034IN - FONDAMENTI DI AUTOMATICA - FUNDAMENTALS OF AUTOMATIC CONTROL
A.Y. 2025-2026
Part VIII: Sinusoidal-Response Analysis & Frequency Response

Gianfranco Fenu, Thomas Parisini

Department of Engineering and Architecture

Transient and Regime (Steady-State) Response

- Consider an **asymptotically stable** LTI continuous-time that is **fully described** (that is, no common factors) by the transfer function:

$$G(s) = \frac{N(s)}{\varphi(s)}$$

- Impose a **generic input** $u(t)$ that admits a rational Laplace transform. Hence (initial conditions are zero):

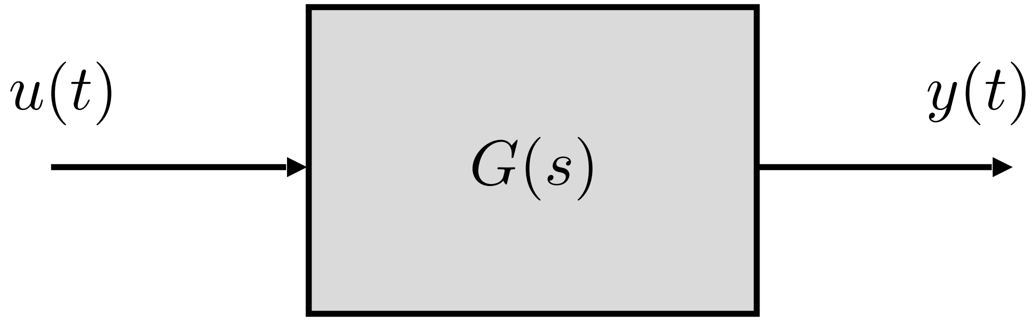
$$Y(s) = \frac{N(s)}{\varphi(s)} \cdot U(s)$$

Example

Consider:

$$G(s) = \frac{2(s+1)}{(s+2)(s+10)}$$

$$u(t) = 4\,t \cdot 1(t)$$



Hence, carrying out the Laplace transform:

$$Y(s) = \frac{2(s+1)}{(s+2)(s+10)} \cdot \frac{4}{s^2}$$

Therefore:

- **Transient Response**:

$$Y_{\text{trans.}}(s) = -\frac{1}{4} \cdot \frac{1}{s+2} + \frac{9}{100} \cdot \frac{1}{s+10}$$

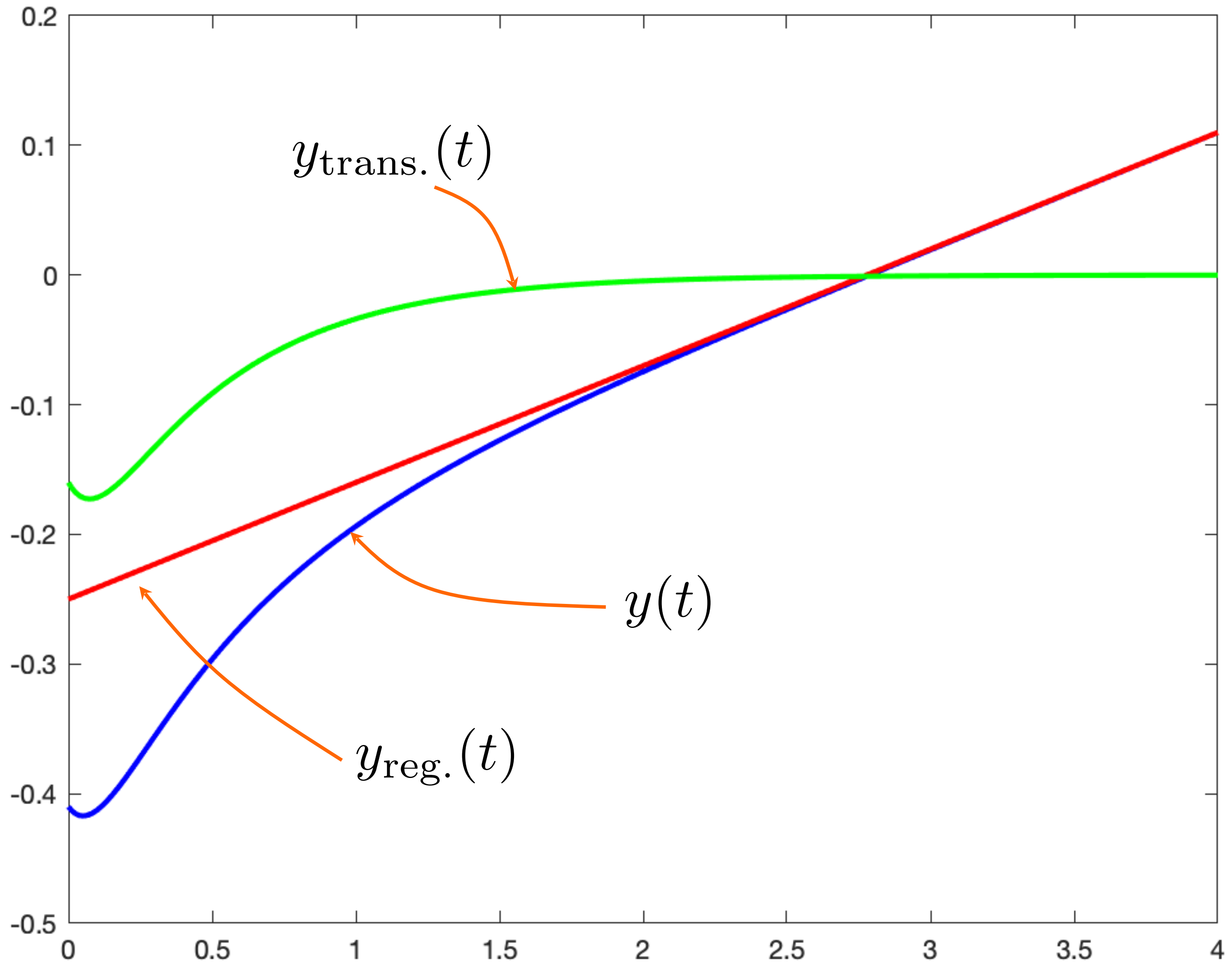
$$\Rightarrow \quad y_{\text{trans.}}(t) = \left[-\frac{1}{4}\, e^{-2t} + \frac{9}{100}\, e^{-10t}\right] \cdot 1(t)$$

- **Regime (Steady-State) Response**:

$$Y_{\text{reg.}}(s) = \frac{4}{25} \cdot \frac{1}{s} + \frac{2}{5} \cdot \frac{1}{s^2}$$

$$\Rightarrow \quad y_{\text{reg.}}(t) = \left[\frac{4}{25} + \frac{2}{5}\, t\right] \cdot 1(t)$$

Example (contd.)



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- Carrying out the partial fraction expansion and the inverse Laplace transform:

$$y(t) = \mathcal{L}^{-1}\left[\sum_{i=1}^{n_{a.s.}} \frac{P_i}{s-p_i}\right] + \dots$$

$$+\mathcal{L}^{-1}\left[\sum_{j=1}^{n_{m.s.}} \frac{Q_j}{s-p_j}\right] + \dots$$

$$+\mathcal{L}^{-1}\left[\sum_{k=1}^{n_{unst.}} \frac{R_k}{s-p_k}\right]$$

Contribution due to asymptotically stable poles of $Y(s)$

Contribution due to marginally stable poles of $Y(s)$

Contribution due to unstable poles of $Y(s)$

Transient and Regime (Steady-State) Response (contd.)

Therefore:

- **Transient Response**:

$$y_{\text{trans.}}(t) = \mathcal{L}^{-1}\left[\sum_{i=1}^{n_{a.s.}} \frac{P_i}{s-p_i}\right] \xrightarrow[t\to\infty]{} 0$$

The transient response contribution **vanishes asymptotically**

- **Regime (Steady-State) Response**:

$$y_{\text{reg.}}(t) = \mathcal{L}^{-1}\left[\sum_{j=1}^{n_{m.s.}} \frac{Q_j}{s-p_j}\right] + \mathcal{L}^{-1}\left[\sum_{k=1}^{n_{unst.}} \frac{R_k}{s-p_k}\right]$$

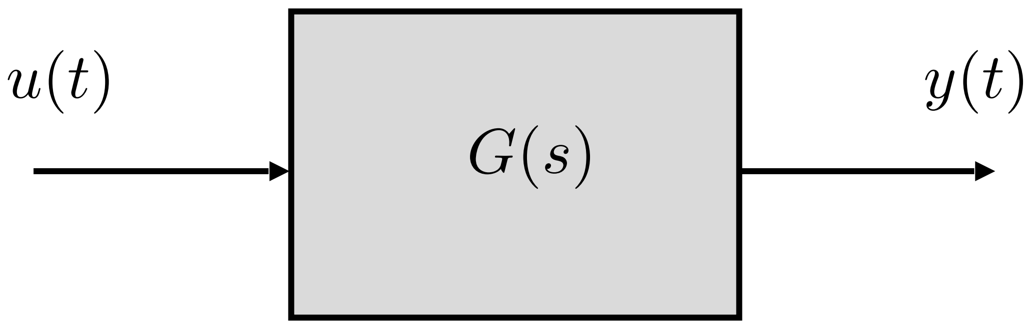
The regime response contribution does not vanish over time and is **only** caused by the **non-vanishing** input

Sinusoidal Response

- Consider an **asymptotically stable** LTI continuous-time that is **fully described** (that is, no common factors) by the transfer function $G(s)$
- Moreover, consider:

$$u(t) = A\sin(\omega t)1(t)$$

$$U(s) = \frac{A\omega}{s^2+\omega^2}$$



- For simplicity, assume that all poles of $G(s)$ are **real** and **distinct**. From the asymptotic stability assumption, it follows that:

$$G(s) = \mu \frac{\prod_{i=1}^{m}(1+sT_i)}{\prod_{i=1}^{n}(1+s\tau_i)}$$

Sinusoidal Response (contd.)

Hence:

$$Y(s) = G(s)\frac{A\omega}{s^2+\omega^2} = G(s)\frac{A\omega}{(s-j\omega)(s+j\omega)}$$

$$\Rightarrow \quad Y(s) = \underbrace{\frac{\alpha_1}{1+s\tau_1} + \frac{\alpha_2}{1+s\tau_2} + \cdots + \frac{\alpha_n}{1+s\tau_n}}_{Y_1(s)} + \underbrace{\frac{\beta s+\gamma}{s^2+\omega^2}}_{Y_2(s)}$$

$\mathcal{L}^{-1}$ $\Rightarrow \quad y(t) = y_1(t) + y_2(t)$

$t \to \infty$: $y_1(t) \to 0$

$\Rightarrow$ for $t \to \infty$ (in practice $t > t_s$) $\Rightarrow$ $y(t) \simeq y_2(t)$

Regime ("steady-state") Sinusoidal Response

$$Y(s) = G(s)\frac{A\omega}{s^2+\omega^2} = G(s)\frac{A\omega}{(s-j\omega)(s+j\omega)}$$

$$\mathcal{L}^{-1} \Rightarrow \quad y(t) \simeq y_2(t) = k_1\, e^{j\omega t} + k_2 e^{-j\omega t}\,, \; t \geq 0$$

$$k_1 = G(s)\frac{A\omega}{s+j\omega}\bigg|_{s=j\omega} = \frac{A}{2j}G(j\omega)$$

$$k_2 = G(s)\frac{A\omega}{s-j\omega}\bigg|_{s=-j\omega} = -\frac{A}{2j}G(-j\omega)$$

Regime ("steady-state") Sinusoidal Response (contd.)

It is easy to show that: $G(s^*) = G^*(s)$ ➡ $G(j\omega) = G^*(-j\omega)$

Since $G(j\omega) = |G(j\omega)|e^{j\varphi(\omega)}$ with $\varphi(\omega) := \arg G(j\omega)$

↳ $G(-j\omega) = G^*(j\omega) = |G(j\omega)|e^{-j\varphi(\omega)}$

Hence

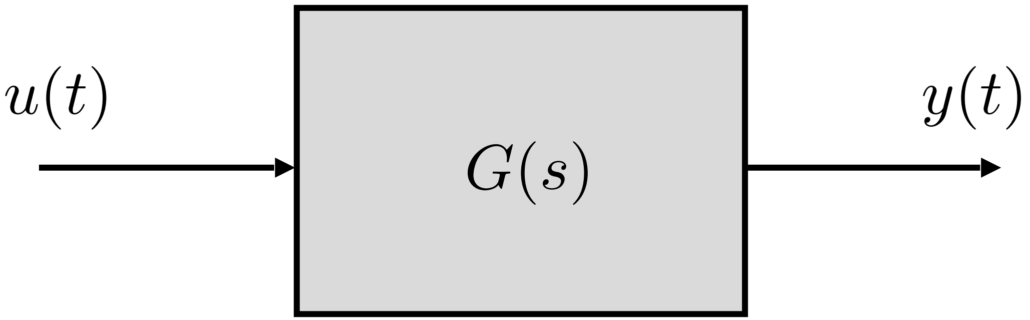
$$
\begin{aligned}
y(t) \simeq y_2(t) \quad &= A|G(j\omega)| \, \frac{e^{j\omega t} \cdot e^{j\varphi(\omega)} - e^{-j\omega t} \cdot e^{-j\varphi(\omega)}}{2j} \\
&= A|G(j\omega)| \, \frac{e^{j(\omega t + \varphi(\omega))} - e^{-j(\omega t + \varphi(\omega))}}{2j} \\
&= A \cdot |G(j\omega)| \, \sin[\omega t + \varphi(\omega)], \;\; t \geq 0
\end{aligned}
$$

Frequency Response

- Consider an **asymptotically stable** LTI continuous-time that is **fully described** (that is, no common factors) by the transfer function $G(s)$
- Moreover, consider:

$$u(t) = A\sin(\omega t)1(t)$$

$$U(s) = \frac{A\omega}{s^2 + \omega^2}$$



The regime sinusoidal response (approximately equal to $y(t)$ for $t > t_s$) is:

$$y_{\text{reg.}}(t) = B\sin(\omega t + \varphi)$$

same angular frequency of the input sinusoid

where: $B = |G(j\omega)| \cdot A$

$$\varphi = \arg G(j\omega)$$

The **frequency response** is defined as: $G(j\omega)\,,\ \omega \geq 0$

Example 1

$$G(s) = \frac{1}{1+s}\,; \qquad u(t) = 10\sin(2t)1(t)$$

$$A = 10;\ \omega = 2\text{rad}/s$$

$$G(j2) = \frac{1}{1+2j} = \frac{1-2j}{(1-2j)(1+2j)} = \frac{1-2j}{5} = \frac{1}{5} - j\frac{2}{5}$$

$$|G(j2)| = \sqrt{\frac{1}{25}+\frac{4}{25}} = \frac{1}{\sqrt{5}} \simeq 0.447$$

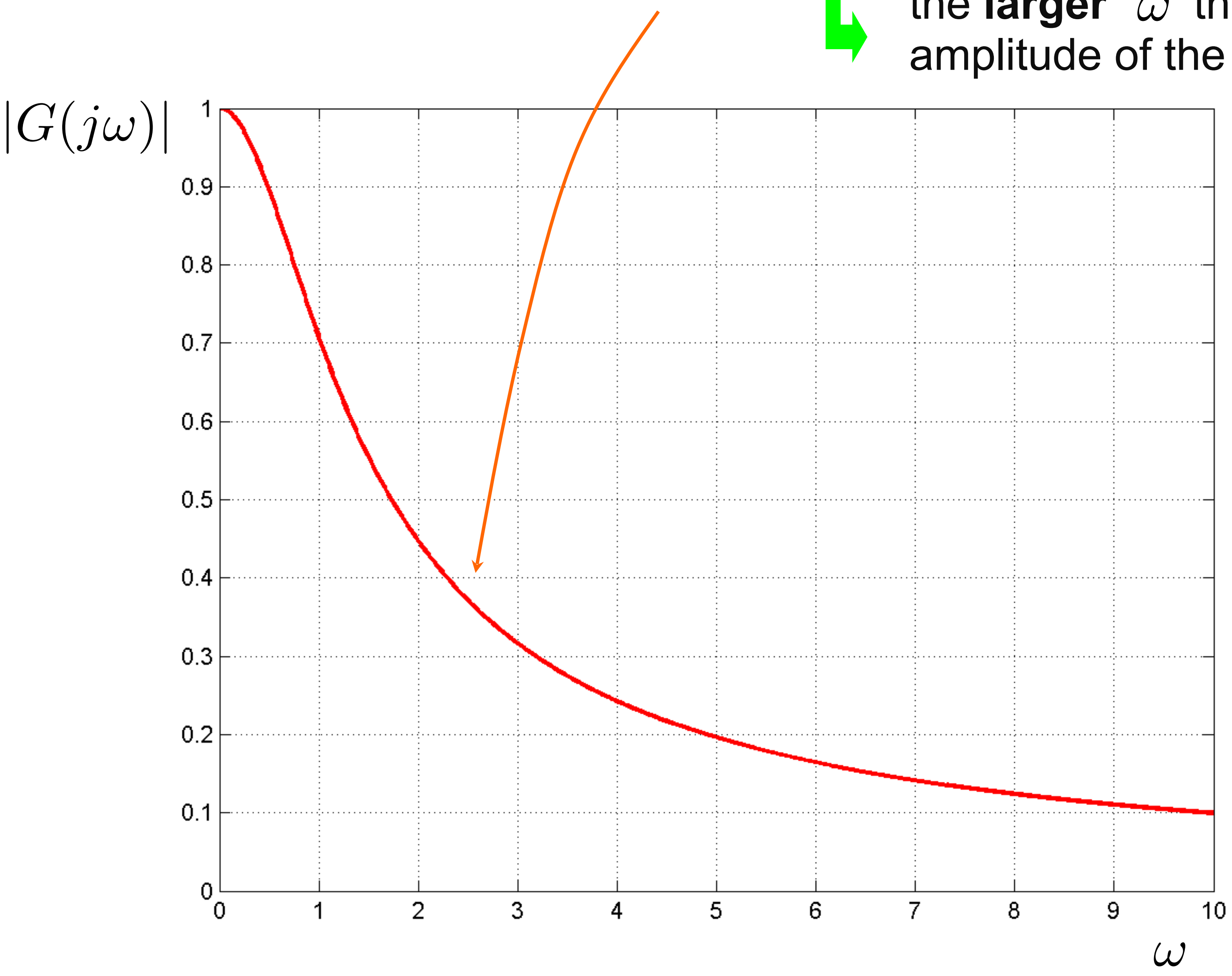
$$\arg G(j2) = \text{arctg}\,(-2) = -63° = -63°\,\tfrac{\pi}{180} \simeq -1.1$$

$$y(t) \simeq \frac{10}{\sqrt{5}}\sin(2t-1.1)\,,\ \ t > t_s \simeq 4.6\text{sec}$$

Example 1 (contd.)

