

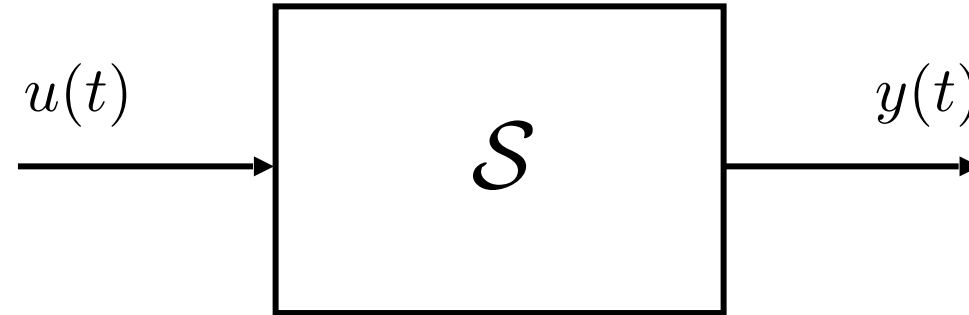


**034IN - FONDAMENTI DI AUTOMATICA -
FUNDAMENTALS OF AUTOMATIC
CONTROL
A.Y. 2025-2026
Part IV: Transfer Function**

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Up to now we have considered an **internal dynamic description** based on **state variables**:



$$u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} \in \mathbb{R}^m \quad x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n \quad y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix} \in \mathbb{R}^p$$

Input State Output

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad \begin{matrix} x \in \mathbb{R}^n \\ u \in \mathbb{R}^m \\ y \in \mathbb{R}^p \end{matrix} \quad (A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m})$$

Recall from Part 2:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$


$$\begin{array}{l} \downarrow \\ \mathcal{L}\{\dot{x}(t)\} = \mathcal{L}\{Ax + Bu\} \end{array} \quad \longrightarrow \quad sX(s) - x(0) = AX(s) + BU(s)$$

$$\begin{array}{l} \downarrow \\ (sI - A)X(s) = x(0) + BU(s) \end{array}$$

$$\begin{array}{l} \downarrow \\ X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s) \end{array}$$

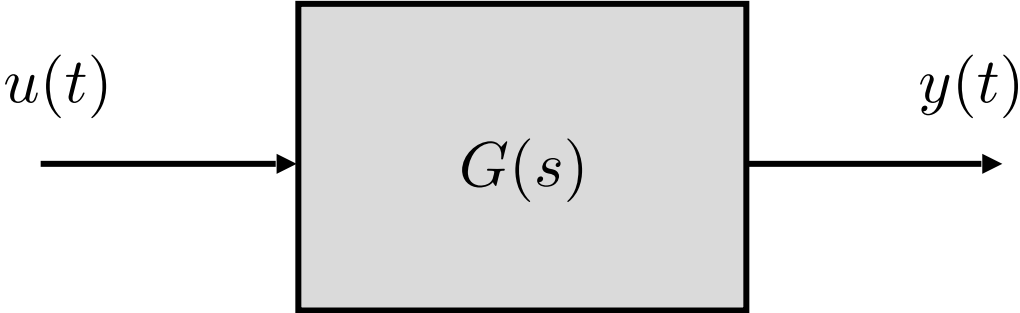
$$\begin{array}{l} \downarrow \\ Y(s) = C(sI - A)^{-1}x(0) + [C(sI - A)^{-1}B + D]U(s) \end{array}$$

Setting $x(0) = 0$:




$$Y(s) = \underbrace{\left[\overset{p \times n}{C} \overset{n \times n}{(sI - A)^{-1}} \overset{n \times m}{B} + \overset{p \times m}{D} \right]}_{p \times m} U(s)$$

$G(s)$
Transfer Function



Transfer Function – General Case

$$G(s) = \begin{bmatrix} G_{11}(s) & \cdots & G_{1m}(s) \\ \vdots & & \vdots \\ G_{i1}(s) & \cdots & G_{im}(s) \\ \vdots & & \vdots \\ G_{p1}(s) & \cdots & G_{pm}(s) \end{bmatrix}$$


$$Y_i(s) = \sum_{j=1}^m G_{ij}(s)U_j(s)$$

$$= G_{i1}(s)U_1(s) + G_{i2}(s)U_2(s) + \cdots$$

Hence:

$$\begin{cases} x(0) = 0 \\ u_k(t) = 0, \quad k \neq j \end{cases} \quad \longrightarrow \quad G_{ij}(s) = \frac{Y_i(s)}{U_j(s)}$$

Transfer Function – SISO Case

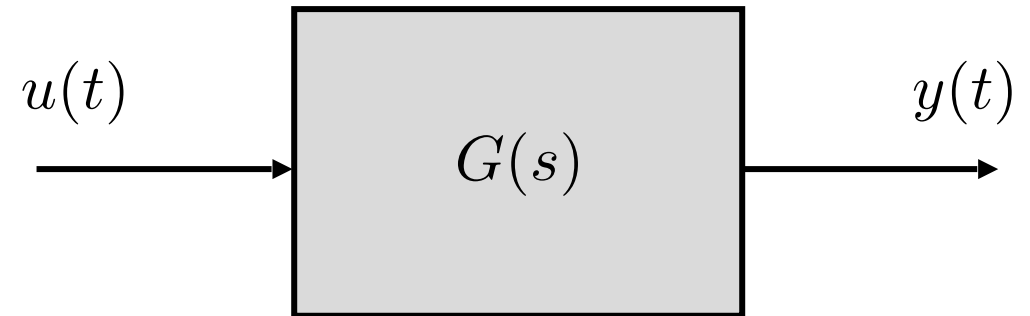
In the Single-Input Single-Output (SISO) case:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad u(t), y(t) \in \mathbb{R} \quad x(0) = 0$$

