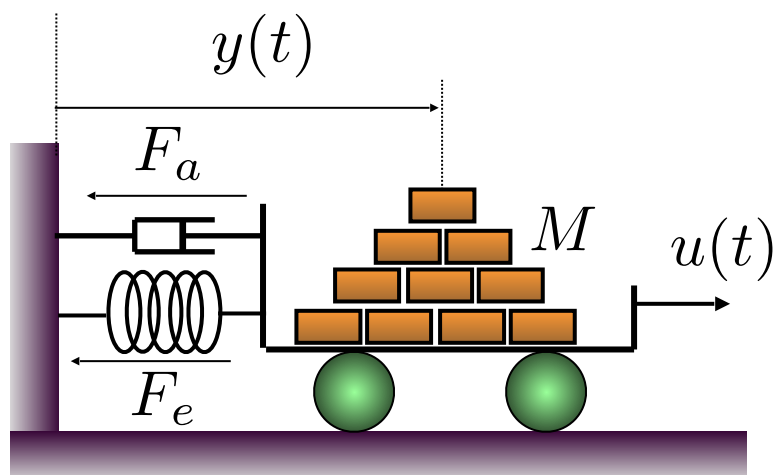


No sign-changes in  $r_0$



asymptotic stability

## Example 4



$$\begin{cases} x_1 := y \\ x_2 := \dot{y} \end{cases}, \quad x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{M}x_1 - \frac{h}{M}x_2 + \frac{1}{M}u \\ y = x_1 \end{cases}$$

$$\downarrow \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{h}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \rightarrow \quad A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{h}{M} \end{bmatrix}$$

$$\downarrow \varphi_A(s) = \det(sI - A) = \det \begin{bmatrix} s & -1 \\ \frac{k}{M} & s + \frac{h}{M} \end{bmatrix} = s^2 + \frac{h}{M}s + \frac{k}{M}$$

where  $M > 0$ ,  $k \geq 0$ ,  $h \geq 0$

- **Case 1 (both elastic and friction forces active):**  $k > 0, h > 0$

↳  $\text{Re}(s_i) < 0, i = 1, 2$     → asymptotic stability

- **Case 2 (only elastic force active):**  $k > 0, h = 0$

↳  $\varphi_A(s) = s^2 + \frac{k}{M}$     →  $s_{1,2} = \pm j \sqrt{\frac{k}{M}}$

↳  $\text{Re}(s_i) = 0, i = 1, 2$

↳ stability (non asymptotic)

- **Case 3 (only friction force active):**  $k = 0, h > 0$

$$\hookrightarrow \varphi_A(s) = s^2 + \frac{h}{M}s \quad \longrightarrow \quad s_1 = 0; \quad s_2 = -\frac{h}{M}$$

$$\hookrightarrow \operatorname{Re}(s_1) = 0, \operatorname{Re}(s_2) < 0$$

$\hookrightarrow$  stability (non asymptotic)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{h}{M}x_2 \end{cases} \longrightarrow x_2(t) = x_{20}e^{-\frac{h}{M}t} \longrightarrow 0$$

$$\hookrightarrow \dot{x}_1 = x_{20}e^{-\frac{h}{M}t}$$

$$\hookrightarrow x_1(t) = x_{10} + x_{20} \int_0^t e^{-\frac{h}{M}\tau} d\tau$$

$$= x_{10} - x_{20} \frac{M}{h} \left( e^{-\frac{h}{M}t} - 1 \right) \longrightarrow x_{10} + x_{20} \frac{M}{h}$$

- **Case 4 (no elastic nor friction force acting):**  $k = 0, h = 0$

$$\downarrow \varphi_A(s) = s^2 \quad \rightarrow \quad s_1 = 0; s_2 = 0$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \rightarrow \quad \text{unstable}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 0 \end{cases} \quad \rightarrow \quad x_2(t) = x_{20}, \forall t$$

$$x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \quad \downarrow \quad x_1(t) = x_{10} + x_{20}t \xrightarrow[t \rightarrow \infty]{} \infty$$

# Stability of Equilibrium of a Nonlinear System via the Linearised System

Recall from Part 2:

$$\begin{cases} \dot{x} = f(x, u) \\ y = g(x, u) \end{cases} \implies 0 = f(x, \bar{u}) \implies \bar{x} \text{ (equilibrium state)}$$

$$\delta \dot{x} \simeq f_x(\bar{x}, \bar{u})\delta x + f_u(\bar{x}, \bar{u})\delta u$$



$$\underbrace{\hspace{2cm}}_A \quad \underbrace{\hspace{2cm}}_B$$



$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} ; \begin{matrix} x \in \mathbb{R}^n \\ u \in \mathbb{R}^m \\ y \in \mathbb{R}^p \end{matrix}$$

$$\delta y \simeq g_x(\bar{x}, \bar{u})\delta x + g_u(\bar{x}, \bar{u})\delta u$$

$$\underbrace{\hspace{2cm}}_C \quad \underbrace{\hspace{2cm}}_D$$

Stability of equilibrium is a **local property** (see Definitions in Part 2, slide 3).