$|G(j\omega)|$ decreases when ω increases

the **larger** ω the **smaller** the amplitude of the regime response



Example 2

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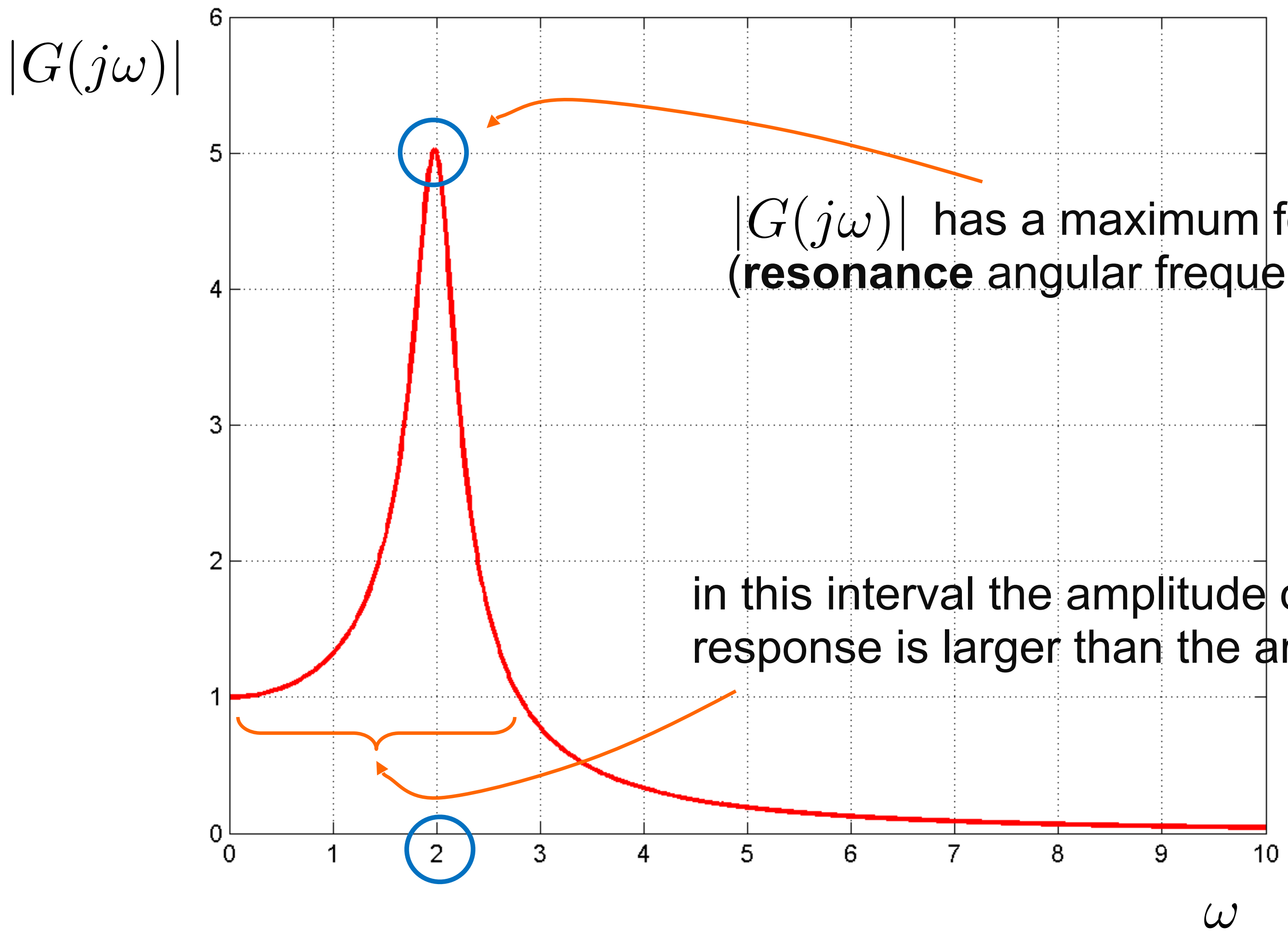
$$G(s) = \frac{1}{1 + 0.1s + \frac{s^2}{4}}$$

$$|G(j\omega)| = \frac{1}{\left|1 + j0.1\omega + \frac{(j\omega)^2}{4}\right|} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{4}\right)^2 + 0.01\omega^2}}$$

Example 2 (contd.)



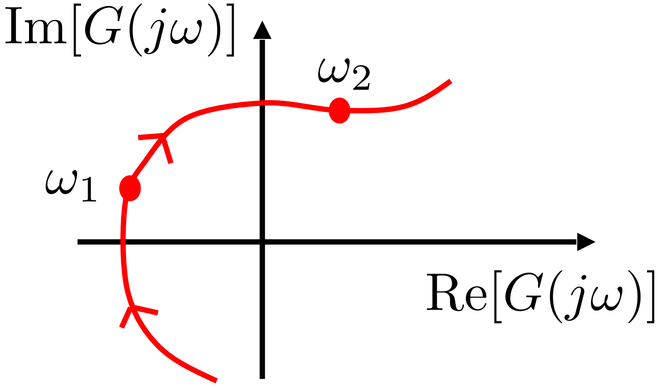
$|G(j\omega)|$ has a maximum for $\omega = 2$ (**resonance** angular frequency)

in this interval the amplitude of the regime response is larger than the amplitude of the input

Graphical Representations of the Frequency Response

- **Polar Diagrams**:

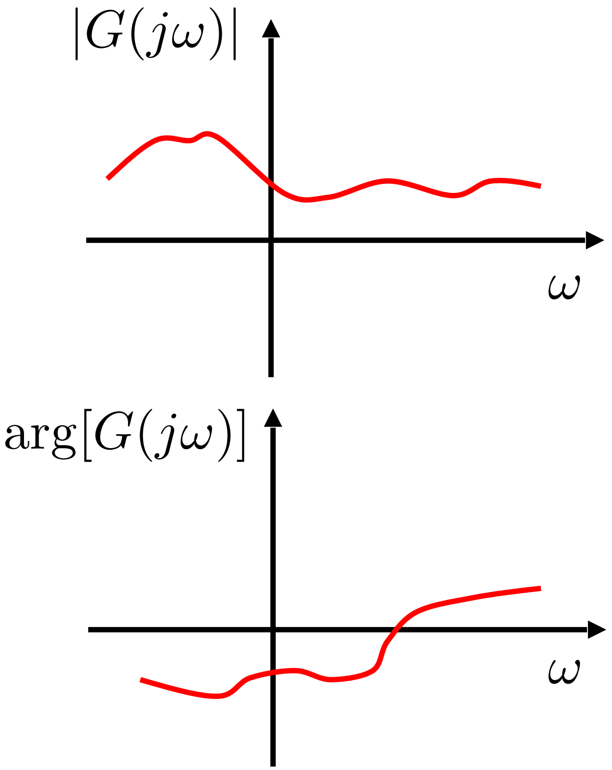
$$G(j\omega)\,,\ \ \omega \geq 0$$



- **Bode Diagrams**:

$$|G(j\omega)|\,,\ \ \omega \geq 0$$

$$\varphi(\omega) = \arg G(j\omega)\,,\ \ \omega \geq 0$$



Polar and Bode diagrams are related:

$$G(j\omega) = |G(j\omega)|e^{j\varphi(\omega)}$$

Bode Diagrams: Conventional Assumptions

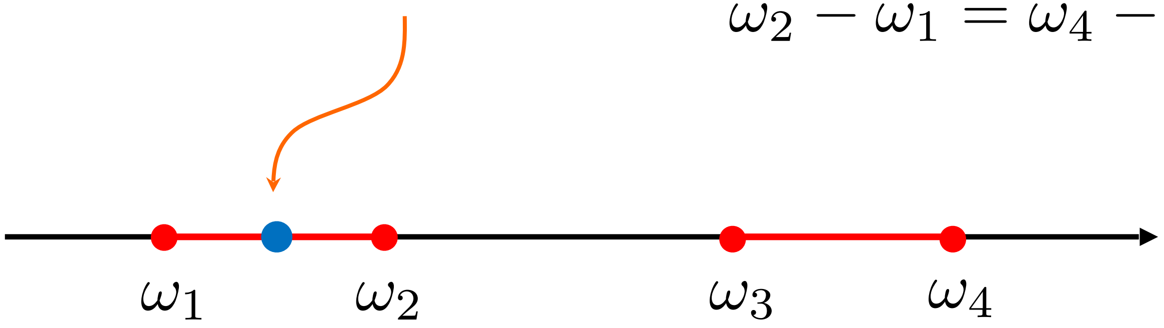
- **Magnitude**:
  - on the "horizontal axis": $\log \omega$
  - on the "vertical axis": $|G(j\omega)|_{\text{dB}}$ $\quad (x_{\text{dB}} := 20 \log x)$
- **Phase**:
  - on the "horizontal axis": $\log \omega$
  - on the "vertical axis": $\arg G(j\omega)$ in degrees

Bode Diagrams: Linear and Logarithmic Scale

- Linear Scale:

$$\frac{\omega_2 + \omega_1}{2}$$

$$\omega_2 - \omega_1 = \omega_4 - \omega_3$$

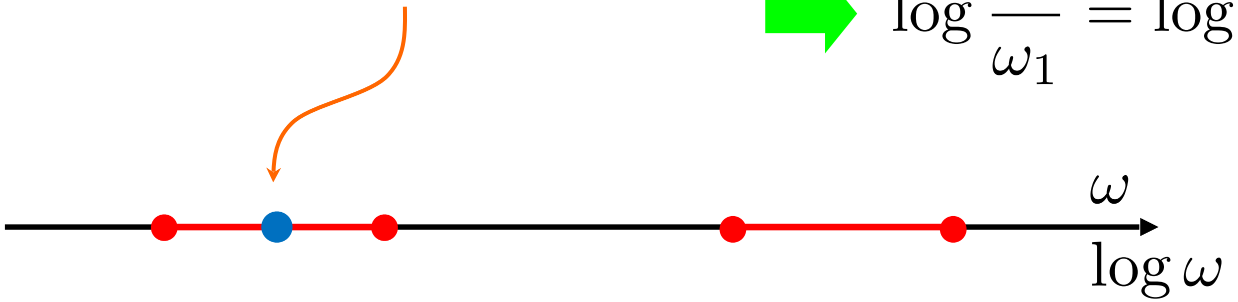


- Logarithmic Scale:

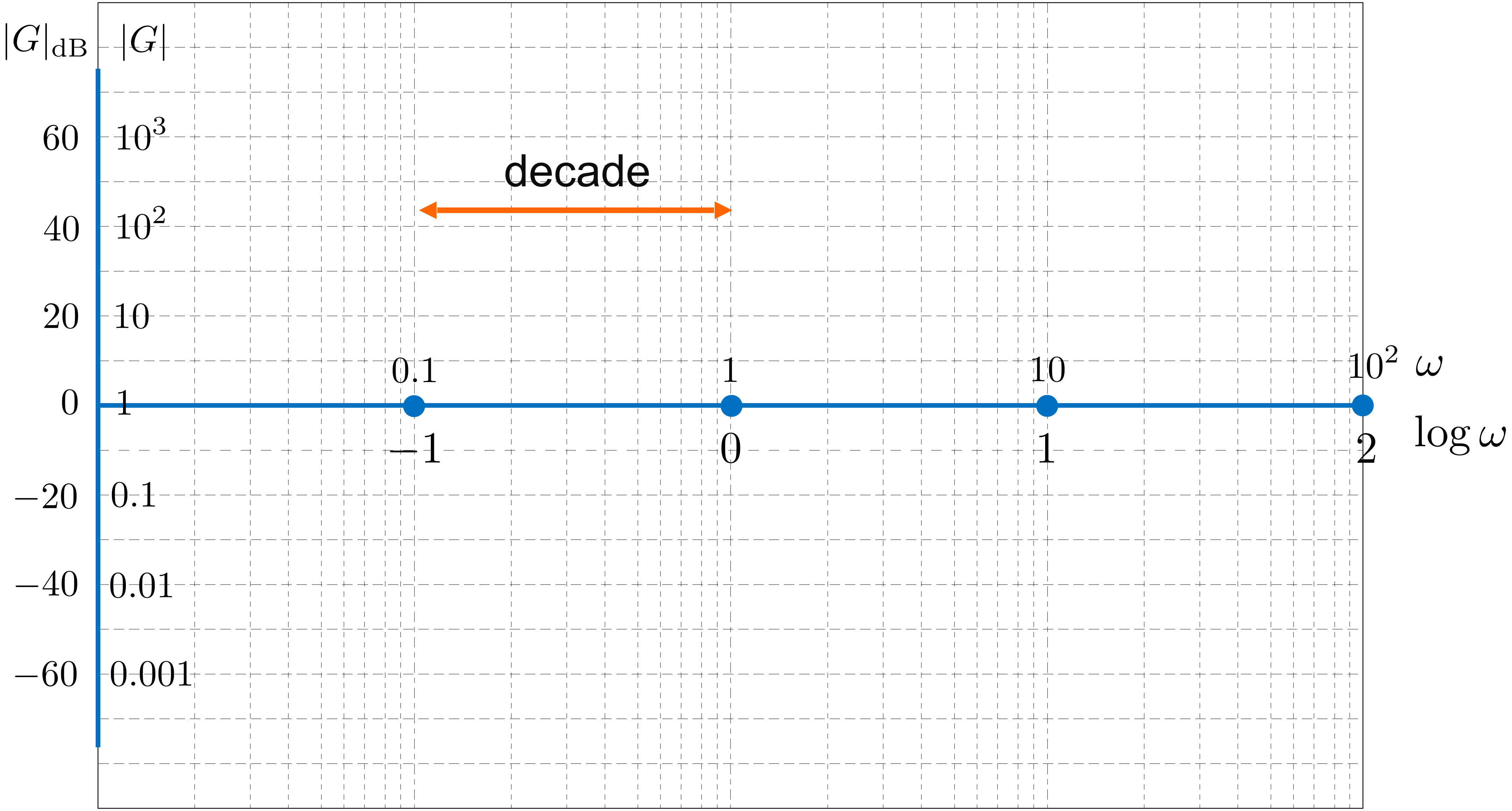
$$\log \omega_2 - \log \omega_1 = \log \omega_4 - \log \omega_3$$

$$\sqrt{\omega_1 \omega_2}$$

$$\log \frac{\omega_2}{\omega_1} = \log \frac{\omega_4}{\omega_3}$$



Bode Semi-Logarithmic Chart



Bode Diagrams: Magnitude

$$G(s) = \frac{\mu}{s^g} \frac{\prod_{i=1}^{m} (1 + sT_i)}{\prod_{i=1}^{n} (1 + s\tau_i)}$$

$$|G(j\omega)| = \frac{|\mu|}{|j\omega|^g} \frac{\prod_{i=1}^{m} |1 + j\omega T_i|}{\prod_{i=1}^{n} |1 + j\omega\tau_i|}$$

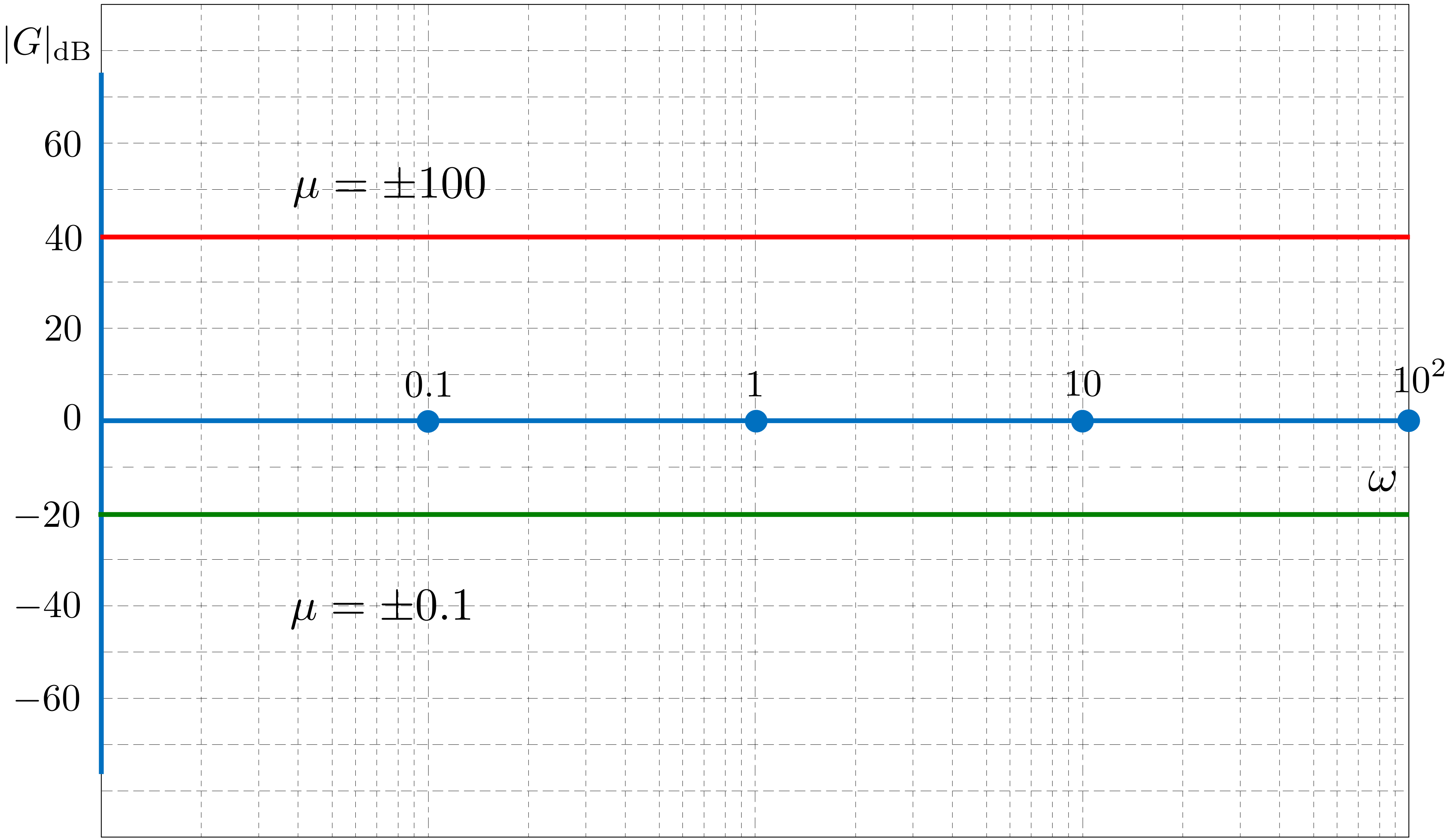
$$|G(j\omega)|_{\mathrm{dB}} = 20 \log |\mu| - 20 \log |j\omega|^g + \sum_i 20 \log |1 + j\omega T_i| - \sum_i 20 \log |1 + j\omega\tau_i|$$

(A): $20 \log |\mu|$

(B): $-20 \log |j\omega|^g$

(C),(D): $+\sum_i 20 \log |1 + j\omega T_i|$, $-\sum_i 20 \log |1 + j\omega\tau_i|$

(A) $20 \log |\mu|$ constant horizontal line



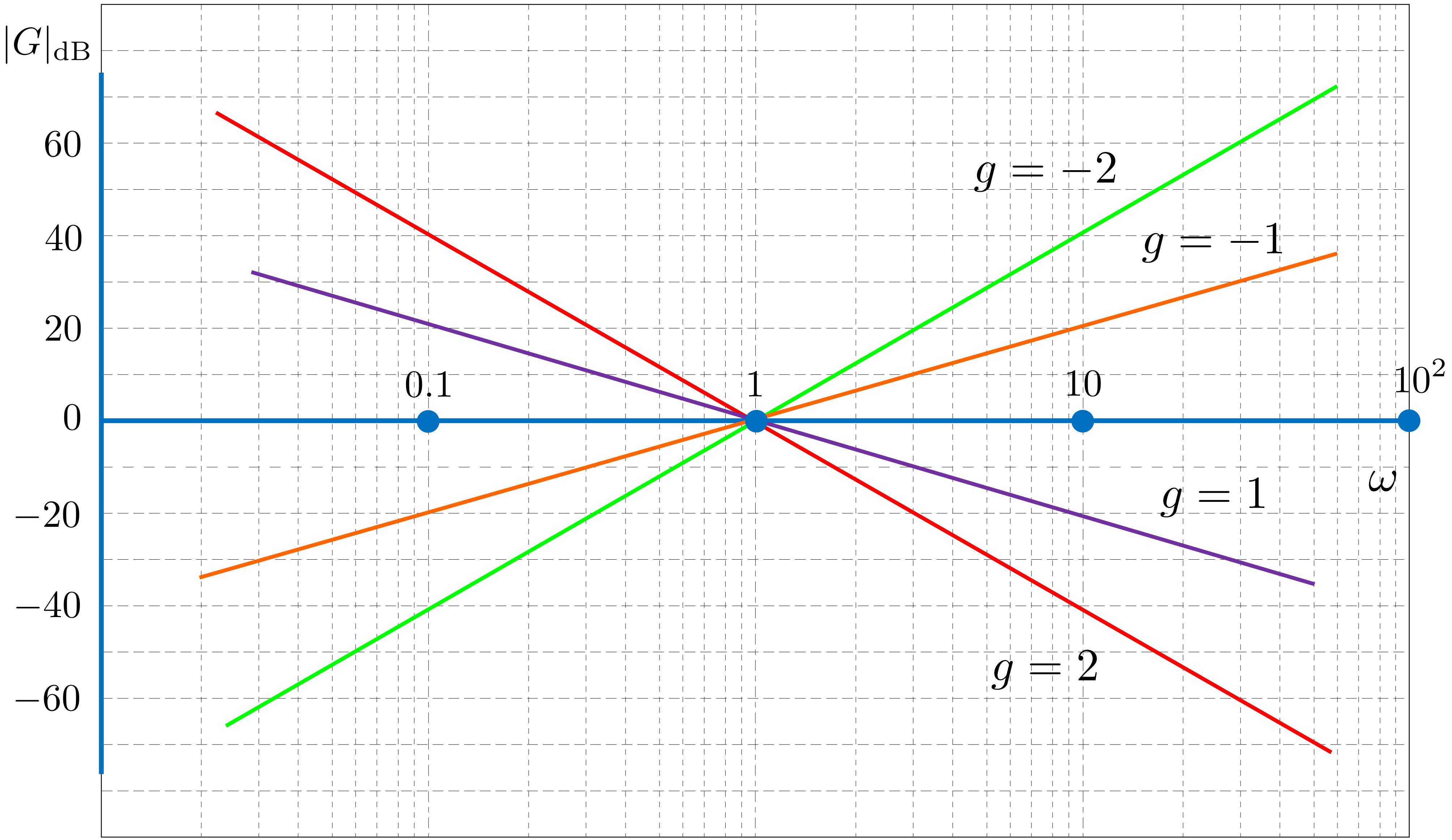
(B) $-20 \log |j\omega|^g$

$$-20 \log |j\omega|^g = -20g \log \omega$$

in logarithmic coordinates it is a straight line with slope $-20 \cdot g \, \mathrm{dB/dec}$