↳

$$Y(s) = \underbrace{\left[\overbrace{C}^{1 \times n} \overbrace{(sI - A)^{-1} B}^{n \times n} + \overbrace{D}^{1 \times 1} \right]}_{1 \times 1} U(s)$$

$G(s)$
Scalar Transfer Function



Transfer Function from Equivalent State Equations

Recall from Slide 2-40:

$$\begin{cases} \dot{x} = Ax + Bu & x \in \mathbb{R}^n \\ y = Cx + Du & u \in \mathbb{R}^m \\ & y \in \mathbb{R}^p \end{cases}$$

Letting:

$$x = T\hat{x}, \quad T \in \mathbb{R}^{n \times n}, \det(T) \neq 0 \quad \longrightarrow \quad \hat{x} = T^{-1}x$$

$$\begin{cases} \dot{\hat{x}} = T^{-1}(Ax + Bu) = \underbrace{T^{-1}AT}_{\hat{A}}\hat{x} + T^{-1}Bu \\ y = \underbrace{CT}_{\hat{C}}\hat{x} + Du \end{cases}$$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad \longleftrightarrow \quad \begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u \\ y = \hat{C}\hat{x} + Du \end{cases}$$


$$\begin{aligned}\hat{G}(s) &= \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D} \\ &= C \left[T^{-1} (sI - TAT^{-1})^{-1} T \right] B + D \\ &= C \left[T^{-1} (sTT^{-1} - TAT^{-1})^{-1} T \right] B + D \\ &= C \left[T^{-1} (T(sI - A)T^{-1})^{-1} T \right] B + D \\ &= C \left[T^{-1}T (sI - A)^{-1} T^{-1}T \right] B + D \\ &= C \left[(sI - A)^{-1} \right] B + D \\ &= G(s) \end{aligned}$$




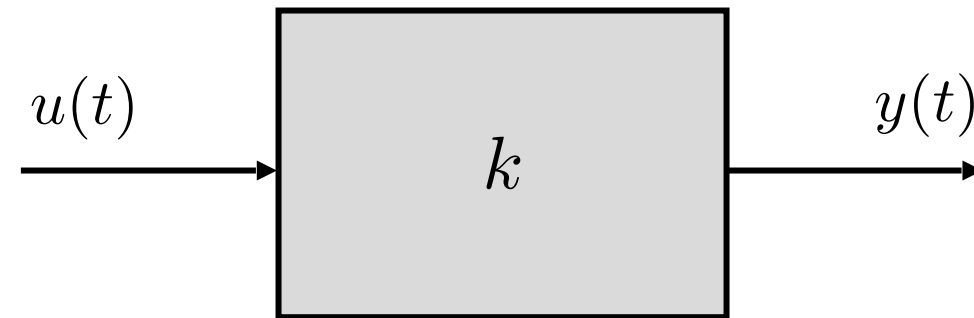
The transfer function is **unique** and is **not related** to the specific internal state representation

Example 1: Constant Gain

$$y(t) = ku(t)$$



$$Y(s) = kU(s)$$

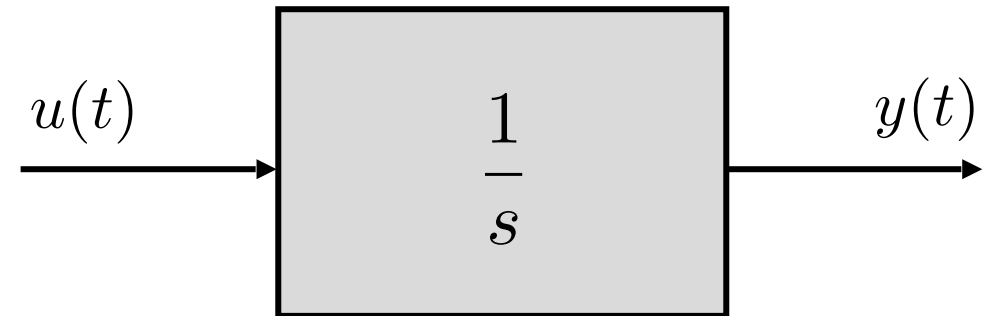

$$G(s) = k$$



Example 2: Integrator

$$\begin{cases} \dot{x} = u \\ y = x \end{cases} \quad A = 0; \quad B = 1; \quad C = 1; \quad D = 0$$

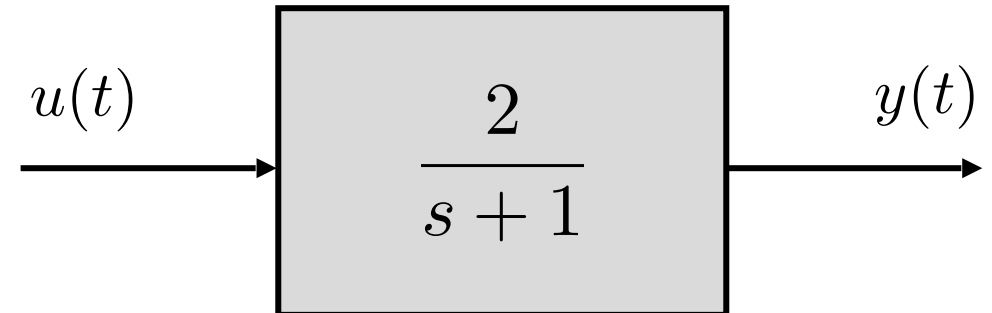
 $G(s) = C \left[(sI - A)^{-1} \right] B + D = \frac{1}{s}$