We can take advantage of the linearised system on the specific equilibrium state to analyse its stability

# Stability of Equilibrium of a Nonlinear System via the Linearised System (contd.)

## Main Result:

Denoting by  $s_i, i = 1, \dots, n$  the eigenvalues of matrix  $A$  :

(A)  $\text{Re}(s_i) < 0, i = 1, \dots, n$    $\bar{x}$  asymptotically stable equilibrium state

(B)  $\exists i$  such that  $\text{Re}(s_i) > 0$    $\bar{x}$  unstable equilibrium state

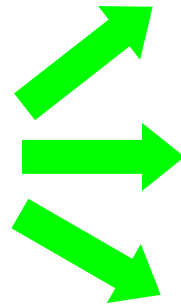
# Stability of Equilibrium of a Nonlinear System via the Linearised System (contd.)

## Critical Case:

$$\operatorname{Re}(s_i) \leq 0, i = 1, \dots, n$$

$$\exists i \text{ such that } \operatorname{Re}(s_i) = 0$$

?



$\bar{x}$  asymptotically stable equilibrium state

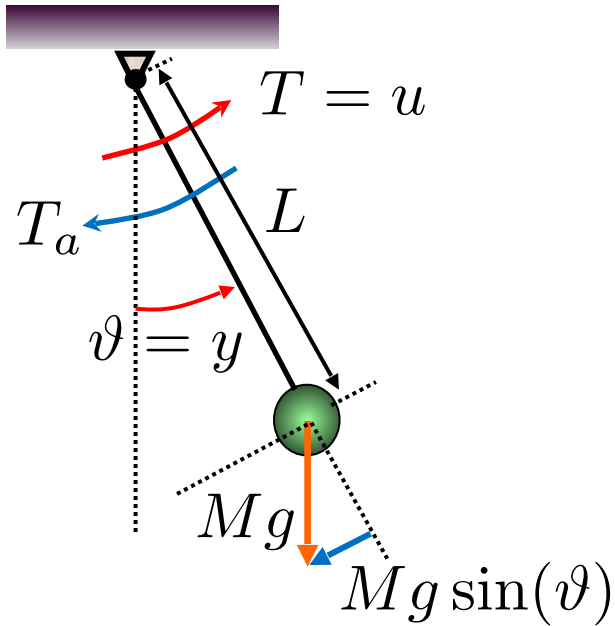
$\bar{x}$  stable equilibrium state

$\bar{x}$  unstable equilibrium state

In this case, **no decision** can be made on the stability of the equilibrium state based on the linearised system

# Example

Recall from Part 2, slides 45, 46:



$$\begin{cases} x_1 := \vartheta \\ x_2 := \dot{\vartheta} \end{cases}, \quad x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{cases} \dot{x}_1 = \dot{\vartheta} = x_2 \\ \dot{x}_2 = -\frac{MgL}{J} \sin(x_1) - \frac{h}{J} x_2 + \frac{1}{J} u \\ y = x_1 \end{cases}$$

$$u(t) = \bar{u} = 0, \forall t \quad \longrightarrow \quad \begin{cases} 0 = x_2 \\ 0 = -\frac{MgL}{J} \sin(x_1) - \frac{h}{J} x_2 \end{cases}$$



$$\begin{cases} \bar{x}_2 = 0 \\ \sin(\bar{x}_1) = \frac{1}{MgL} \bar{u} \end{cases} \quad \longrightarrow \quad \begin{cases} \bar{x}_2 = 0 \\ \bar{x}_1 = k\pi, \quad \forall k \in \mathbb{Z} \end{cases}$$

We pick the two "physical" solutions:



$$\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \tilde{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

The state matrices of the linearised system on the two equilibrium states are:

$$f_x(\bar{x}, \bar{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{\bar{x}, \bar{u}} = \begin{bmatrix} 0 & 1 \\ -\frac{MgL}{J} \cos(x_1) & -\frac{h}{J} \end{bmatrix}_{\bar{x}, \bar{u}} = \begin{bmatrix} 0 & 1 \\ -\frac{MgL}{J} & -\frac{h}{J} \end{bmatrix} = \bar{A}$$


 $\det(sI - \bar{A}) = s^2 + \frac{h}{J}s + \frac{MgL}{J}$ 

 $\bar{x}$  asymptotically stable equilibrium state

$$f_x(\tilde{x}, \bar{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{\tilde{x}, \bar{u}} = \begin{bmatrix} 0 & 1 \\ -\frac{MgL}{J} \cos(x_1) & -\frac{h}{J} \end{bmatrix}_{\tilde{x}, \bar{u}} = \begin{bmatrix} 0 & 1 \\ +\frac{MgL}{J} & -\frac{h}{J} \end{bmatrix} = \tilde{A}$$


 $\det(sI - \tilde{A}) = s^2 + \frac{h}{J}s - \frac{MgL}{J}$ 

 $\tilde{x}$  unstable equilibrium state