Conventionally, the straight line is set to pass through the point $\omega = 1$ with 0dB gain

(B) $-20\log|j\omega|^g$



(C), (D) $20\log|1+j\omega T|\,,\ \ T\in\mathbb{R}$

$$20\log|1+j\omega T| = 20\log\sqrt{1+\omega^2T^2}$$

- If $\omega^2T^2 \ll 1$, that is $\omega \ll \dfrac{1}{|T|}$

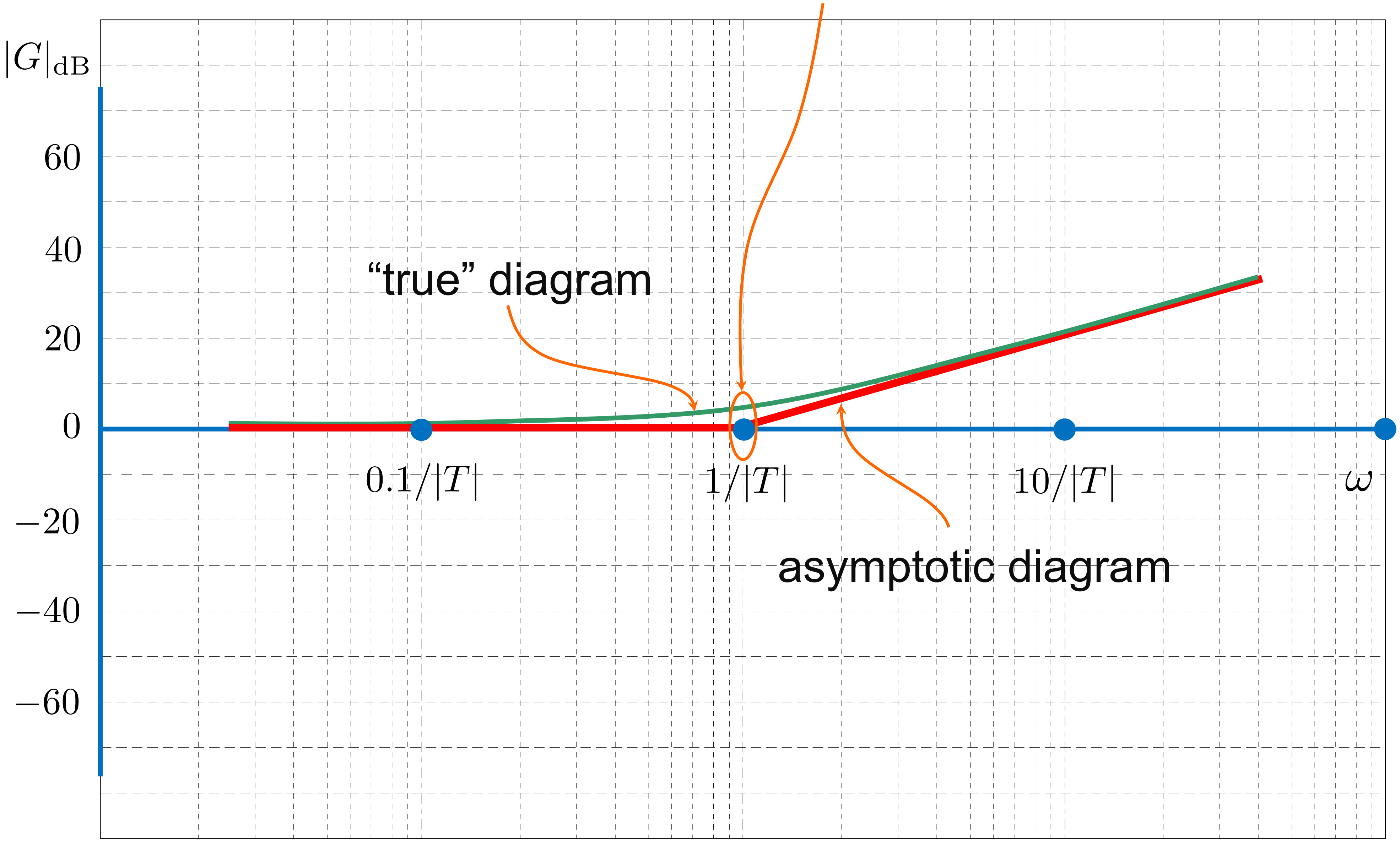
$$20\log|1+j\omega T| \simeq 0$$

- If $\omega^2T^2 \gg 1$, that is $\omega \gg \dfrac{1}{|T|}$

$$\begin{aligned}20\log|1+j\omega T| &\simeq 20\log|j\omega T|\\ &= 20\log\omega|T|\\ &= 20\log\omega + 20\log|T|\end{aligned}$$

in logarithmic coordinates it is a straight line with slope $20\cdot\,\mathrm{dB/dec}$

maximum approximation error for $\omega = 1/|T|$: $20 \log \sqrt{2} = 10 \log 2 \simeq 3\text{dB}$



(C),(D) $20\log|1+j\omega T| + 20\log|1+j\omega T^*|,\ T, T^* \in \mathbb{C}$

Recall:

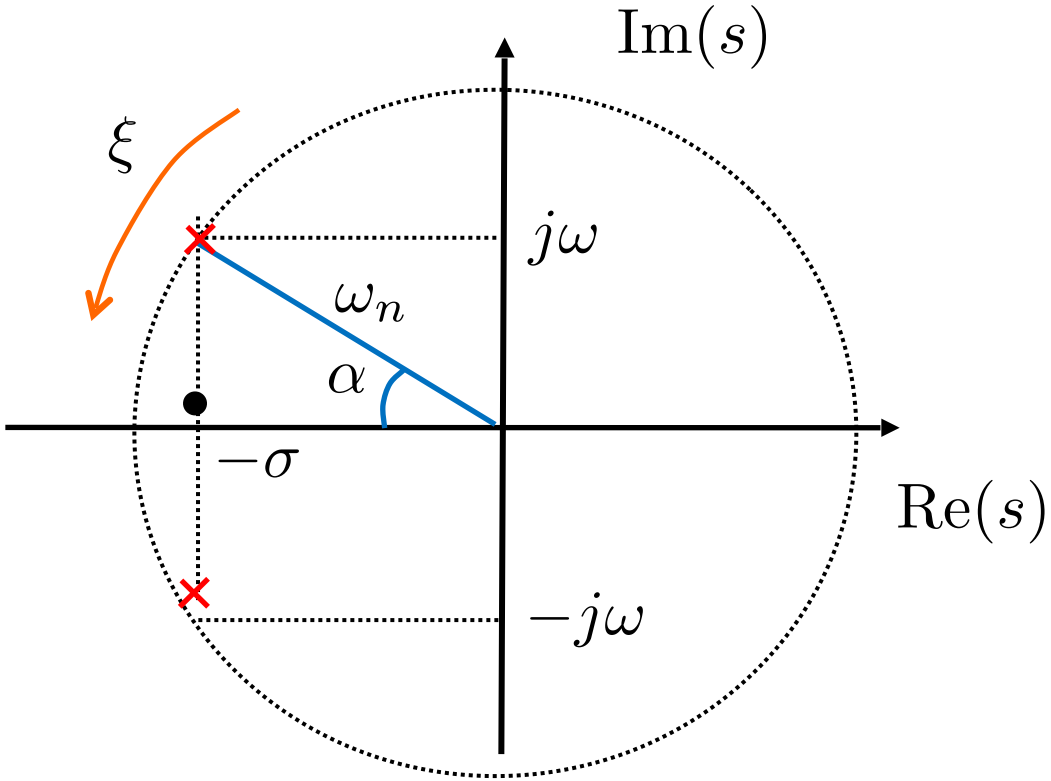
$$(1+sT)(1+sT^*)$$

$$= 1 + \frac{2\xi}{\omega_n}s + \frac{1}{\omega_n^2}s^2$$

where:

$0 < \xi < 1$ → left half-plane

$-1 < \xi < 0$ → right half-plane



$$20 \log |1 + j\omega T| + 20 \log |1 + j\omega T^*|$$

$$= 20 \log \left| 1 + \frac{2\xi}{\omega_n} j\omega - \frac{1}{\omega_n^2}\omega^2 \right| = 20 \log \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \frac{4\xi^2\omega^2}{\omega_n^2}}$$

- If $\omega \to 0$

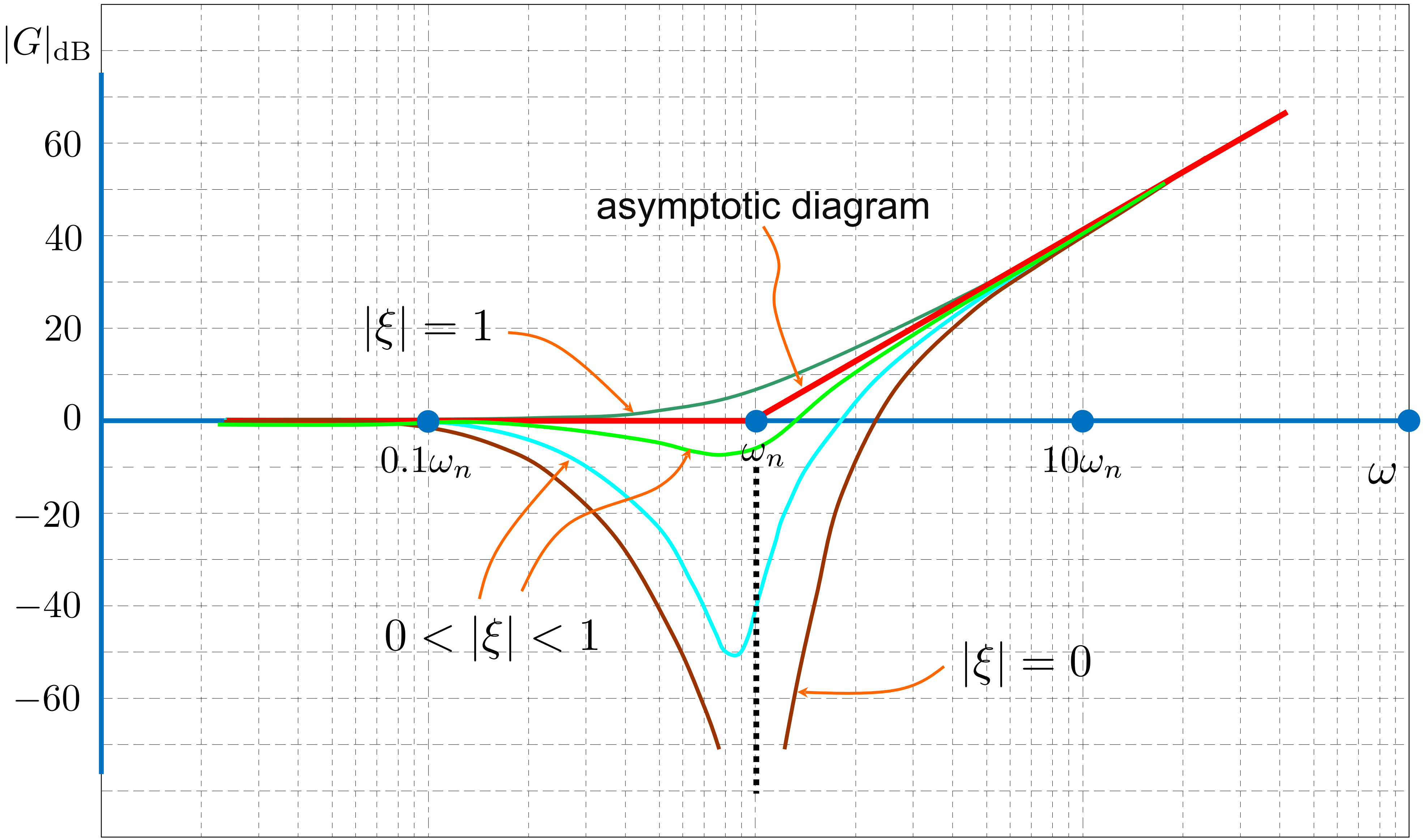
$$\Rightarrow \quad 20 \log \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \frac{4\xi^2\omega^2}{\omega_n^2}} \simeq 0$$

- If $\omega \to \infty$

$$\Rightarrow \quad 20 \log \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \frac{4\xi^2\omega^2}{\omega_n^2}}$$

$$\simeq 20 \log \sqrt{\left(\frac{\omega^2}{\omega_n^2}\right)^2} = 20 \log \frac{\omega^2}{\omega_n^2} = 40 \log \frac{\omega}{\omega_n} = 40 \log \omega - 40 \log \omega_n$$

in logarithmic coordinates it is a straight line with slope $40 \cdot \mathrm{dB/dec}$



$|G|_{\mathrm{dB}}$
60
40
20
0
-20
-40
-60
asymptotic diagram
$|\xi| = 1$
$0.1\omega_n$
ω_n
$10\omega_n$
ω
$0 < |\xi| < 1$
$|\xi| = 0$

Maximum approximation error for $\omega = \omega_n$:

$$20 \log \sqrt{4\xi^2} = 20 \log(2|\xi|)$$

- If $|\xi| = 1$

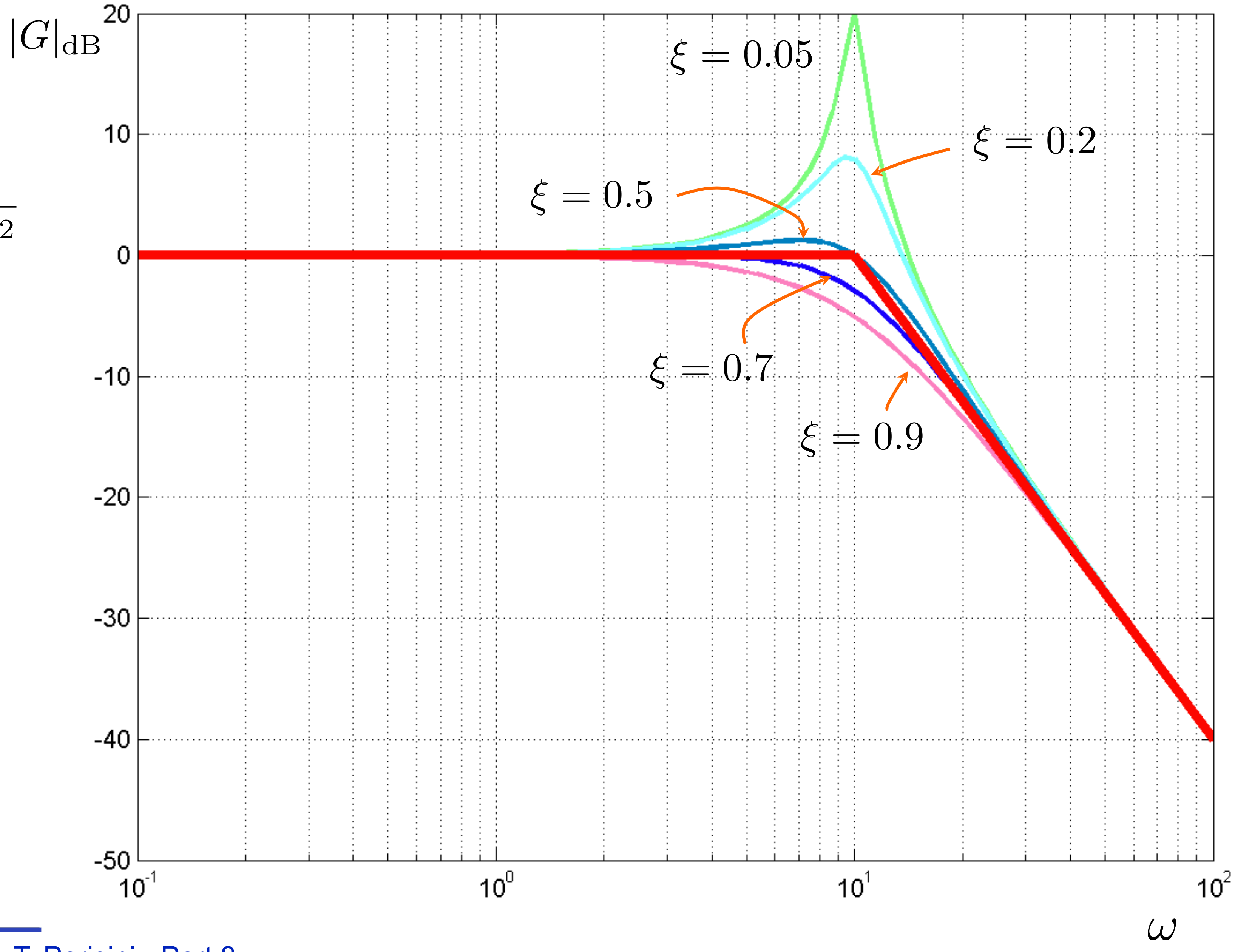
$$20 \log(2) \simeq 6\text{dB}$$

- If $|\xi| \to 0$

$$\to -\infty$$

Example 1

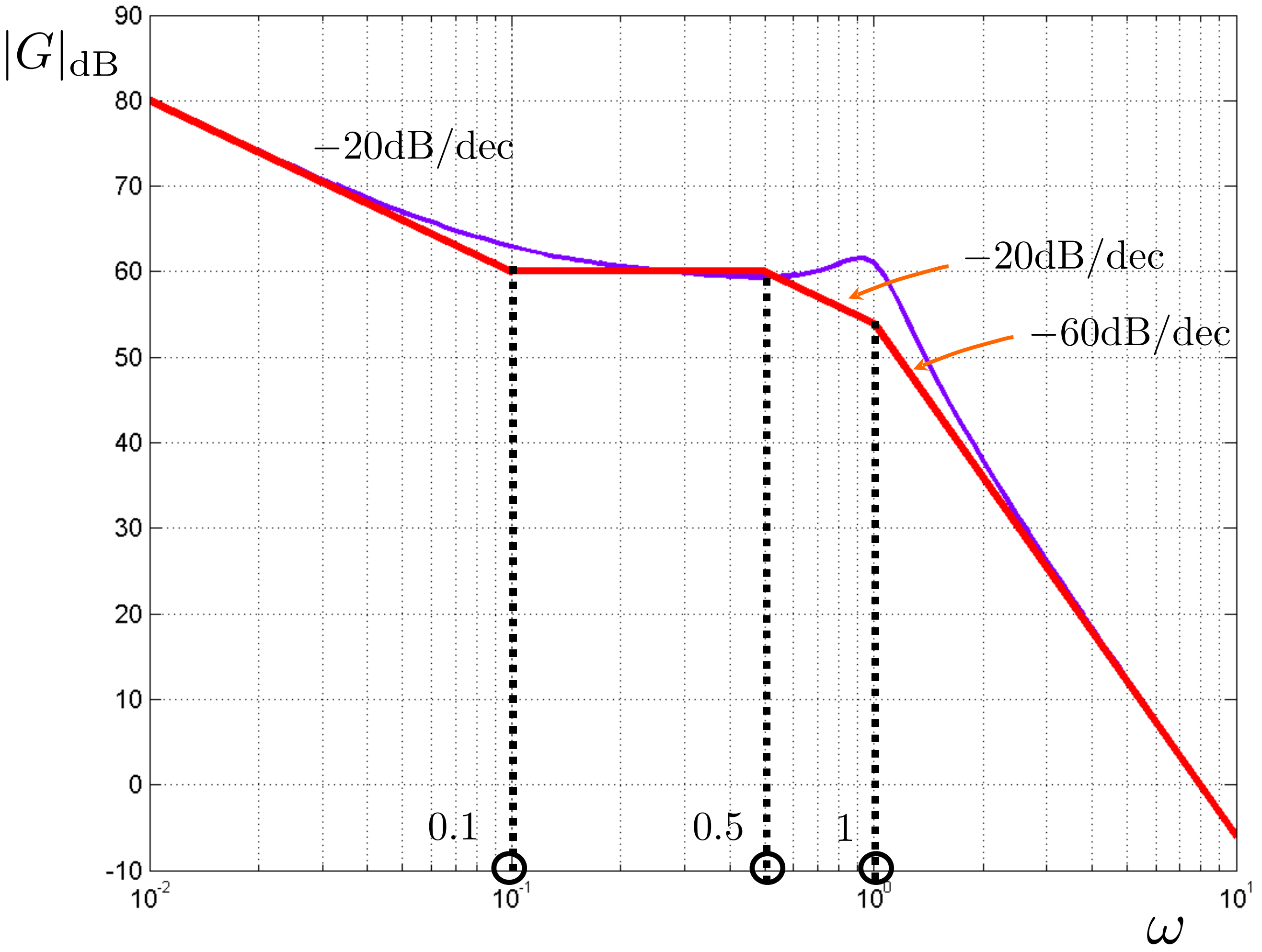
$$G(s) = \frac{1}{1 + \frac{2\xi}{10}s + \frac{1}{10^2}s^2}$$