Example 3: First-Order System


$$\begin{cases} \dot{x} = -x + u \\ y = 2x \end{cases} \quad A = -1; \quad B = 1; \quad C = 2; \quad D = 0$$

↳ $G(s) = C \left[(sI - A)^{-1} \right] B + D = \frac{2}{s + 1}$



Example 4: Double Integrator

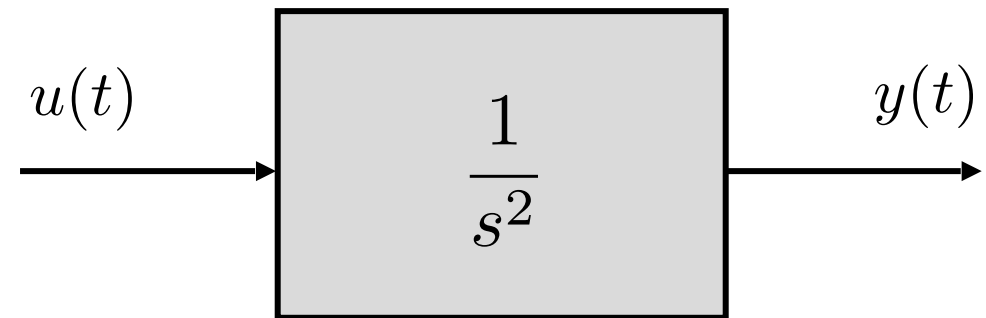
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \\ y = x_1 \end{cases} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad C = [1 \quad 0]; \quad D = 0$$


$$G(s) = C \left[(sI - A)^{-1} \right] B + D$$

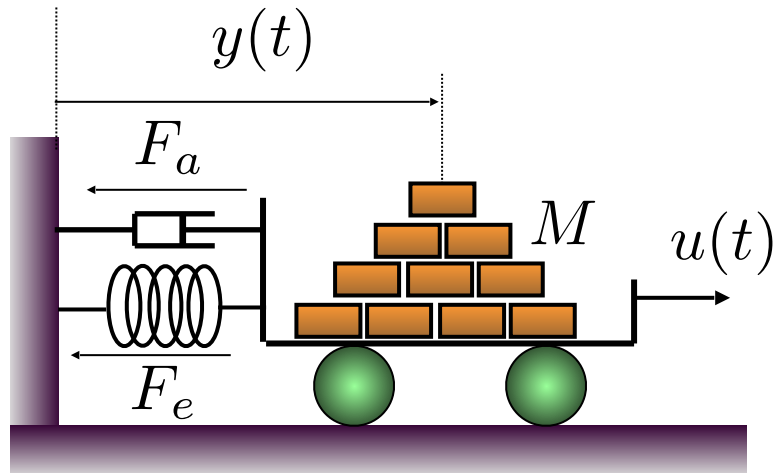
$$= [1 \quad 0] \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= [1 \quad 0] \frac{1}{s^2} \begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2}$$



Example 5: Mechanical System

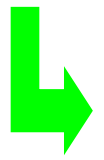


[Livescripts in MS Teams](#): see Part 4

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{M}x_1 - \frac{h}{M}x_2 + \frac{1}{M}u \\ y = x_1 \end{cases}$$



$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{h}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u \\ y = [1 \ 0] x \end{cases}$$



$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ \frac{k}{M} & s + \frac{h}{M} \end{bmatrix}^{-1} = \frac{1}{s^2 + \frac{h}{M}s + \frac{k}{M}} \begin{bmatrix} s + \frac{h}{M} & 1 \\ -\frac{k}{M} & s \end{bmatrix}$$



$$G(s) = C \left[(sI - A)^{-1} \right] B + D$$

$$= \frac{1}{s^2 + \frac{h}{M}s + \frac{k}{M}} [1 \ 0] \begin{bmatrix} s + \frac{h}{M} & 1 \\ -\frac{k}{M} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} = \frac{1}{Ms^2 + hs + k}$$

From the definition:

$$G(s) = C \left[(sI - A)^{-1} \right] B + D$$

We have:

$$(sI - A)^{-1} = \begin{bmatrix} s - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & s - a_{22} & & \vdots \\ \vdots & & \ddots & \\ -a_{n1} & \cdots & & s - a_{nn} \end{bmatrix}^{-1}$$
$$= \frac{1}{\det(sI - A)} K(s)$$

Let's analyse the polynomial $\det(sI - A)$ and the matrix $K(s)$:

- The polynomial $\varphi(s) = \det(sI - A)$ has degree equal to n
- The matrix $K(s)$ is given by:

$$K(s) = \text{adj}(sI - A) = [k_{ij}(s), i, j = 1, \dots, n]$$

where $k_{ij}(s), i, j = 1, \dots, n$ are polynomials of degree $< n$
and $\text{adj}(sI - A)$ is the transpose of the cofactor matrix

Hence:

$$C(sI - A)^{-1}B = \frac{1}{\det(sI - A)} \underbrace{CK(s)B}_{M(s)} = \frac{M(s)}{\varphi(s)}$$

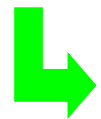
where $M(s)$ is a polynomial of degree $< n$

$$\begin{aligned}\text{Therefore: } G(s) &= C (sI - A)^{-1} B + D = \frac{M(s)}{\varphi(s)} + D \\ &= \frac{M(s) + D\varphi(s)}{\varphi(s)} = \frac{N(s)}{\varphi(s)}\end{aligned}$$

where, in the **absence of common factors** among $N(s)$ and $\varphi(s)$:

- if $D = 0$ (**strictly proper system**) the polynomial $N(s)$ has degree $< n$
- otherwise (**non strictly proper system**) the polynomial $N(s)$ has degree n

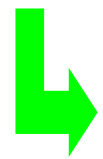
In the presence of **common factors** among $N(s)$ and $\varphi(s)$ they "cancel out" and the degrees of these polynomials **decrease** accordingly.