Example 2

$$G(s) = \frac{100(1+10s)}{s(1+2s)(1+0.4s+s^2)}$$

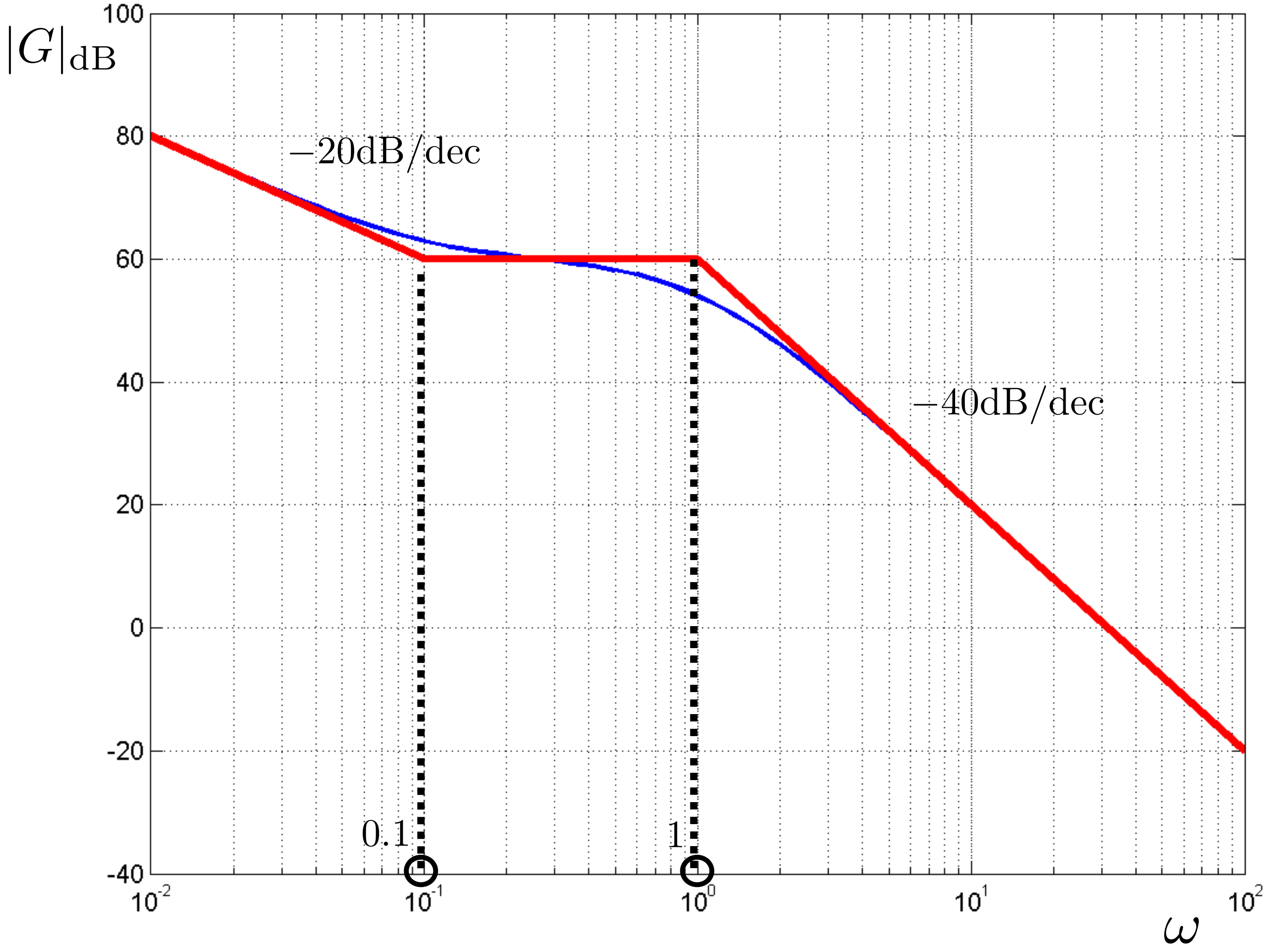
$$g = 1$$
$$\mu = 100 \implies \mu_{\text{dB}} = 40\text{dB}$$
$$z_1 = -0.1$$
$$p_1 = 0$$
$$p_2 = -0.5$$
$$p_{3,4} = -0.2 \pm j\sqrt{0.96}$$
$$\omega_n = 1;\ \xi = 0.2$$



Example 3

$$G(s) = \frac{100(1+10s)}{s(1+s)^2}$$

$$g = 1$$
$$\mu = 100 \implies \mu_{\mathrm{dB}} = 40\mathrm{dB}$$
$$z_1 = -0.1$$
$$p_1 = 0$$
$$p_2 = p_3 = -1$$



Example 4

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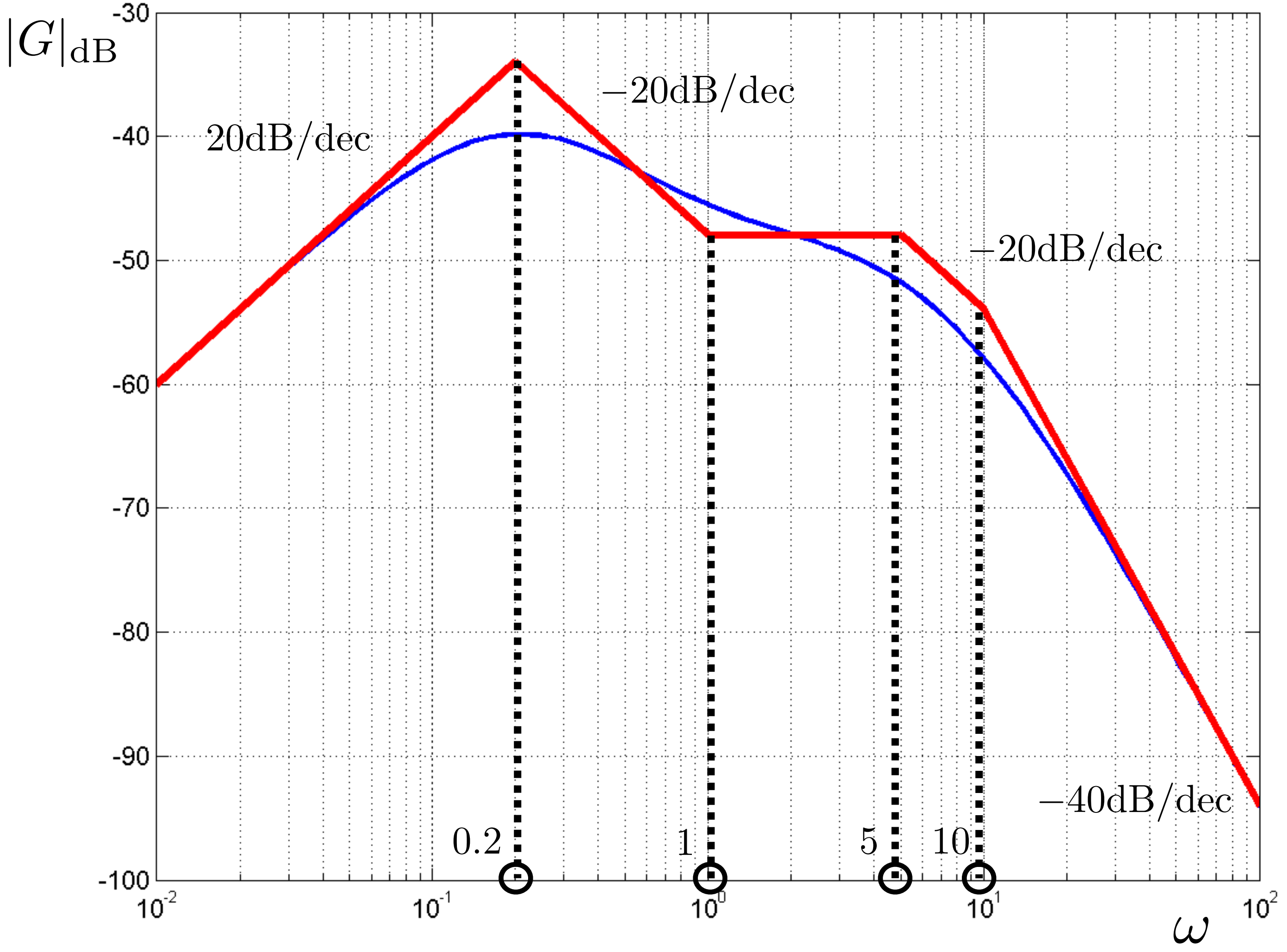
$$G(s) = \frac{0.1s(1+s)}{(1+5s)^2(1+0.2s)(1-0.1s)}$$

$$g = -1$$
$$\mu = 0.1 \implies \mu_{\mathrm{dB}} = -20\mathrm{dl}$$
$$z_1 = 0$$
$$z_2 = -1$$
$$p_1 = p_2 = -0.2$$
$$p_3 = -5$$
$$p_4 = 10$$

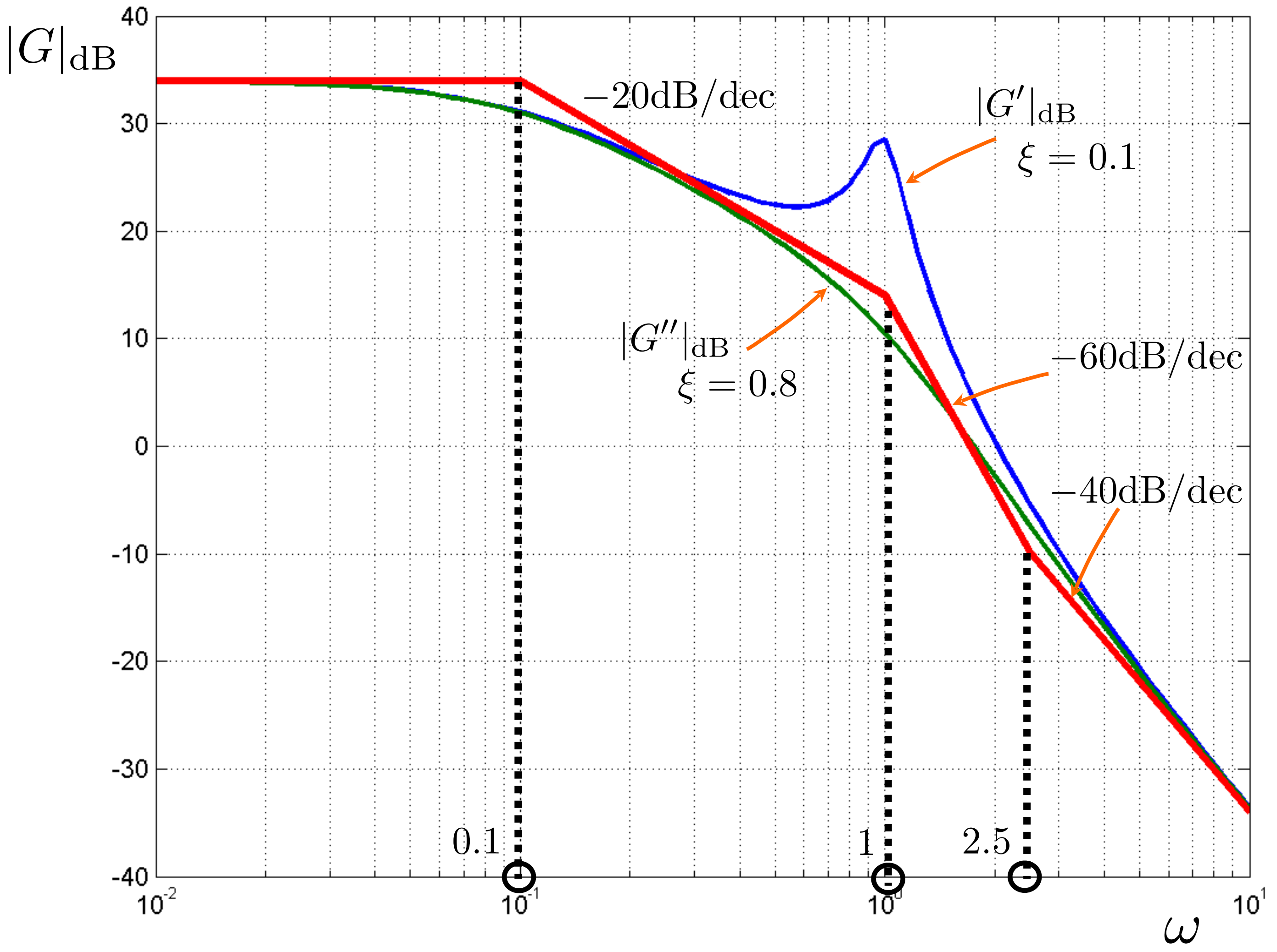


Example 5

$$G'(s) = \frac{50(1+0.4s)}{(1+10s)(1+0.2s+s^2)}$$

$$G''(s) = \frac{50(1+0.4s)}{(1+10s)(1+1.6s+s^2)}$$

$$g = 0$$
$$\mu = 50 \implies \mu_{\mathrm{dB}} \simeq 34\mathrm{dB}$$
$$z_1 = -2.5$$
$$p_1 = -0.1$$
$$p'_{2,3} = -0.1 \pm j\sqrt{0.99}$$
$$\omega_n = 1,\ \xi = 0.1$$
$$p''_{2,3} = -0.8 \pm j\sqrt{0.36}$$
$$\omega_n = 1,\ \xi = 0.8$$



Bode Diagrams: Phase

First, recall the **conventions** and **assumptions** for the frequency response diagram of the phase:

- **Phase**:
  - on the “horizontal axis”: $\log \omega$
  - on the “vertical axis”: $\arg G(j\omega)$ in degrees

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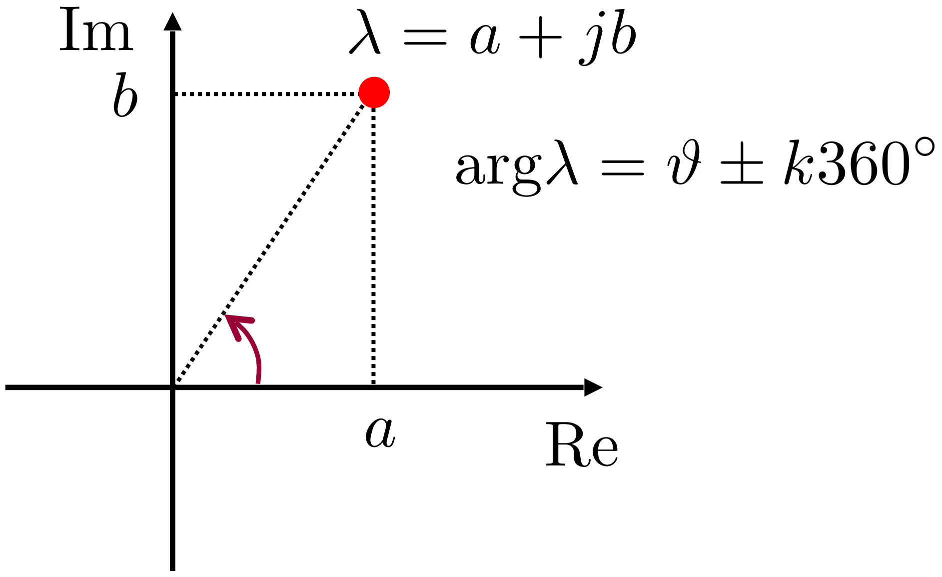
Care has to be exercised on the **calculation of the phase of a complex number**:

Conventions:



- $\vartheta > 0$: counterclockwise rotations
- $\arg\lambda \in [-180°, 180°)$
  - If $a > 0$

    $\arg\lambda = \text{arctg}\,\dfrac{b}{a}$
  - If $a < 0\,;\ b > 0$

    $\arg\lambda = \text{arctg}\,\dfrac{b}{a} + 180°$
  - If $a < 0\,;\ b < 0$

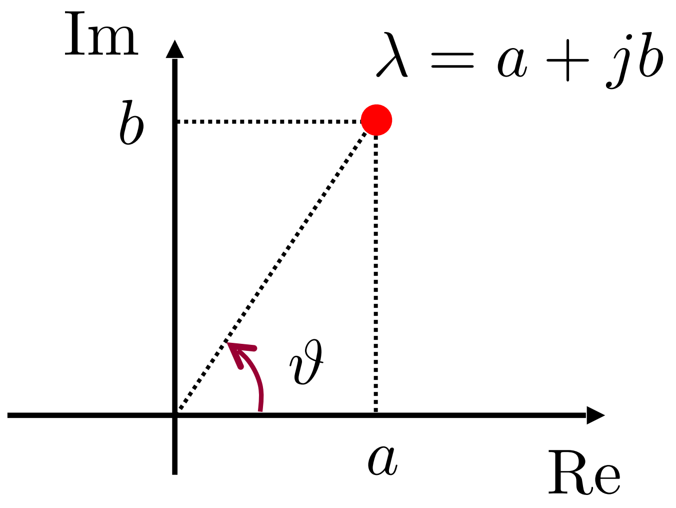
    $\arg\lambda = \text{arctg}\,\dfrac{b}{a} - 180°$

Remark: the Matlab Control Systems Toolbox uses a different convention since it represents the phase in the Bode diagrams in the interval $[0, +360°)$

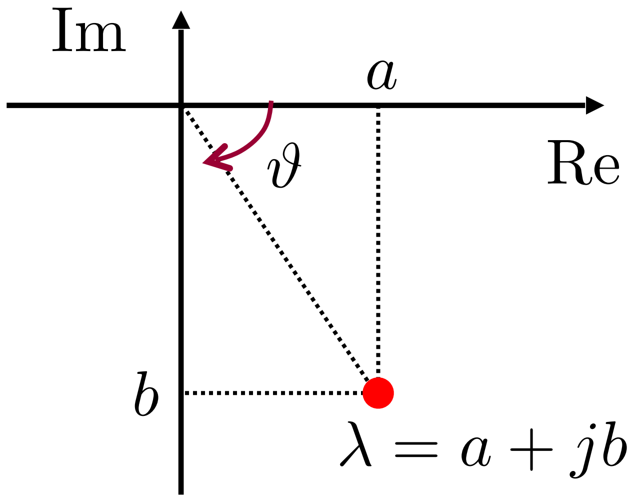
Bode Diagrams: Phase (contd.)

More specifically:

- If $a > 0,\ b \neq 0$ ⟹ $\arg \lambda = \text{arctg}\, \dfrac{b}{a}$



$0^\circ < \vartheta < +90^\circ$

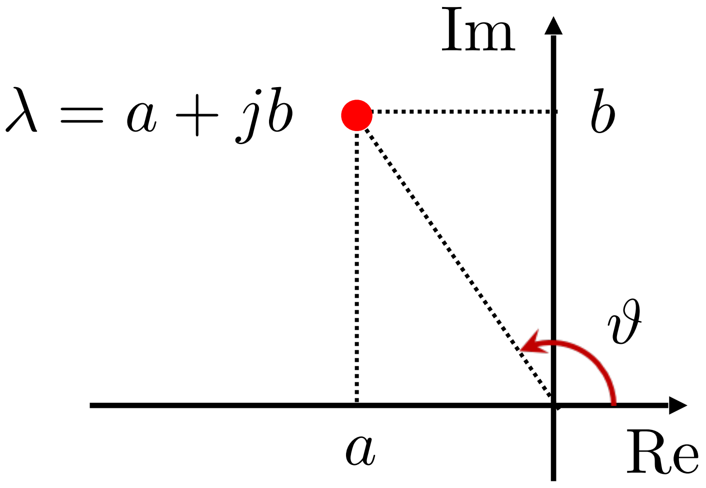


$-90^\circ < \vartheta < 0^\circ$

Bode Diagrams: Phase (contd.)

- If $a < 0\,;\ b > 0$

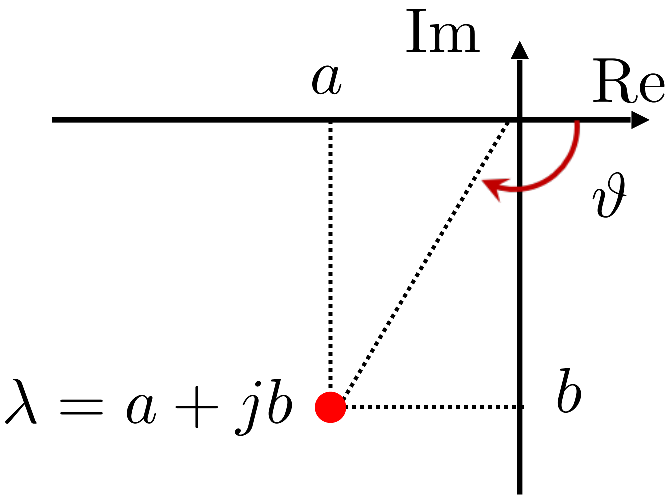
$$\arg\lambda = \operatorname{arctg}\frac{b}{a} + 180^\circ$$



$$+90^\circ < \vartheta < +180^\circ$$

- If $a < 0\,;\ b < 0$

$$\arg\lambda = \operatorname{arctg}\frac{b}{a} - 180^\circ$$

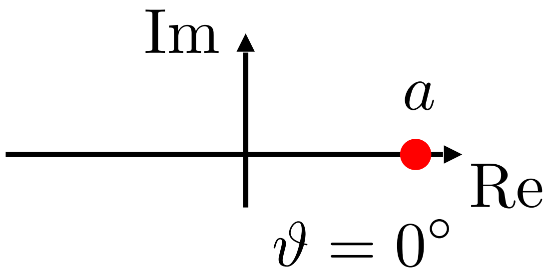


$$-180^\circ < \vartheta < -90^\circ$$

Bode Diagrams: Phase (contd.)

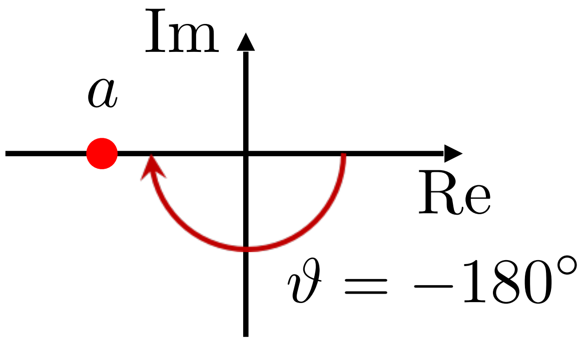
- If $a > 0,\ b = 0$

  $\Rightarrow \arg \lambda = 0^\circ$



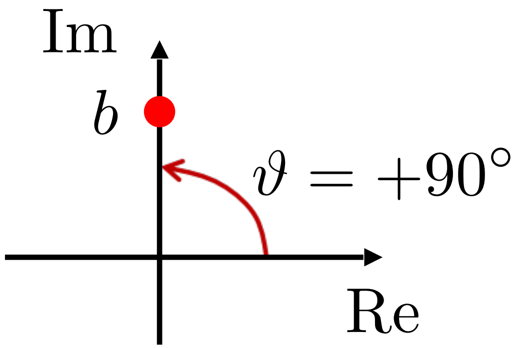
- If $a < 0,\ b = 0$

  $\Rightarrow \arg \lambda = -180^\circ$



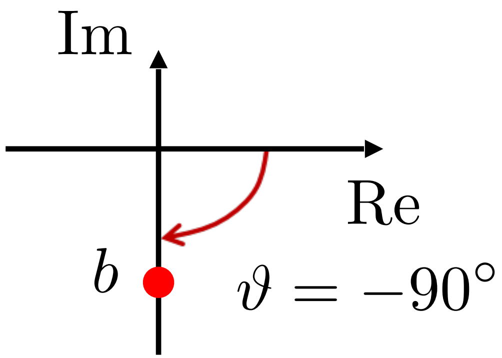
- If $a = 0,\ b > 0$

  $\Rightarrow \arg \lambda = +90^\circ$



- If $a = 0,\ b < 0$

  $\Rightarrow \arg \lambda = -90^\circ$



Bode Diagrams: Phase (contd.)

The phase of a complex number satisfies some useful properties:

- $$\arg(\lambda\eta) = \arg(\lambda) + \arg(\eta)$$
- $$\arg(\lambda^k) = k \arg(\lambda)$$
- $$\arg\left(\frac{\lambda}{\eta}\right) = \arg(\lambda) - \arg(\eta)$$

The calculation of the phase of $G(j\omega)$ satisfies properties that are analogous to the ones of the logarithm in the case of the calculation of $20\log|G(j\omega)|$

Bode Diagrams: Phase (contd.)

$$G(s) = \frac{\mu}{s^g} \frac{\prod_{i=1}^{m} (1 + sT_i)}{\prod_{i=1}^{n} (1 + s\tau_i)}$$

$$\arg G(j\omega) = \underbrace{\arg \mu}_{(A)} \underbrace{- \arg (j\omega)^g}_{(B)} \underbrace{+ \sum_i \arg (1 + j\omega T_i) - \sum_i \arg (1 + j\omega \tau_i)}_{(C),(D)}$$

- The conventions have to be applied separately on each single element (A), (B), (C), and (D)
- All such phase contributions have to be added along with their sign
- The total phase at each value of the angular frequency ω may well take on a value outside $[-180°, 180°]$

(A) $\arg \mu$ constant horizontal line

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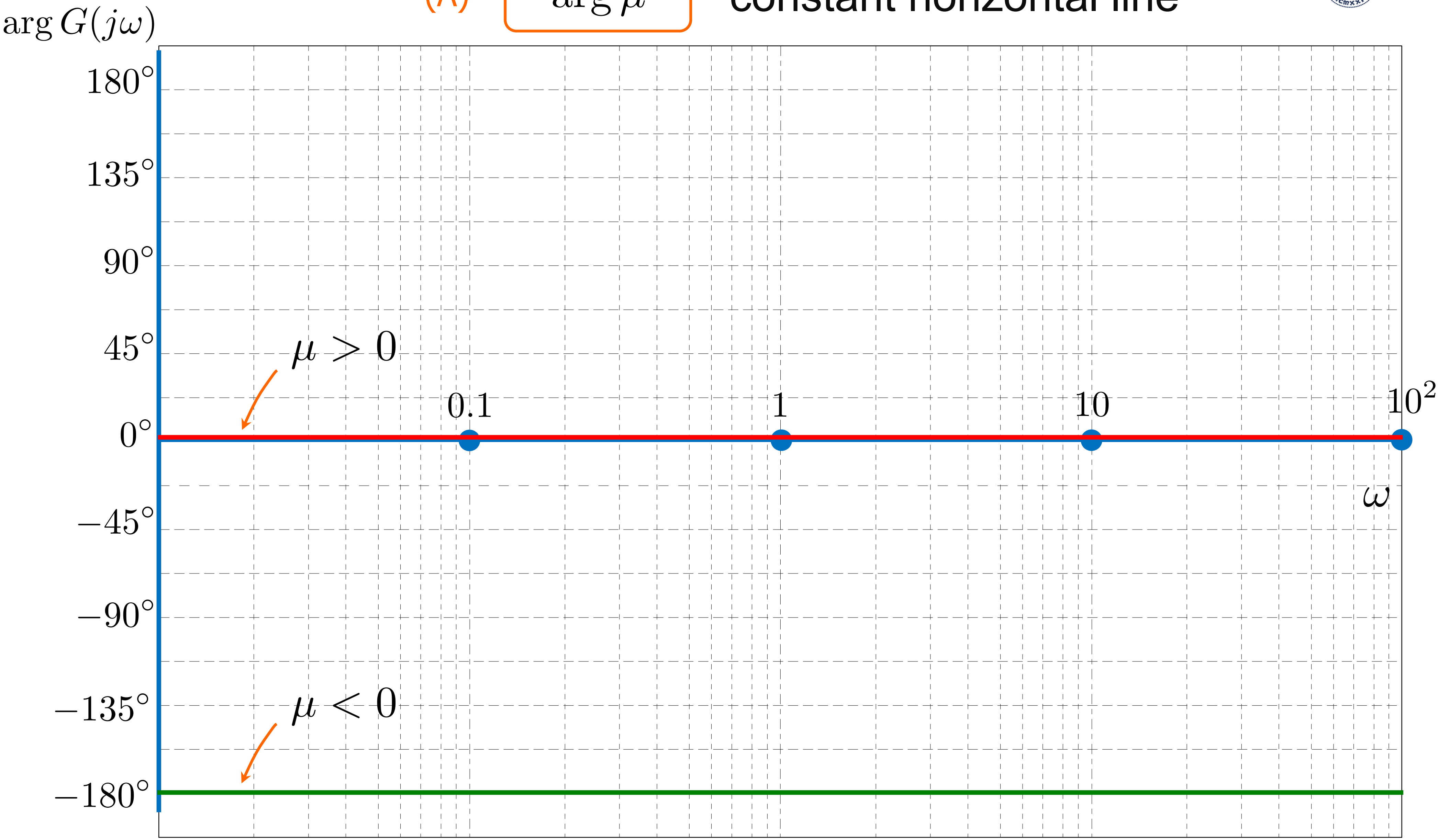
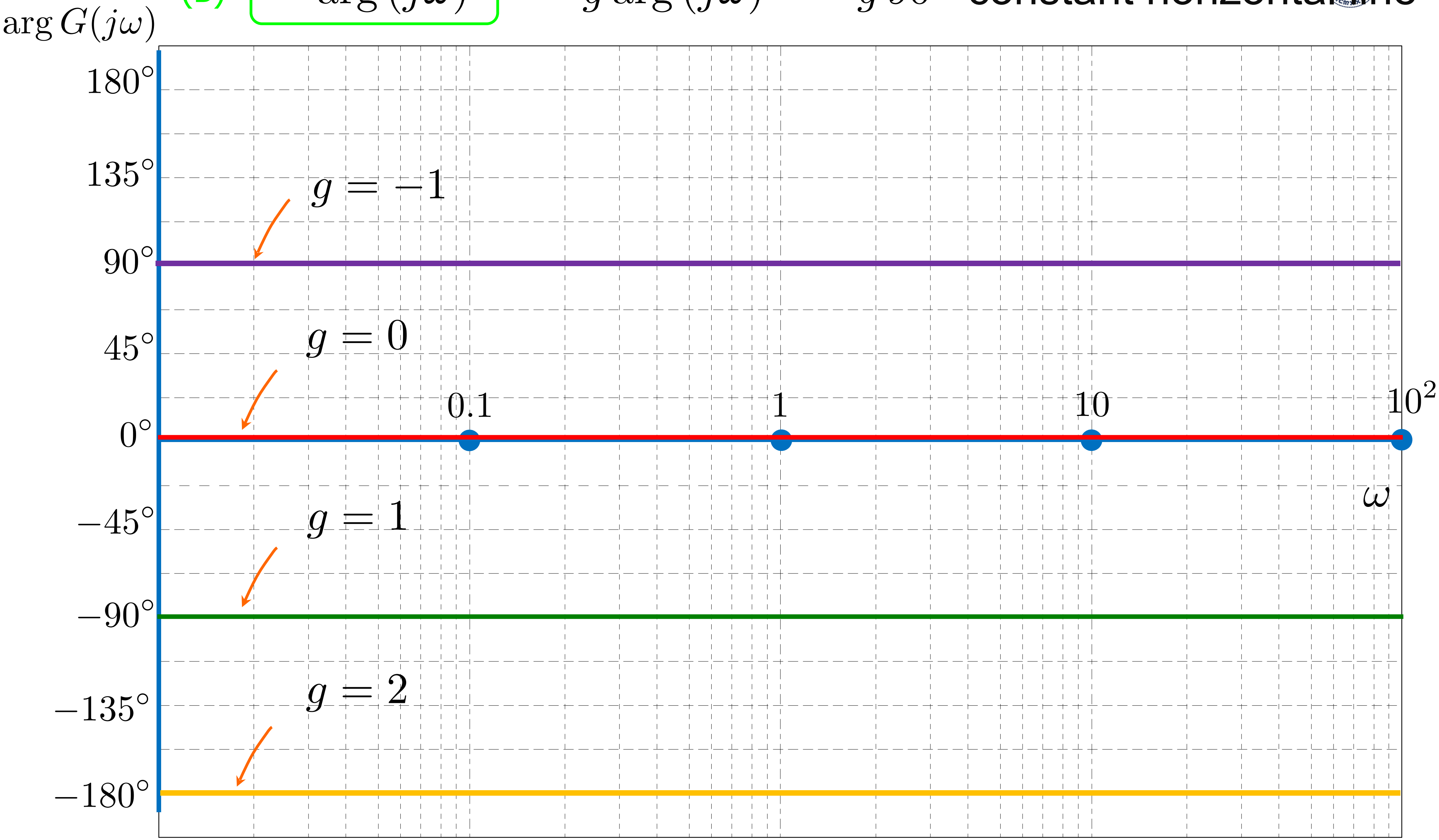


(B) $-\arg(j\omega)^g = -g\arg(j\omega) = -g\,90^\circ$ constant horizontal line



(C) $\arg(1+j\omega T)\,,\ T \in \mathbb{R}$

$$\arg(1+j\omega T) = \operatorname{arctg} \omega T$$

- If $\omega \to 0$

$$\arg(1+j\omega T) \to 0^\circ$$

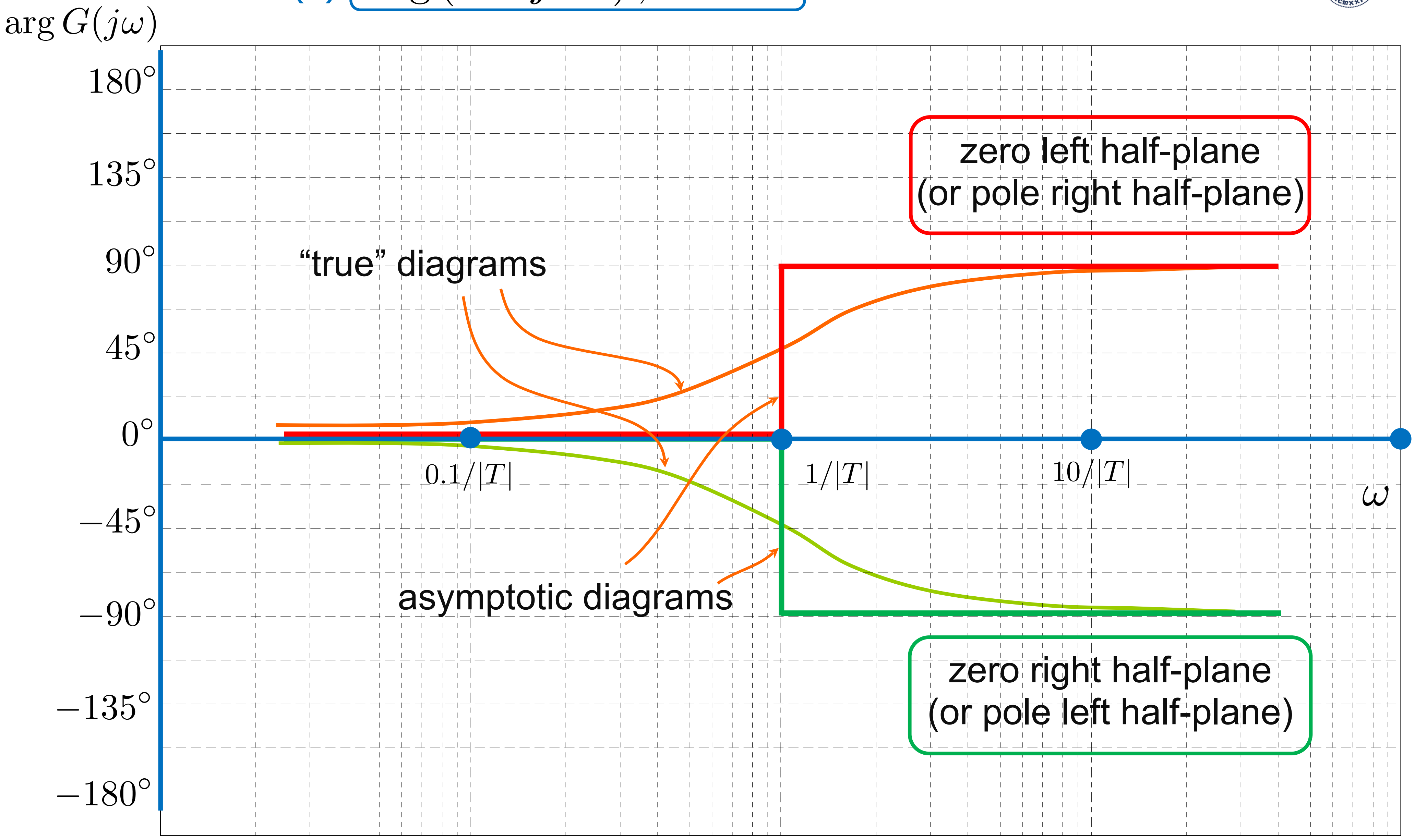
- If $\omega \to \infty$

$$\arg(1+j\omega T) \begin{cases} \to 90^\circ \text{ if } T > 0 \\ \to -90^\circ \text{ if } T < 0 \end{cases}$$

- If $\omega = \dfrac{1}{|T|}$

$$\arg(1+j\omega T) = \begin{cases} 45^\circ \text{ if } T > 0 \\ -45^\circ \text{ if } T < 0 \end{cases}$$

(C) $\arg(1 + j\omega T)\,,\ T \in \mathbb{R}$



(D) $\arg(1+j\omega T)+\arg(1+j\omega T^*)\,,\ T\in\mathbb{C}$

↳ $$\arg(1+j\omega T)+\arg(1+j\omega T^*)=\arg\left(1+\frac{2\xi}{\omega_n}j\omega-\frac{1}{\omega_n^2}\omega^2\right)$$

- If $\omega\to 0$

↳ $$\arg\left(1+\frac{2\xi}{\omega_n}j\omega-\frac{1}{\omega_n^2}\omega^2\right)\simeq\arg(1)=0^\circ$$

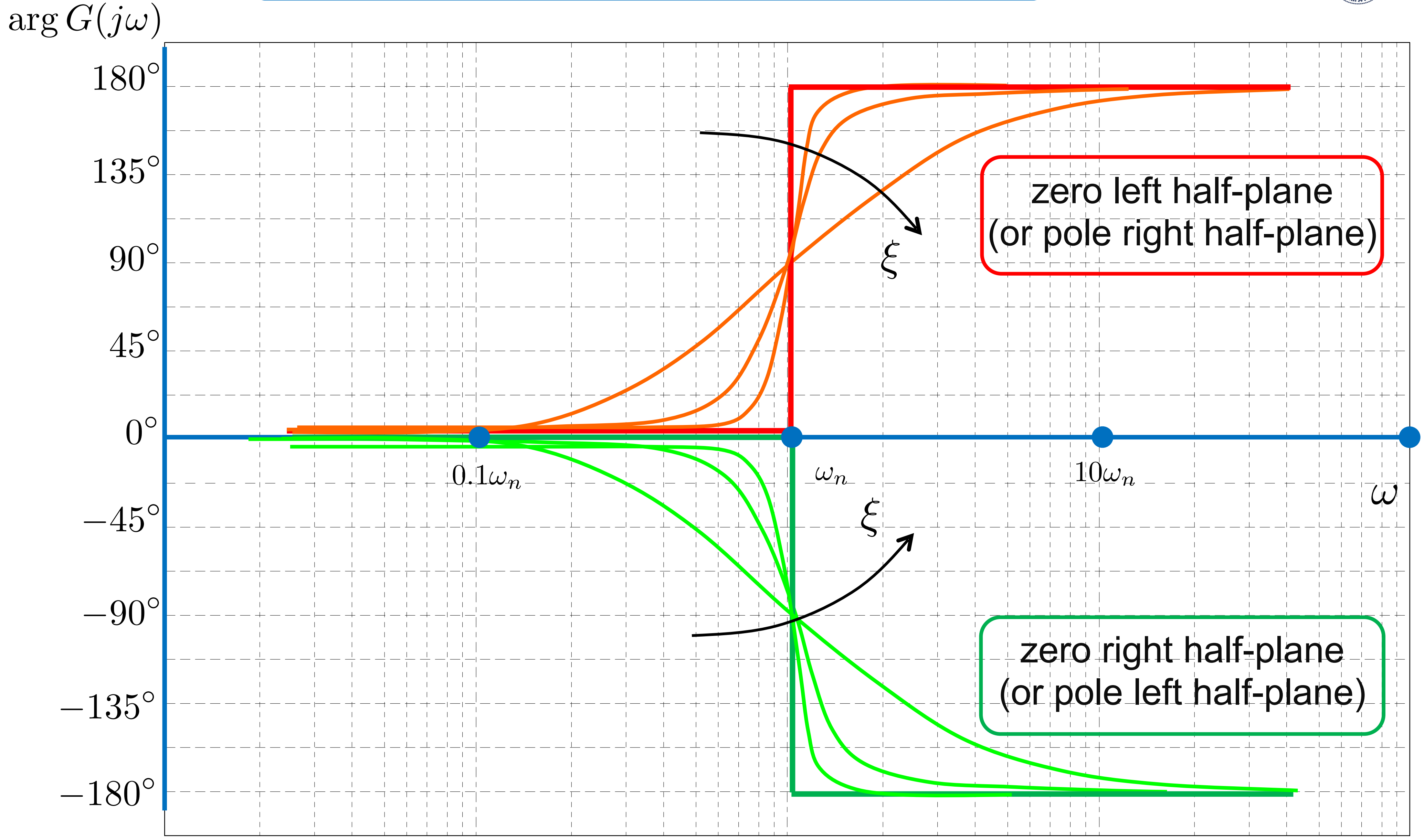
- If $\omega\to\infty$

↳ $$\lim_{\omega\to\infty}\ \text{arctg}\,\frac{2\xi\omega/\omega_n}{1-\omega^2/\omega_n^2}=0^\circ\begin{cases}+180^\circ\ \text{if}\ \xi>0\\ -180^\circ\ \text{if}\ \xi<0\end{cases}$$