Hidden internal dynamics that is not represented by $G(s)$

"Hidden-Dynamics": Example 1

$$\begin{cases} \dot{x} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = [1 \ 1] x \end{cases} \quad n = 2$$



$$G(s) = [1 \ 1] \begin{bmatrix} s - 1 & 0 \\ -1 & s + 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= [1 \ 1] \frac{1}{(s - 1)(s + 1)} \begin{bmatrix} s + 1 & 0 \\ 1 & s - 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{\cancel{(s - 1)}}{\cancel{(s - 1)}(s + 1)} = \frac{1}{s + 1}$$





the denominator of $G(s)$ has degree $1 < 2 = n$

"Hidden-Dynamics": Example 1 (contd.)

Moreover:

$$\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = x_1 - x_2 + u \\ y = x_1 + x_2 \end{cases} \quad \text{with} \quad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since $x_1(0) = 0$ from $\dot{x}_1 = x_1$  $x_1(t) = 0, \forall t \geq 0$


$$\begin{cases} \dot{x}_2 = \cancel{x_1} - x_2 + u = -x_2 + u \\ y = \cancel{x_1} + x_2 = x_2 \end{cases}$$

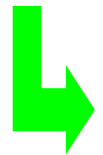


The dynamics of $x_1(t)$ **does not show up** (it is “**hidden**”) and this is consistent with the fact that

$$G(s) = \frac{1}{s + 1}$$

"Hidden-Dynamics": Example 2

$$\begin{cases} \dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y = [0 \ 1] x \end{cases} \quad n = 2$$



$$G(s) = [0 \ 1] \begin{bmatrix} s - 1 & -1 \\ 0 & s + 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= [0 \ 1] \frac{1}{(s - 1)(s + 1)} \begin{bmatrix} s + 1 & 1 \\ 0 & s - 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{\cancel{(s - 1)}}{\cancel{(s - 1)}(s + 1)} = \frac{1}{s + 1}$$



the denominator of $G(s)$ has degree $1 < 2 = n$

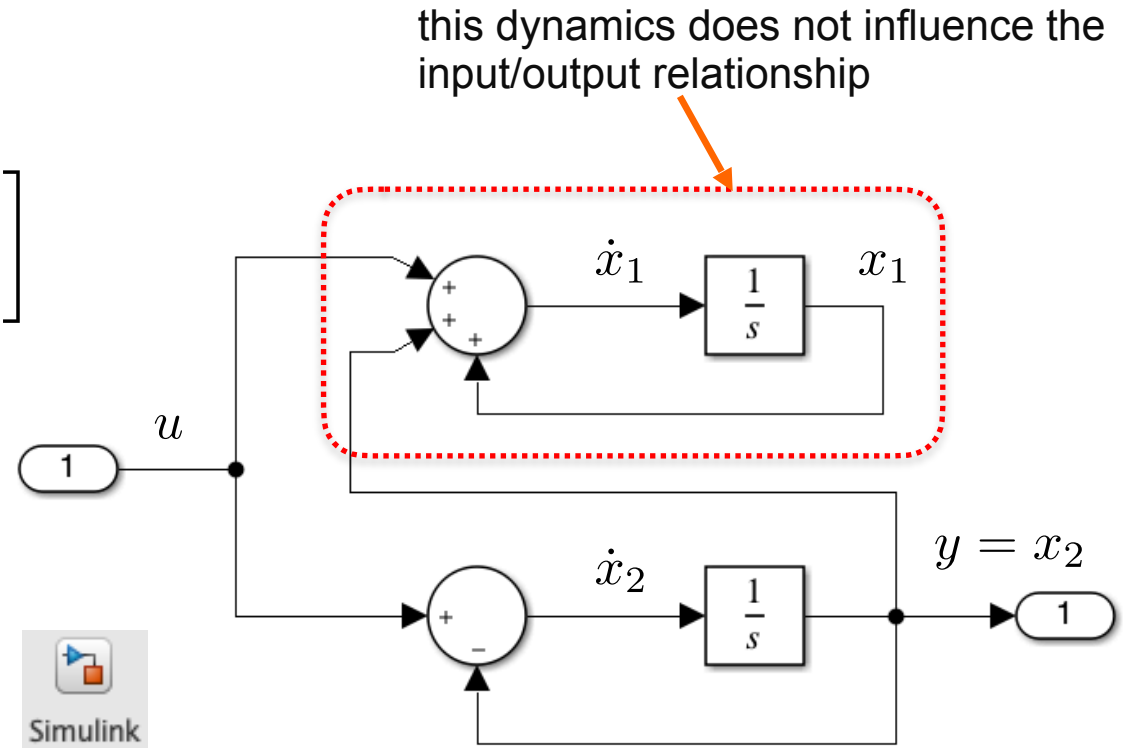
"Hidden-Dynamics": Example 2 (contd.)