- If $\omega=\omega_n$

↳ $$\arg\left(j\frac{2\xi\omega_n}{\omega_n}\right)=\begin{cases}90^\circ\ \text{if}\ \xi>0\\ -90^\circ\ \text{if}\ \xi<0\end{cases}$$

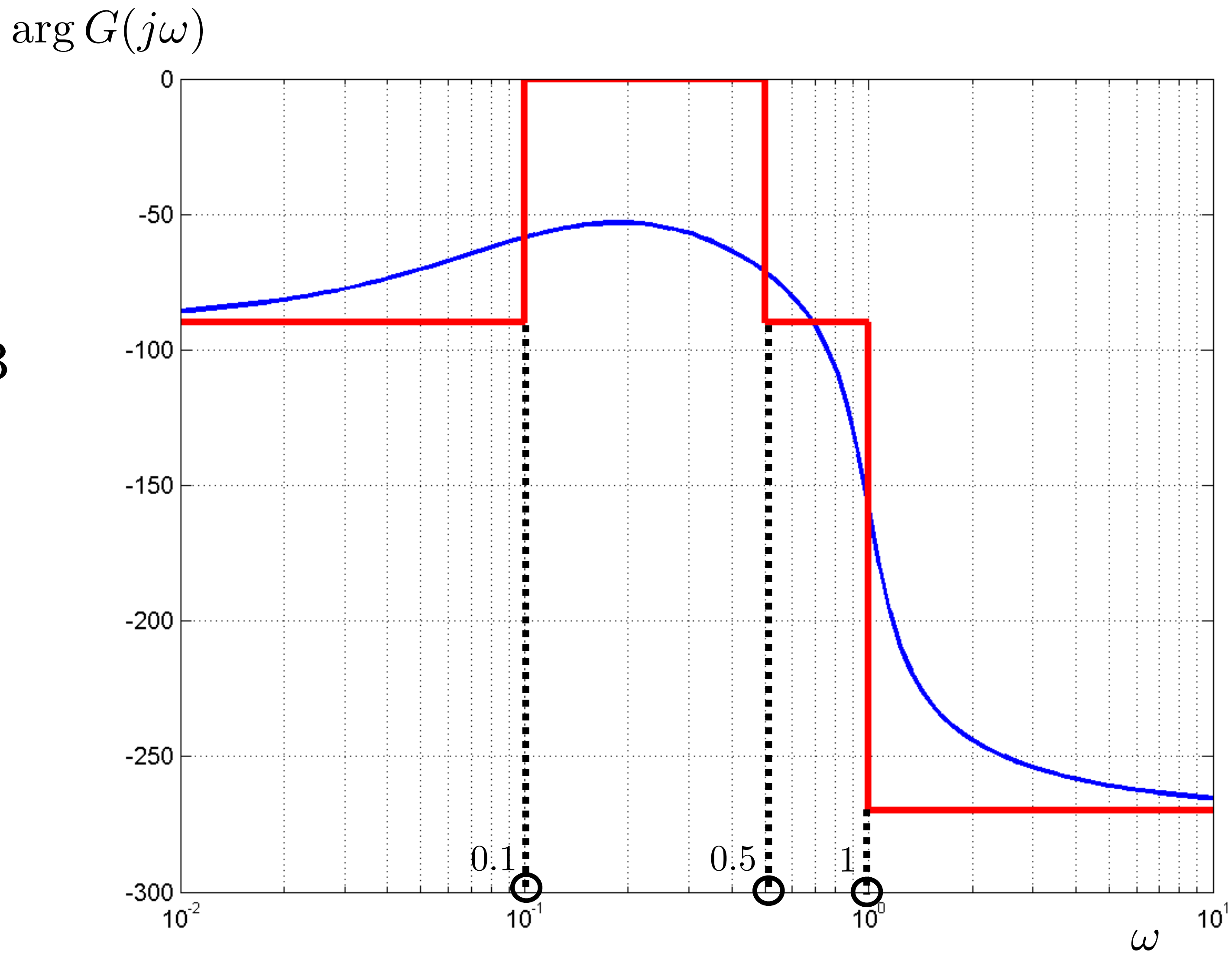
(D) $\arg(1 + j\omega T) + \arg(1 + j\omega T^*),\ T \in \mathbb{C}$



Example 2 (contd.)

$$G(s) = \frac{100(1+10s)}{s(1+2s)(1+0.4s+s^2)}$$

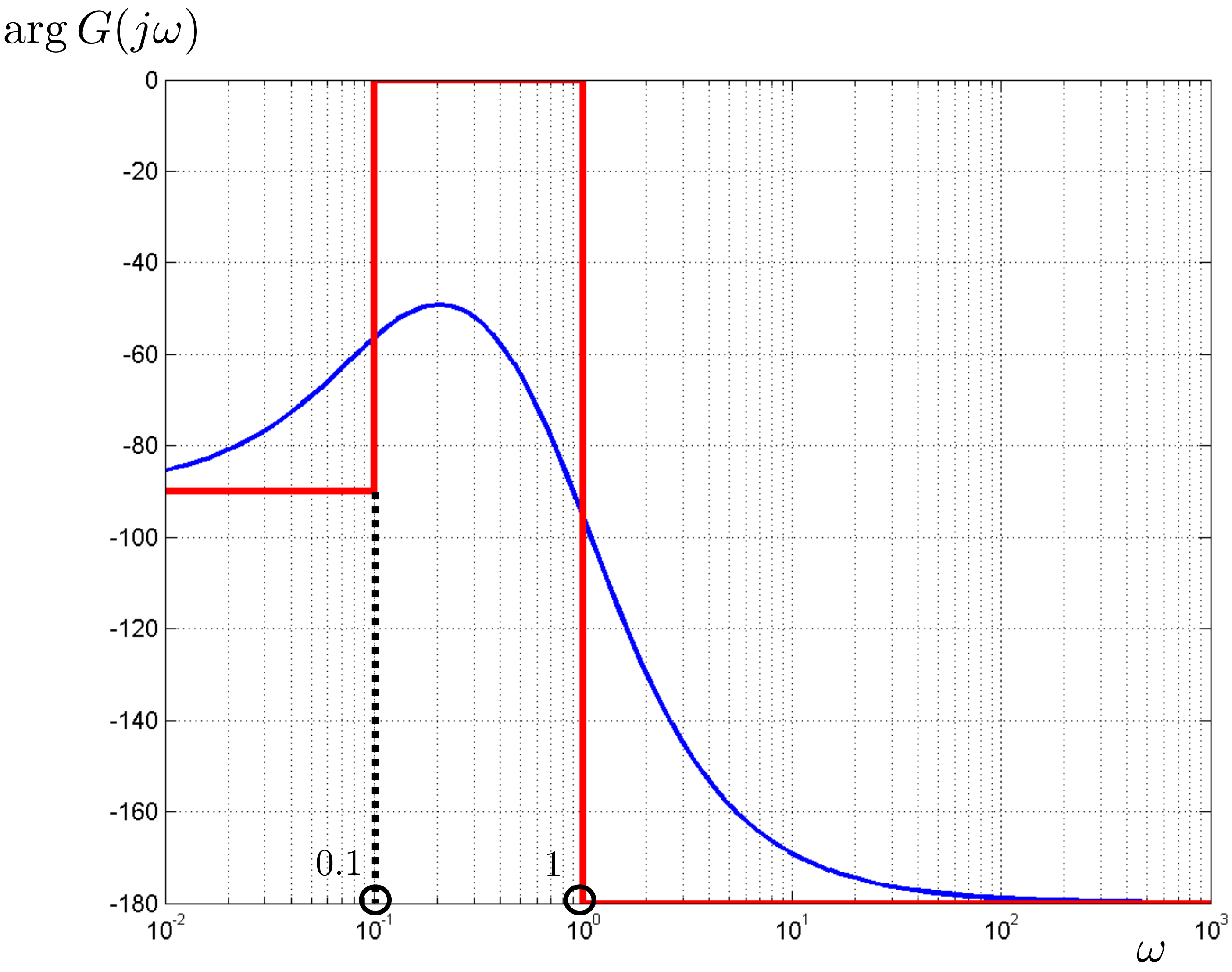
$$g = 1$$
$$\mu = 100 \implies \mu_{\text{dB}} = 40\text{dB}$$
$$z_1 = -0.1$$
$$p_1 = 0$$
$$p_2 = -0.5$$
$$p_{3,4} = -0.2 \pm j\sqrt{0.96}$$
$$\omega_n = 1;\ \xi = 0.2$$



Example 3 (contd.)

$$G(s) = \frac{100(1+10s)}{s(1+s)^2}$$

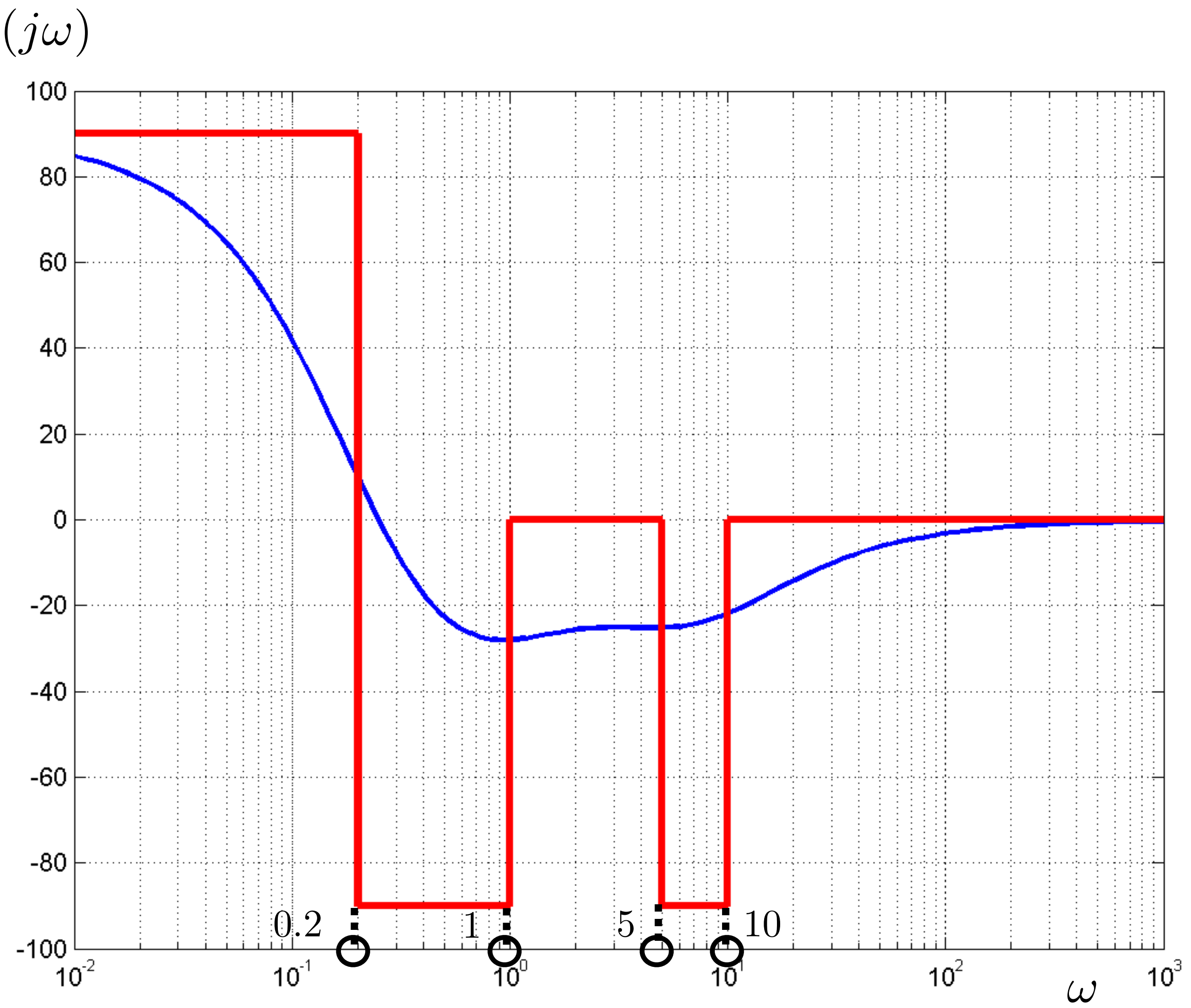
$$g = 1$$
$$\mu = 100 \implies \mu_{\text{dB}} = 40\text{dB}$$
$$z_1 = -0.1$$
$$p_1 = 0$$
$$p_2 = p_3 = -1$$



Example 4 (contd.)

$$G(s) = \frac{0.1s(1+s)}{(1+5s)^2(1+0.2s)(1-0.1s)}$$

$$g = -1$$
$$\mu = 0.1 \implies \mu_{\text{dB}} = -20\text{dB}$$
$$z_1 = 0$$
$$z_2 = -1$$
$$p_1 = p_2 = -0.2$$
$$p_3 = -5$$
$$p_4 = 10$$

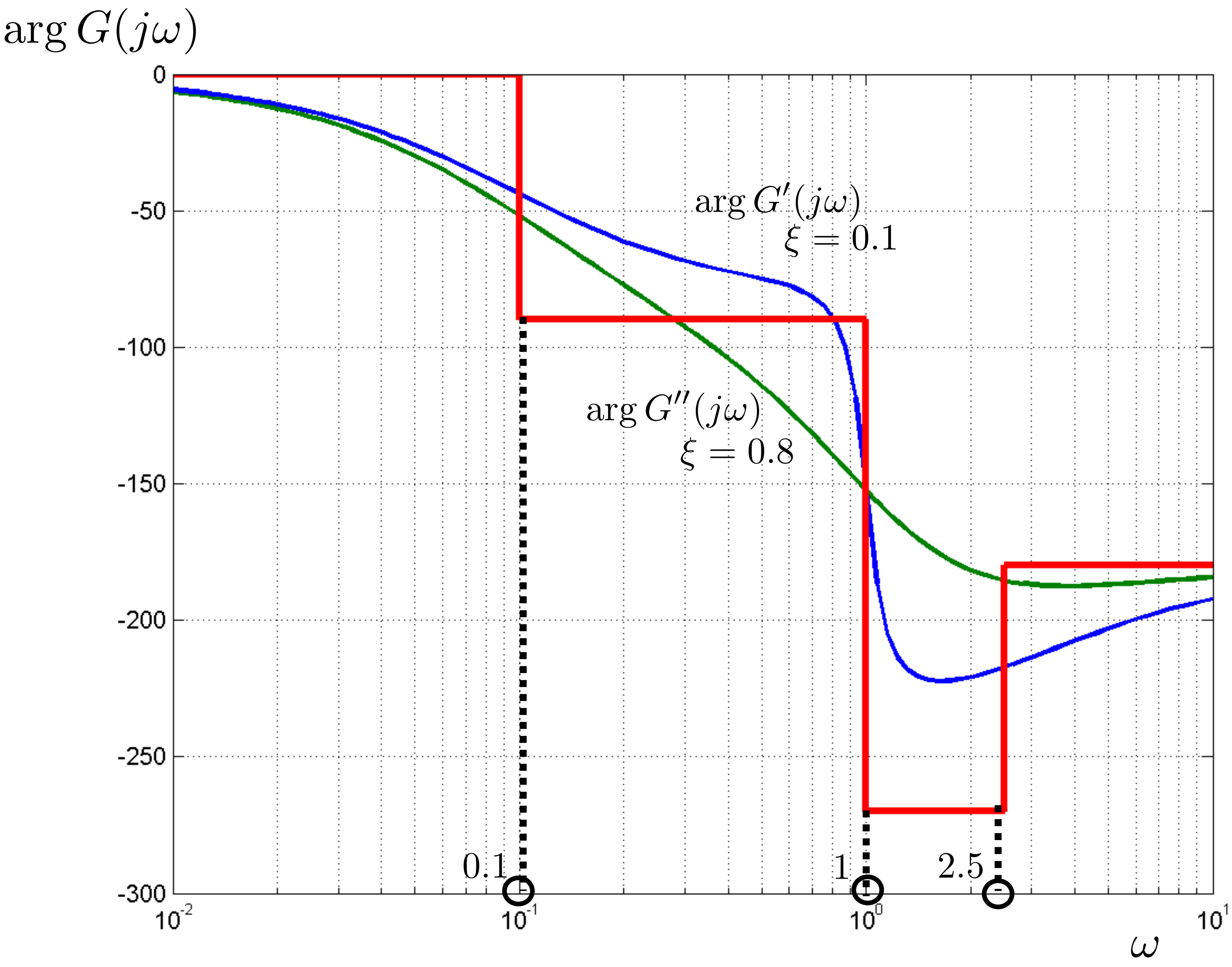


Example 5 (contd.)

$$G'(s) = \frac{50(1+0.4s)}{(1+10s)(1+0.2s+s^2)}$$

$$G''(s) = \frac{50(1+0.4s)}{(1+10s)(1+1.6s+s^2)}$$

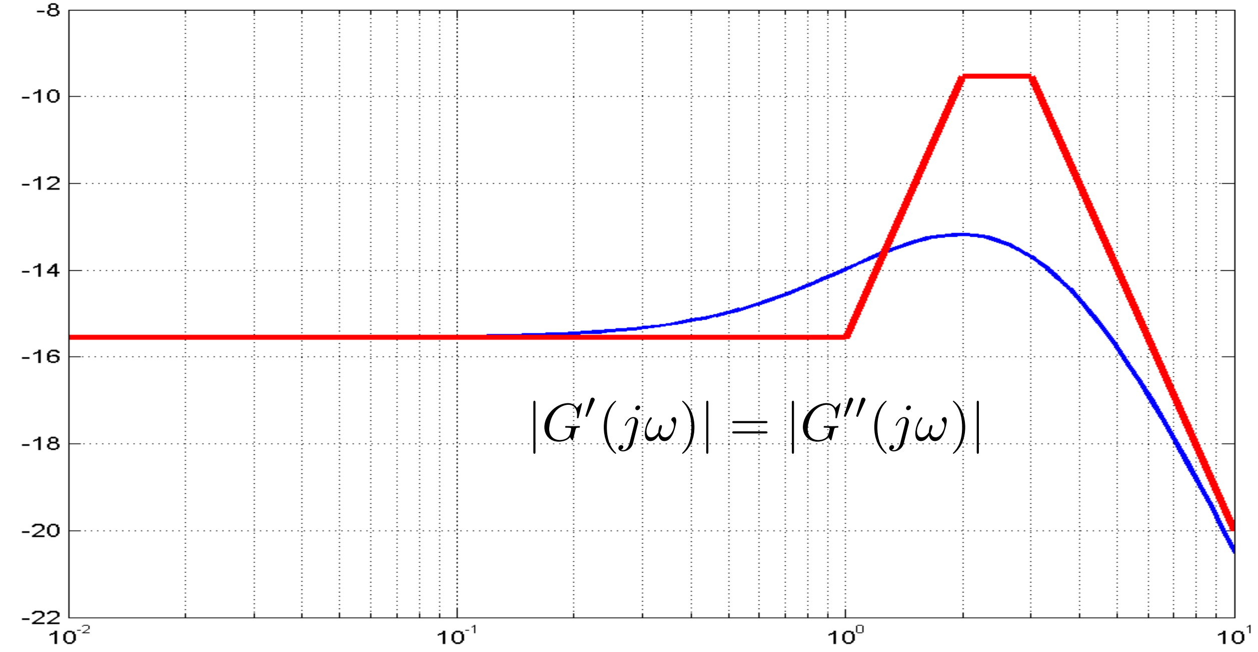
$$g = 0$$
$$\mu = 50 \implies \mu_{\mathrm{dB}} \simeq 34\mathrm{dB}$$
$$z_1 = -2.5$$
$$p_1 = -0.1$$
$$p'_{2,3} = -0.1 \pm j\sqrt{0.99}$$
$$\omega_n = 1,\ \xi = 0.1$$
$$p''_{2,3} = -0.8 \pm j\sqrt{0.36}$$
$$\omega_n = 1,\ \xi = 0.8$$