Moreover:

$$\begin{cases} \dot{x}_1 = x_1 + x_2 + u \\ \dot{x}_2 = -x_2 + u \\ y = x_2 \end{cases}$$

with $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$G(s) = \frac{1}{s + 1}$$

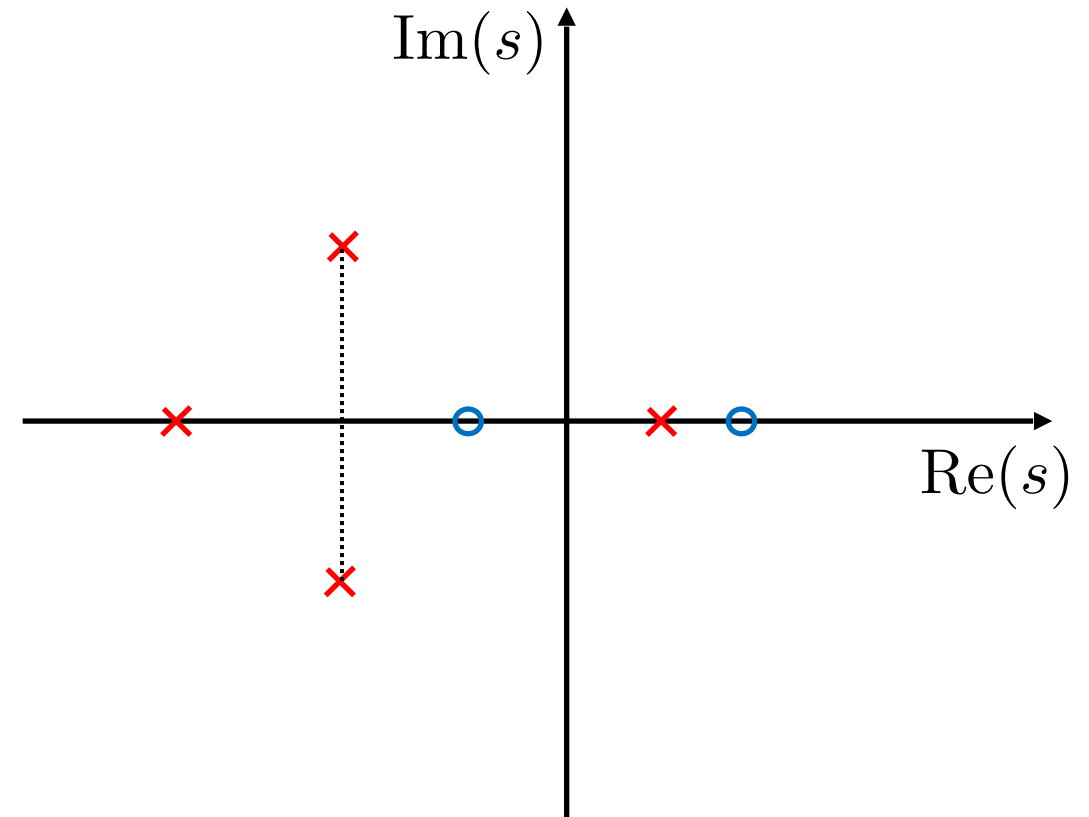


The dynamics of $x_2(t)$ **is not influenced** by the time-evolution of $x_1(t)$ (which is “**hidden**”) and this is consistent with the fact that


$$G(s) = \frac{1}{s + 1}$$

$$G(s) = \frac{N(s)}{\varphi(s)}$$

- Poles: roots of polynomial $\varphi(s)$ ×
- Zeros: roots of polynomial $N(s)$ ○



- Poles of $G(s)$ are also eigenvalues of system's matrix A
- An eigenvalue of system's matrix A might not be a pole of $G(s)$ in case of **common factors** among polynomials $N(s)$ and $\varphi(s)$ (as shown by the examples)
- Stability depends on the poles of $G(s)$:

Asymptotic Stability  $\text{Re}(\text{poles}) < 0$
in the absence of
common factors

- The number of zeros is less or equal to the number of poles

(A) Example with real zeros and poles:

$$\begin{aligned} G(s) &= \frac{4s^2 + 12s}{s^4 + 3s^3 + 2s^2} \\ &= 4 \frac{s(s + 3)}{s^2(s + 1)(s + 2)} \\ &= \frac{1}{s} \frac{4 \cdot 3}{1 \cdot 2} \frac{\left(1 + \frac{s}{3}\right)}{\left(1 + s\right) \left(1 + \frac{s}{2}\right)} \end{aligned}$$

Parameters:

$$\begin{aligned} \beta_2 &= 4, \beta_1 = 12, \beta_0 = 0 \\ \alpha_4 &= 1, \alpha_3 = 3, \alpha_2 = 2, \alpha_1 = \alpha_0 = 0 \end{aligned}$$

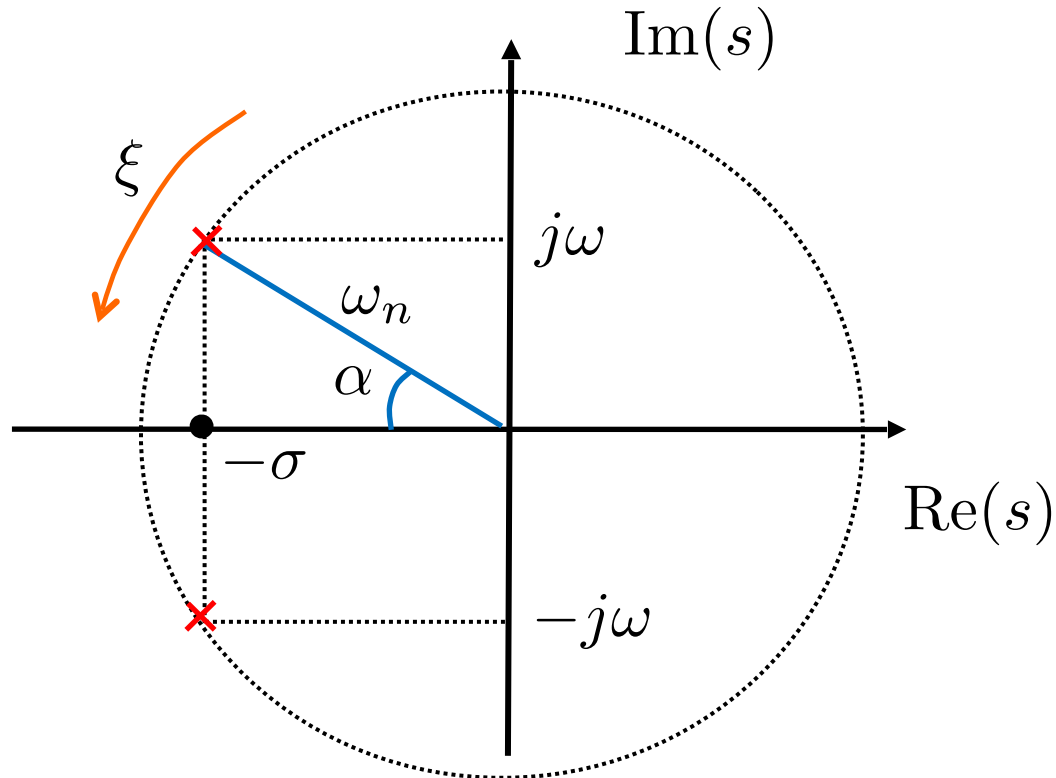
Parameters:

$$\begin{aligned} \rho &= 4, z_1 = -3 \\ p_1 &= 0, p_2 = -1, p_3 = -2 \end{aligned}$$

Parameters:

$$\begin{aligned} \mu &= 6, T_1 = 1/3 \\ \tau_1 &= 1, \tau_2 = 1/2 \end{aligned}$$

(B) The parametric form in the presence of **complex** zeros/poles is **different**:



$$\omega_n^2 = \sigma^2 + \omega^2$$

$$\omega_n \xi = \sigma$$

$$\omega_n \sqrt{1 - \xi^2} = \omega$$

Parameters:

ω_n natural angular frequency:

$\xi = \cos(\alpha)$ damping ratio

Moreover, we have:

$$\begin{aligned} G(s) &= \frac{\rho}{(s + \sigma + j\omega)(s + \sigma - j\omega)} = \frac{\rho}{(s + \sigma)^2 + \omega^2} \\ &= \frac{\rho}{s^2 + \underbrace{2\sigma s}_{2\xi\omega_n} + \underbrace{\sigma^2 + \omega^2}_{\omega_n^2}} = \boxed{\frac{\rho}{s^2 + 2\xi\omega_n s + \omega_n^2}} \end{aligned}$$



$$G(s) = \frac{\rho/\omega_n^2}{1 + \frac{2\xi}{\omega_n}s + \frac{1}{\omega_n^2}s^2} = \frac{\mu}{1 + \frac{2\xi}{\omega_n}s + \frac{1}{\omega_n^2}s^2}$$

where: $\mu := \frac{\rho}{\omega_n^2}$

(B) Example with real & complex-conjugate zeros/poles:

$$G(s) = \frac{5s + 15}{s^3 + 6s^2 + 10s + 8}$$

$$= 5 \frac{s + 3}{(s + 4)[(s + 1 - j)(s + 1 + j)]}$$

$$= \frac{5 \cdot 3}{4 \cdot 2} \frac{\left(1 + \frac{s}{3}\right)}{\left(1 + \frac{s}{4}\right) \left(1 + s + \frac{s^2}{2}\right)}$$

Parameters:

$$\beta_1 = 5, \beta_0 = 15$$

$$\alpha_3 = 1, \alpha_2 = 6, \alpha_1 = 10, \alpha_0 = 8$$

Parameters:

$$\rho = 5, z_1 = -3$$

$$p_1 = -4, p_2 = -1 + j, p_3 = -1 - j$$

Parameters:

$$\mu = \frac{15}{8}, T_1 = \frac{1}{3}$$

$$\tau_1 = \frac{1}{4}, \xi = \frac{1}{\sqrt{2}}, \omega_n = \sqrt{2}$$

In general:

(a) Parameterisation using the coefficients of the polynomials $N(s)$, $\varphi(s)$

$$G(s) = \frac{\beta_m s^m + \beta_{n-1} s^{m-1} + \cdots + \beta_1 s + \beta_0}{\alpha_n s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0}$$

The parameters are the coefficients α_i , $i = 0, \dots, n$; β_j , $j = 0, \dots, m$

(b) Parameterisation using poles and zeros of $G(s)$

$$G(s) = \varrho \frac{(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)}$$

The parameters are the transfer constant ϱ , the zeros (with opposite sign) $z_i, i = 1, \dots, m$ and the poles (with opposite sign) $p_j, j = 1, \dots, n$

(c) Parameterisation using **time-constants** when

$$z_i \in \mathbb{R}, i = 1, \dots, m; p_j \in \mathbb{R}, j = 1, \dots, n$$

$$\begin{aligned}
 G(s) &= \varrho \frac{(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)} \\
 &= \varrho \frac{1}{s^g} \frac{\prod_{i=1}^m z_i \left(\frac{s}{z_i} + 1 \right)}{\prod_{i=1}^n p_i \left(\frac{s}{p_i} + 1 \right)} = \frac{1}{s^g} \varrho \frac{\prod_{i=1}^m z_i}{\prod_{i=1}^n p_i} \frac{\prod_{i=1}^m \left(1 + \frac{s}{z_i} \right)}{\prod_{i=1}^n \left(1 + \frac{s}{p_i} \right)} = \mu \frac{1}{s^g} \frac{\prod_{i=1}^m (1 + T_i s)}{\prod_{i=1}^n (1 + \tau_i s)}
 \end{aligned}$$