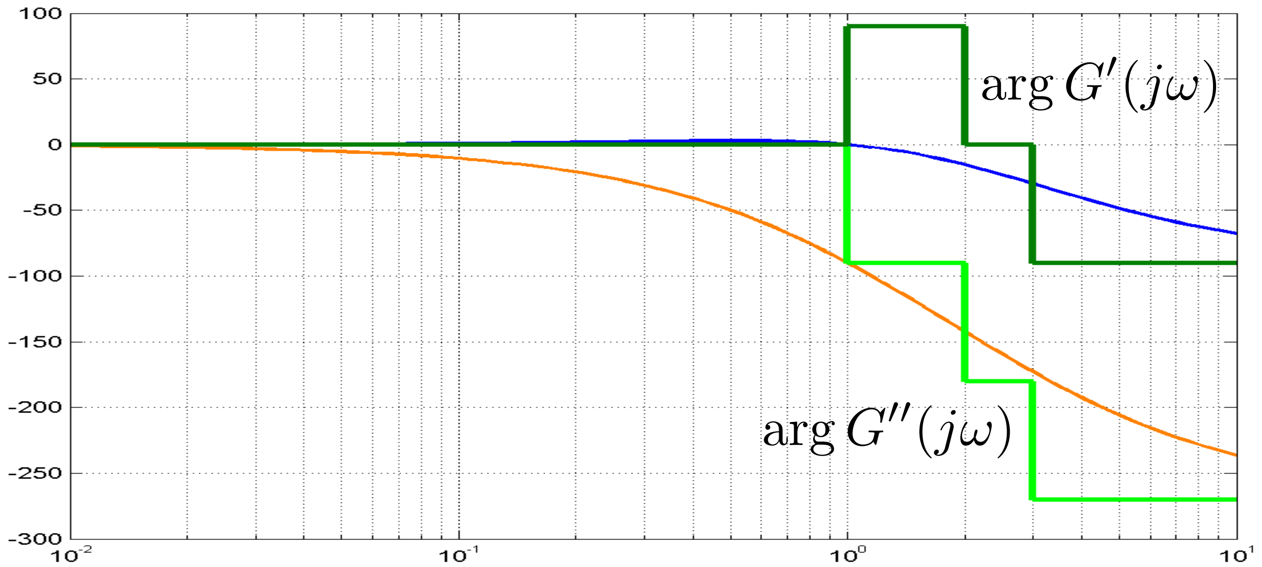


Example - Non Minimum Phase

$$G'(s) = \frac{1+s}{s^2+5s+6}$$

$$G''(s) = \frac{1-s}{s^2+5s+6}$$





Simple Rules for Plotting Asymptotic Bode Diagrams

- **Magnitude**:
  - initial slope: $-g \cdot 20\mathrm{dB/dec}$
  - initial segment intersects the point $\omega = 1, |\mu|_{\mathrm{dB}}$
  - slope changes are located in correspondence of poles and zeros:
    - zero: $+20\mathrm{dB/dec}$
    - pole: $-20\mathrm{dB/dec}$
  - final slope: $(\text{nr. zeros - nr. poles}) \cdot 20\mathrm{dB/dec}$

- **Phase**:
  - initial value: $\arg(\mu) - g\,90^\circ$
  - value changes are located in correspondence of poles and zeros:
    - zero in the left half-plane: $+90^\circ$
    - zero in the right half-plane: -90°
    - pole in the left half-plane: -90°
    - pole in the right half-plane: $+90^\circ$

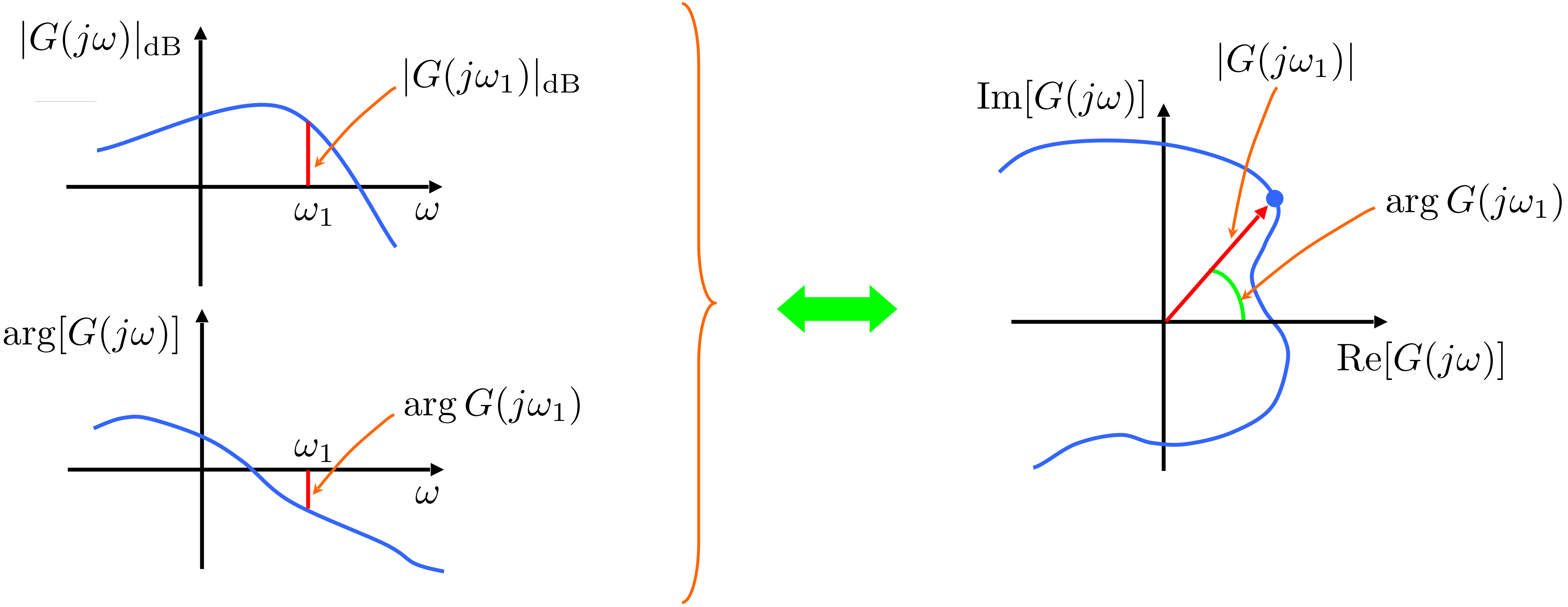
Relationship Between Magnitude and Phase

- In general there is no relation between the magnitude $|G(j\omega)|_{\mathrm{dB}}$ and the phase $\arg G(j\omega)$
- An exception is represented by **Minimum Phase Systems**:
  - positive gain: $\mu > 0$
  - all poles and zeros located in the left half-plane
- For minimum phase systems there is a direct relation among the approximate Bode diagrams of magnitude and phase:

	Slope of $|G(j\omega)|_{\mathrm{dB}}$	Value of $\arg G(j\omega)$
pole	$-20\mathrm{dB/dec}$	-90°
zero	$20\mathrm{dB/dec}$	90°

Polar Diagram of the Frequency Response

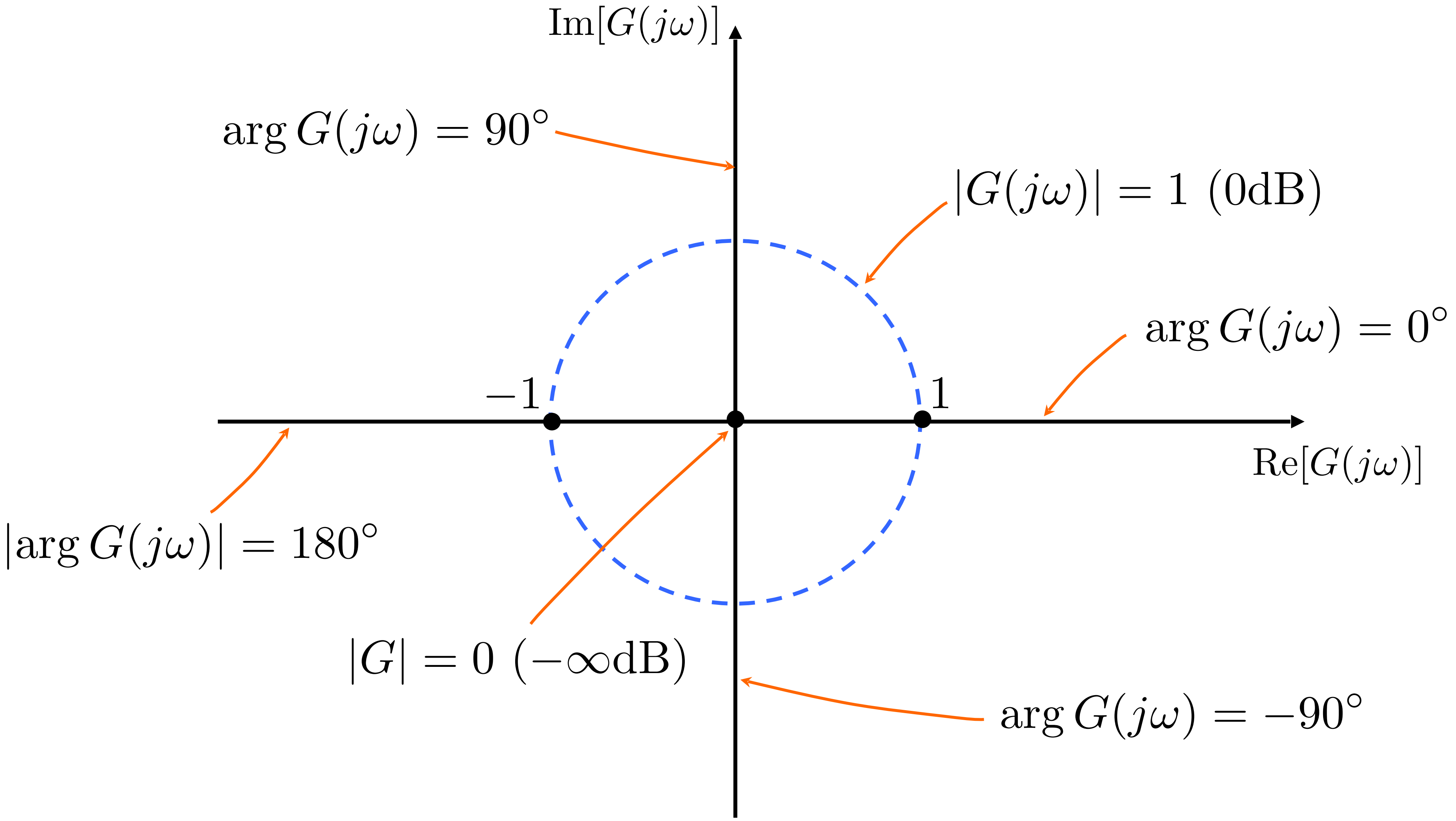
Polar Diagram: $G(j\omega)\,,\ \omega \geq 0$



Polar and Bode diagrams are related:

$$G(j\omega) = |G(j\omega)|e^{j\arg G(j\omega)}$$

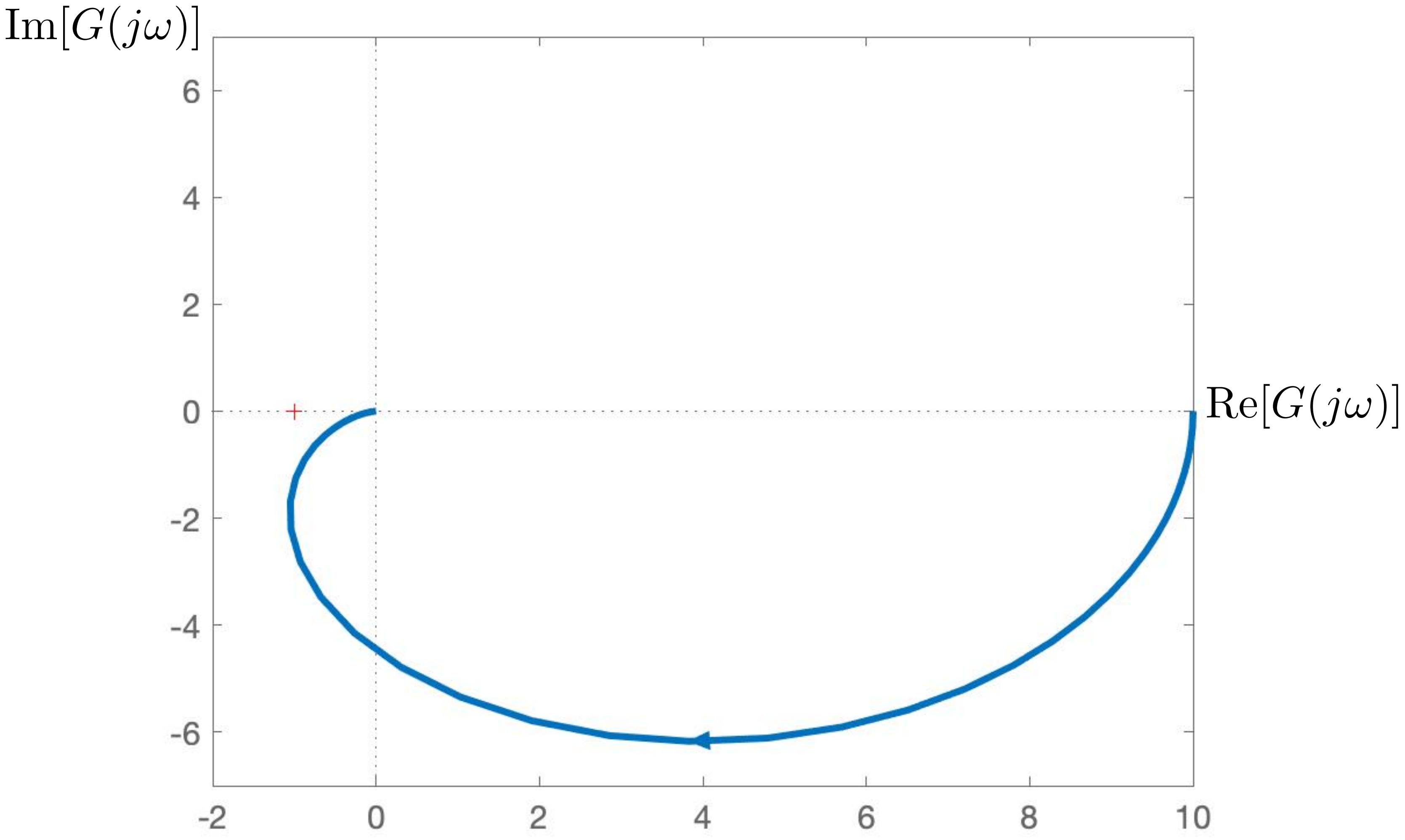
Polar Diagram: Key Elements



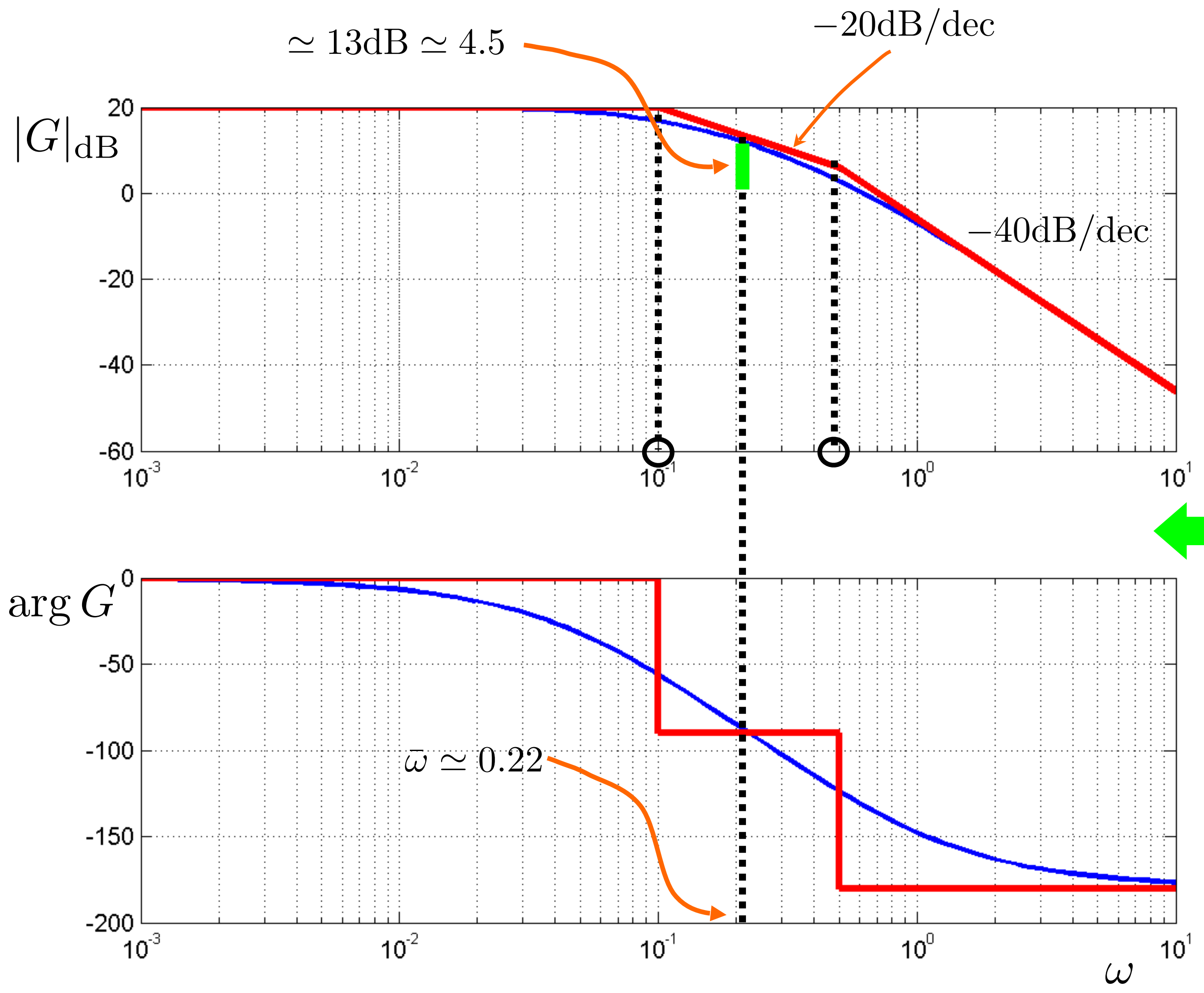
Example 1

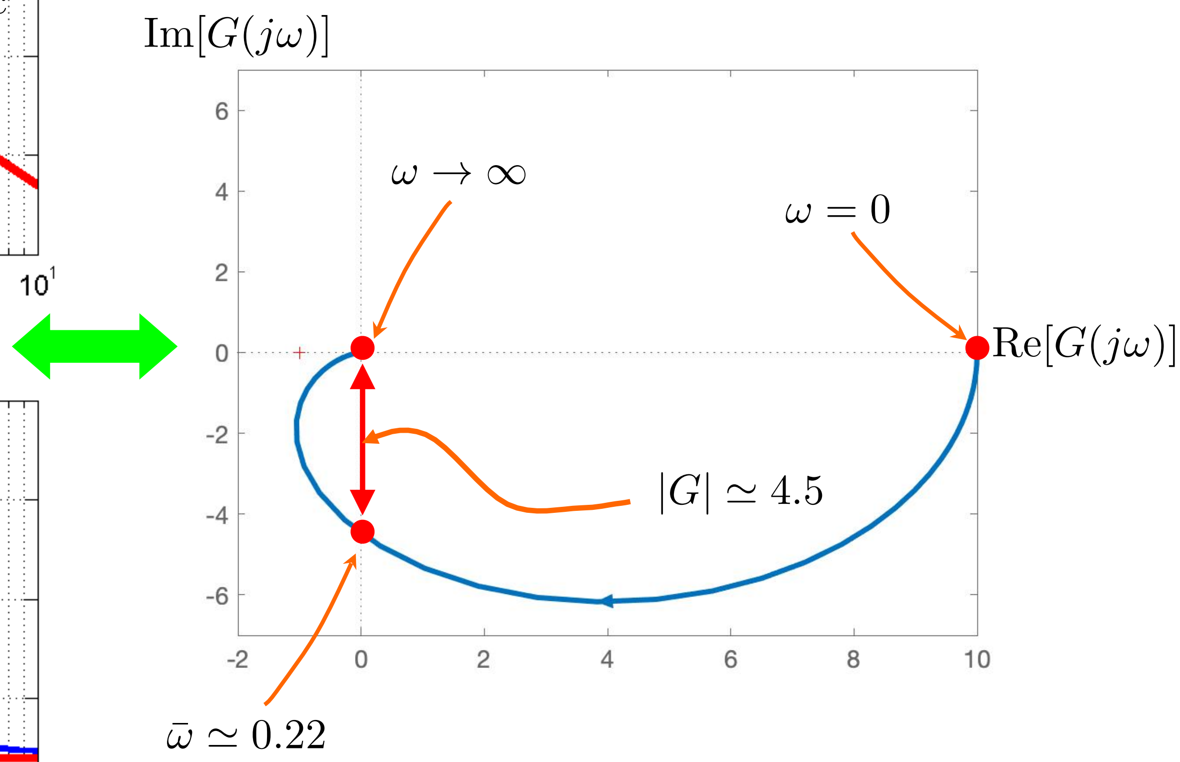
$$G(s) = \frac{10}{(1+10s)(1+2s)}$$

$$g = 0$$
$$\mu = 10 \implies \mu_{\mathrm{dB}} = 20\mathrm{dB}$$
$$\tau_1 = 10$$
$$\tau_2 = 2$$
$$\omega_1 = 0.1$$
$$\omega_2 = 0.5$$



Example 1 (contd.)