Hence, in the case $z_i \in \mathbb{R}, i = 1, \dots, m; p_j \in \mathbb{R}, j = 1, \dots, n$:

$$G(s) = \mu \frac{1}{s^g} \frac{\prod_{i=1}^m (1 + T_i s)}{\prod_{i=1}^n (1 + \tau_i s)}$$

The parameters are:

- **Gain:** $\mu := \varrho \frac{\prod_{i=1}^m z_i}{\prod_{i=1}^n p_i}$
- **Time constants:** $\frac{1}{z_i} = T_i; \frac{1}{p_i} = \tau_i$
- **Type of the system:** $g = (\text{number of poles in } s = 0) - (\text{number of zeros in } s = 0)$

Hence, the parameterisation using **time-constants** when some/all zero(s)/pole(s) are complex, that is, when

$$z_l \in \mathbb{R}, l = 1, \dots, m_R; z_h \in \mathbb{C}, h = 1, \dots, m_C;$$

$$p_j \in \mathbb{R}, i = 1, \dots, n_R; p_k \in \mathbb{C}, k = 1, \dots, n_C$$

takes on the form:

$$G(s) = \frac{1}{s^g} \varrho \frac{\prod_l z_l}{\prod_i p_i} \frac{\prod_h \alpha_{nh}^2}{\prod_k \omega_{nk}^2} \frac{\prod_l \left(1 + \frac{s}{z_l}\right)}{\prod_i \left(1 + \frac{s}{p_i}\right)} \frac{\prod_h \left(1 + \frac{2\zeta_h}{\alpha_{nh}} s + \frac{1}{\alpha_{nh}^2} s^2\right)}{\prod_k \left(1 + \frac{2\xi_k}{\omega_{nk}} s + \frac{1}{\omega_{nk}^2} s^2\right)}$$

$$= \mu \frac{1}{s^g} \frac{\prod_l \left(1 + \frac{s}{z_l}\right)}{\prod_i \left(1 + \frac{s}{p_i}\right)} \frac{\prod_h \left(1 + \frac{2\zeta_h}{\alpha_{nh}} s + \frac{1}{\alpha_{nh}^2} s^2\right)}{\prod_k \left(1 + \frac{2\xi_k}{\omega_{nk}} s + \frac{1}{\omega_{nk}^2} s^2\right)}$$

Hence, in the case $z_i \in \mathbb{R}, l = 1, \dots, m_R; z_i \in \mathbb{C}, h = 1, \dots, m_C;$
 $p_j \in \mathbb{R}, i = 1, \dots, n_R; p_j \in \mathbb{C}, k = 1, \dots, n_C$

$$G(s) = \mu \frac{1}{s^g} \frac{\prod_l \left(1 + \frac{s}{z_l}\right) \prod_h \left(1 + \frac{2\zeta_h}{\alpha_{nh}} s + \frac{1}{\alpha_{nh}^2} s^2\right)}{\prod_i \left(1 + \frac{s}{p_i}\right) \prod_k \left(1 + \frac{2\xi_k}{\omega_{nk}} s + \frac{1}{\omega_{nk}^2} s^2\right)}$$

and the parameters are:

• **Gain:**

$$\mu := \varrho \frac{\prod_l z_l}{\prod_i p_i} \frac{\prod_h \alpha_{nh}^2}{\prod_k \omega_{nk}^2}$$

• **Time constants:** $\frac{1}{z_i} = T_i; \frac{1}{p_i} = \tau_i$

• **Damping ratios:** ζ_h, ξ_k


• **Natural angular frequencies:** $\alpha_{nh}^2, \omega_{nk}^2$


• **Type of the system:**

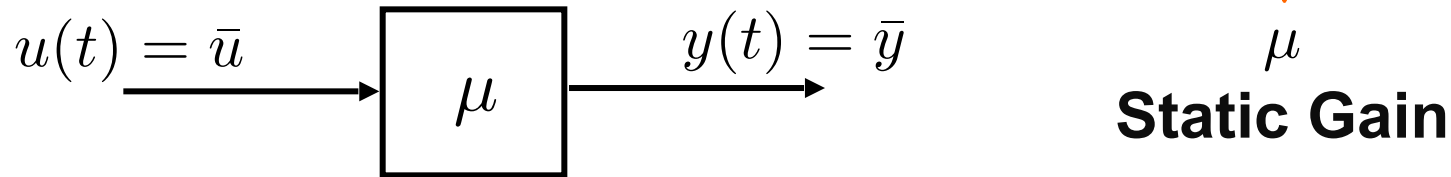
$$g = (\text{number of poles in } s = 0) - (\text{number of zeros in } s = 0)$$

Recall from Part 2:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad \begin{array}{l} x \in \mathbb{R}^n \\ u \in \mathbb{R}^m \\ y \in \mathbb{R}^p \end{array} \quad u(t) = \bar{u}, t \geq 0$$

$\det(A) \neq 0$  $\bar{x} = -A^{-1}B\bar{u}$ one and only one equilibrium state

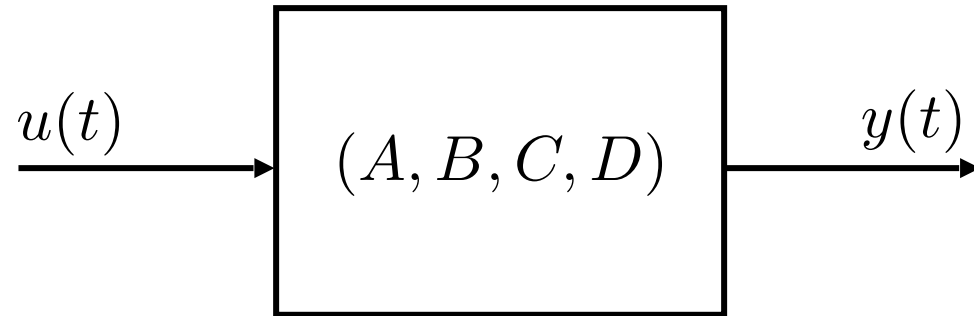
 $\bar{y} = C\bar{x} + D\bar{u} = -CA^{-1}B\bar{u} + D\bar{u} = \underbrace{(-CA^{-1}B + D)}_{\mu} \bar{u}$



Hence, for **systems of 0-type**, that is when $g = 0$:

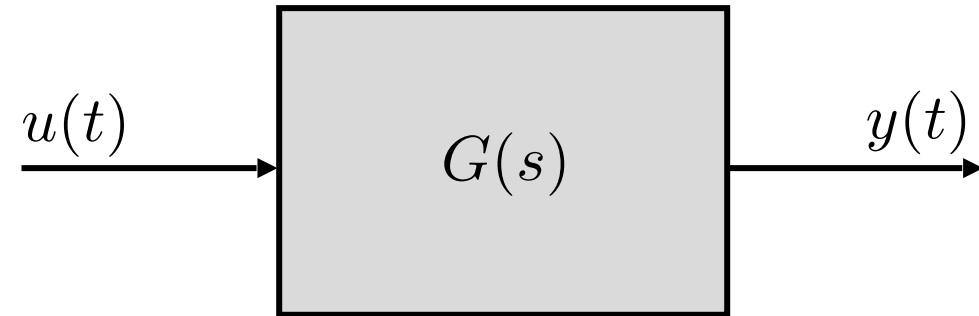
$$\mu = (-CA^{-1}B + D) \bar{u} = G(s)|_{s=0} = G(0)$$

Internal – State Equations



$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

External – Transfer Functions



$$G(s) = [C(sI - A)^{-1}B + D]$$
$$Y(s) = G(s)U(s) \quad \text{with} \quad x(0) = 0$$

? Realisation Problem

Recall from Slide 2-40:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad \begin{array}{l} x \in \mathbb{R}^n \\ u \in \mathbb{R}^m \\ y \in \mathbb{R}^p \end{array}$$

[Livescripts in MS Teams](#): see Part 4

Letting: $x = T\hat{x}$, $T \in \mathbb{R}^{n \times n}$, $\det(T) \neq 0$ \longrightarrow $\hat{x} = T^{-1}x$

$$\begin{cases} \dot{\hat{x}} = T^{-1}(Ax + Bu) = \underbrace{T^{-1}AT}_{\hat{A}}\hat{x} + T^{-1}Bu \\ y = \underbrace{CT}_{\hat{C}}\hat{x} + Du \end{cases}$$

$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \longleftrightarrow \begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u \\ y = \hat{C}\hat{x} + Du \end{cases} \longrightarrow$ **Infinite** number of **equivalent** state equations representing the **same** dynamic system

Realisation Problem for SISO Systems: Control Form

Consider the generic **strictly proper** transfer function

$$G(s) = \frac{\beta_m s^m + \beta_{m-1} s^{m-1} + \dots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1} s^{n-1} + \dots + \alpha_1 s + \alpha_0} \quad \text{with } m < n$$

Then:

$$\left\{ \begin{array}{l} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \dots & -\alpha_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \\ \\ y = [\beta_0 \quad \beta_1 \quad \dots \quad \beta_m \quad 0 \quad \dots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \end{array} \right.$$

Realisation Problem for SISO Systems: Observation Form

Consider the generic **strictly proper** transfer function

$$G(s) = \frac{\beta_m s^m + \beta_{m-1} s^{m-1} + \dots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1} s^{n-1} + \dots + \alpha_1 s + \alpha_0} \quad \text{with } m < n$$

Then:

$$\left\{ \begin{array}{l} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & -\alpha_0 \\ 1 & 0 & \dots & 0 & -\alpha_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & -\alpha_{n-2} \\ 0 & \dots & 0 & 1 & -\alpha_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} u \\ \\ y = [0 \quad \dots \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \end{array} \right.$$

Realisation Problem for SISO Systems: Remarks

- Control and Observation Forms are perfectly equivalent in our context
- Many other solutions to the realisation problem are available
- The solution of the realisation problem in the MIMO case is more complicated and is **not dealt with in this introductory course**
- When the transfer function is **not strictly proper**, that is $m = n$, we have:

$$G(s) = \frac{\beta_n s^n + \beta_{n-1} s^{n-1} + \dots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1} s^{n-1} + \dots + \alpha_1 s + \alpha_0} = \beta_n + \tilde{G}(s)$$

where

$$\tilde{G}(s) = \frac{\gamma_{n-1} s^{n-1} + \gamma_{n-2} s^{n-2} + \dots + \gamma_1 s + \gamma_0}{s^n + \alpha_{n-1} s^{n-1} + \dots + \alpha_1 s + \alpha_0}$$

where $\gamma_0, \gamma_1, \dots, \gamma_{n-2}, \gamma_{n-1}$ are obtained via polynomial division. Then, the scalar D of the state equations is given by

$$D = \beta_n$$

and the realisation of $\tilde{G}(s)$ can be obtained as in the Control and Observation Forms