Example 2

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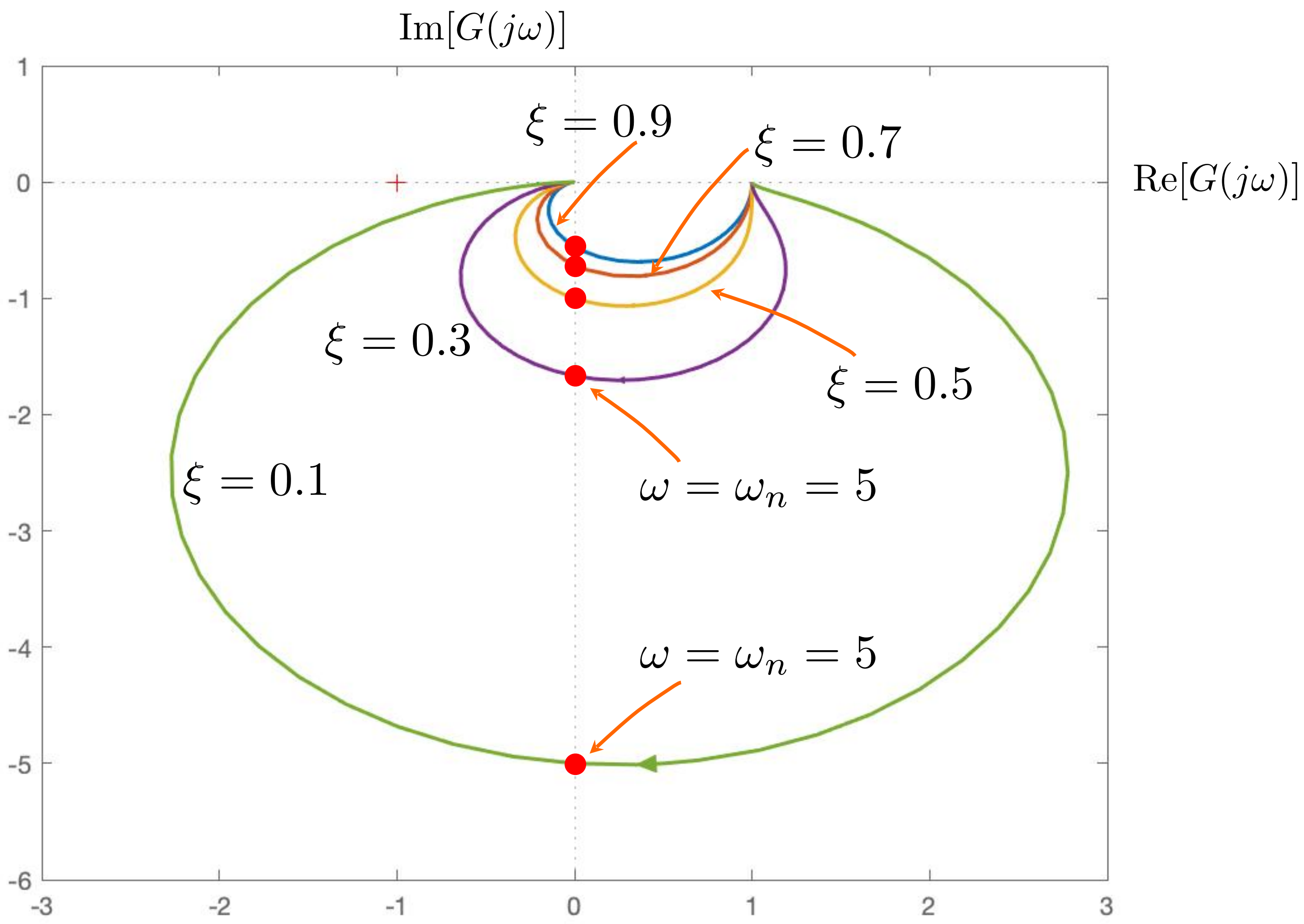
$$G(s) = \frac{25}{s^2 + 10\xi s + 25}$$

$$g = 0$$
$$\mu = G(0) = 1 \implies \mu_{\text{dB}} = 0\text{dB}$$

Consider five possible values of the damping factor:

$$\xi = 0.9$$
$$\xi = 0.7$$
$$\xi = 0.5$$
$$\xi = 0.3$$
$$\xi = 0.1$$

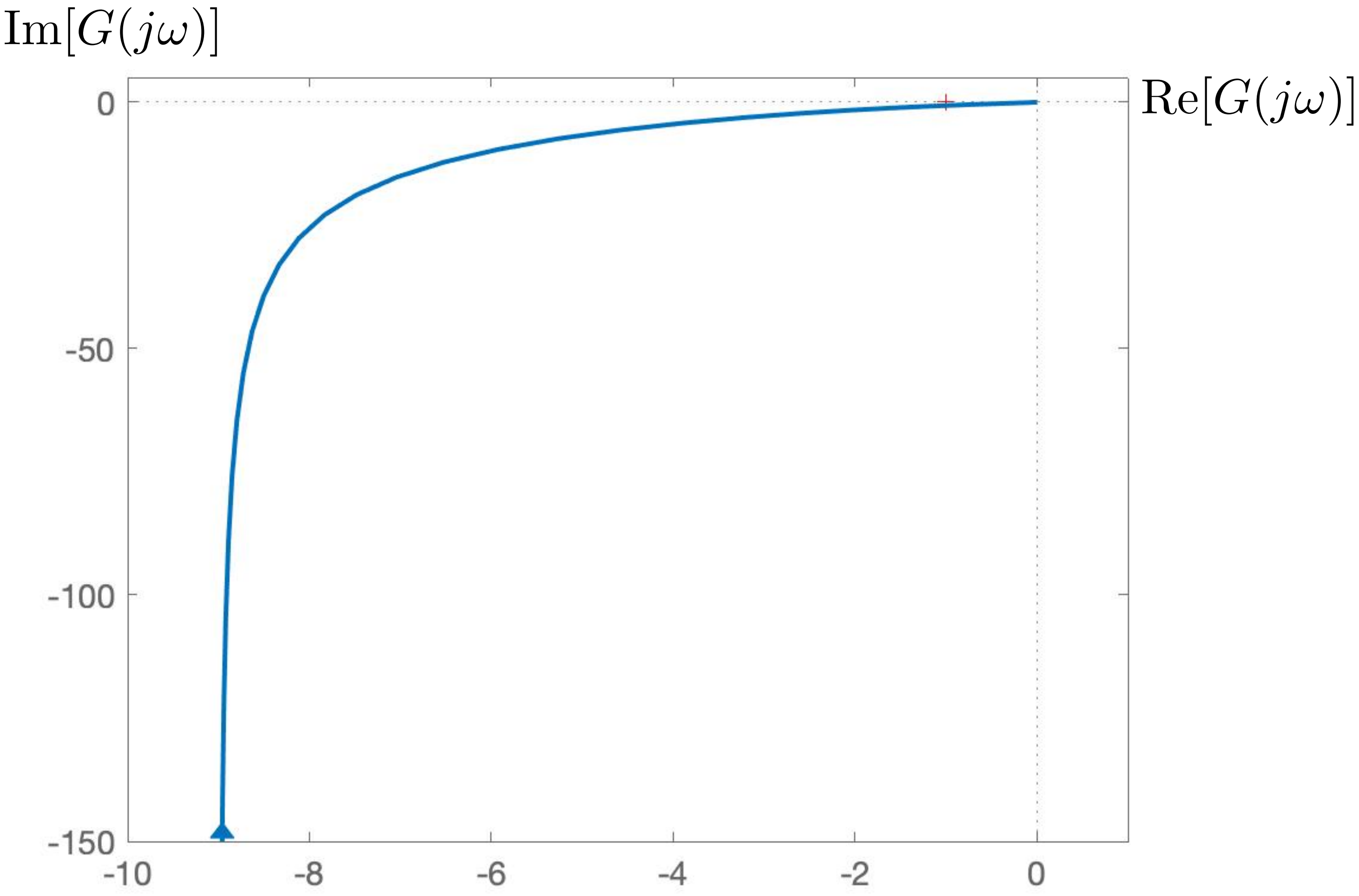
Example 2 (contd.)



Example 3

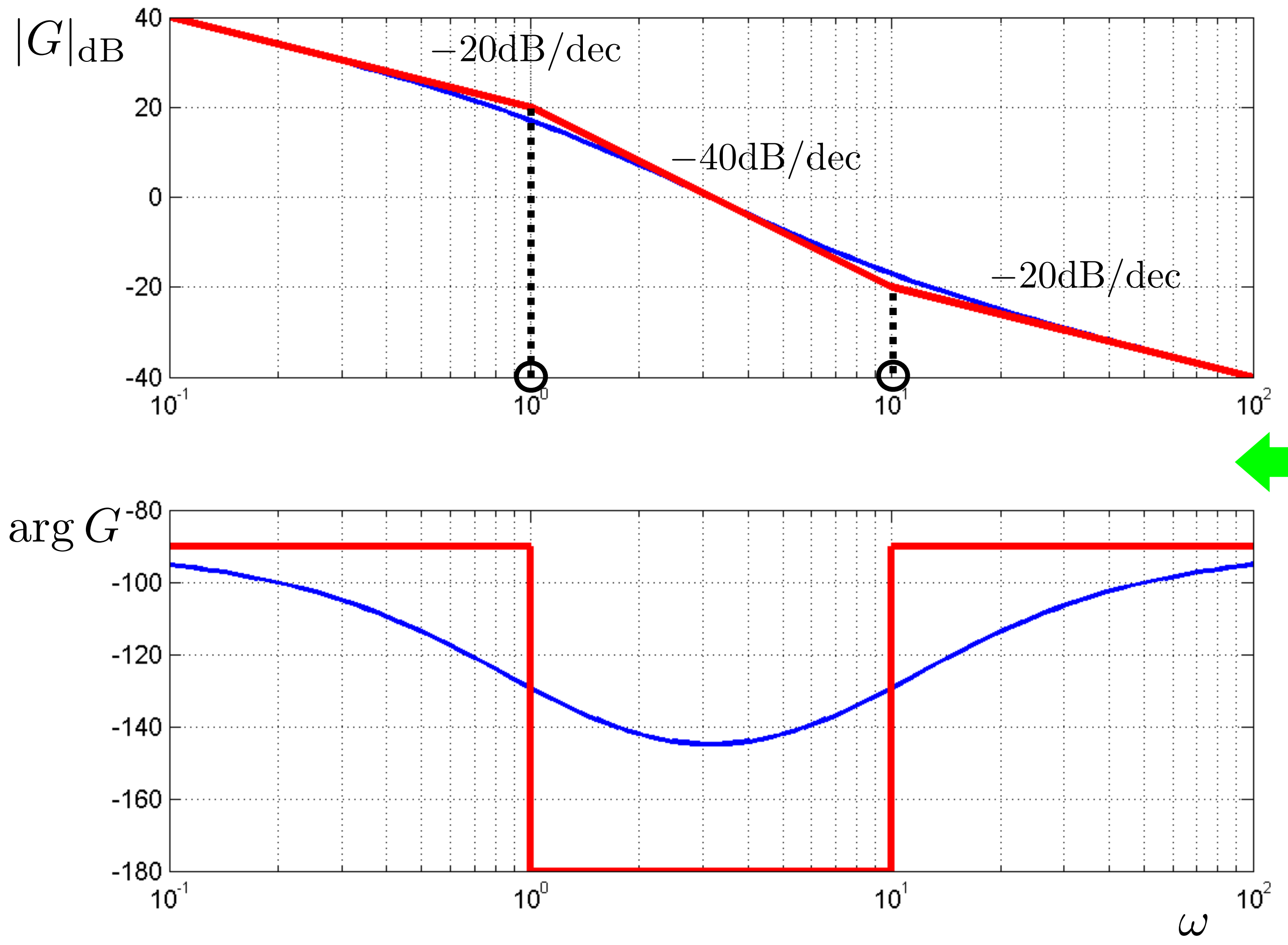
$$G(s) = \frac{s+10}{s(s+1)} = \frac{10(1+\frac{s}{10})}{s(1+s)}$$

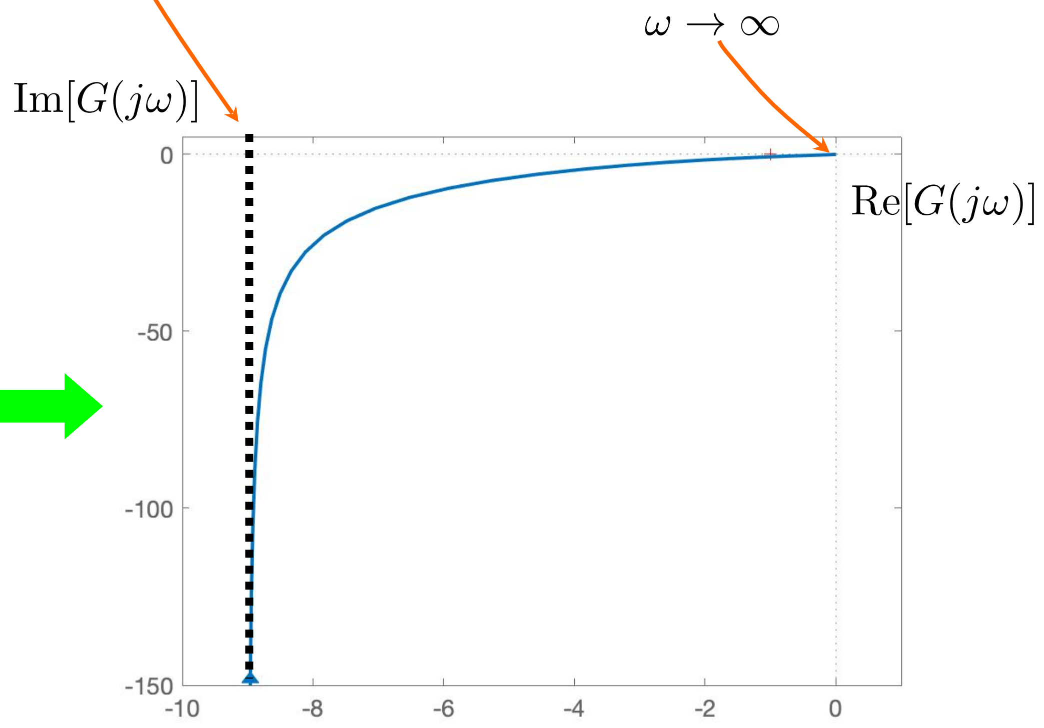
$$g = 1$$
$$\mu = 10 \implies \mu_{\text{dB}} = 20\text{dB}$$
$$T_1 = 1/10$$
$$\tau_1 = 1$$
$$\omega_1 = 1$$
$$\omega_2 = 10$$



$$G(j\omega) = \frac{j\omega + 10}{j\omega(j\omega+1)} = \frac{(-\omega + 10j)(1 - j\omega)}{-\omega(1+\omega^2)} = \frac{9}{-(1+\omega^2)} + j\frac{10+\omega^2}{-\omega(1+\omega^2)}$$

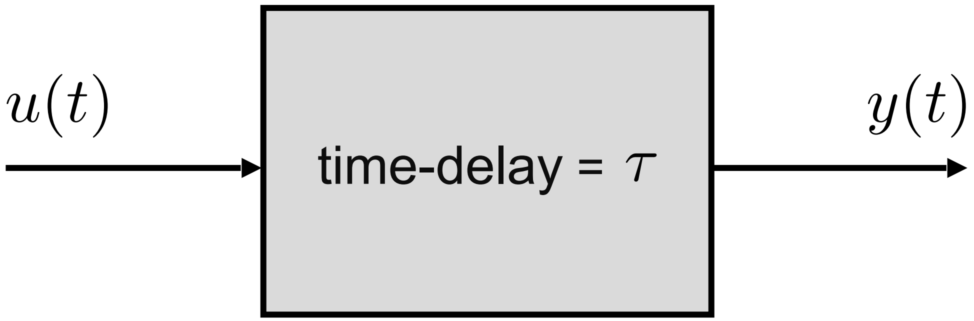
$$\mathrm{Re}\left[G(j\omega)\right] = \frac{9}{-(1+\omega^2)} \underset{\omega \to 0^+}{\longrightarrow} -9$$



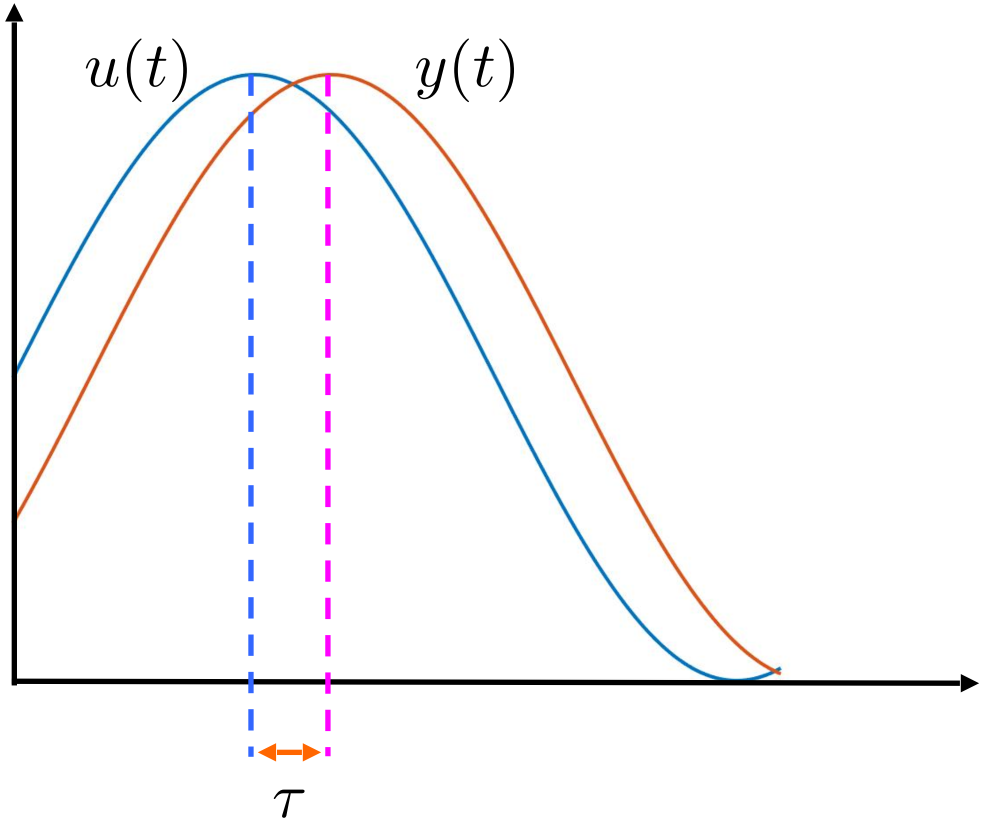


Time-Delay and Frequency Response

Consider:



$$y(t) = u(t - \tau)$$



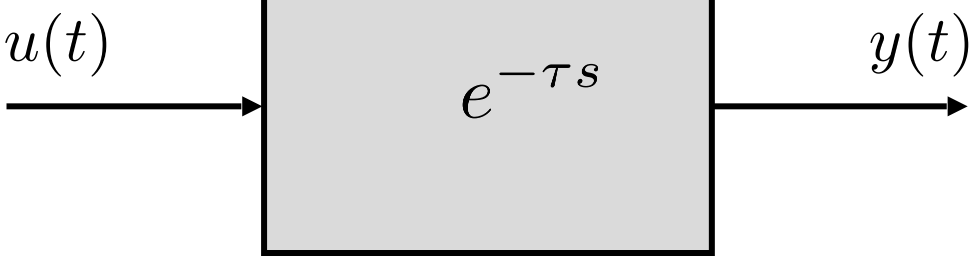
Time-Delay and Frequency Response (contd.)

Hence:

$$Y(s) = \mathcal{L}[y(t)] = \mathcal{L}[u(t-\tau)] = e^{-\tau s} \cdot U(s)$$

Letting:

$$G(s) := e^{-\tau s}$$



we are able to treat $G(s)$ as the usual transfer function (not rational though) with $G(0) = 1$

Moreover, if $u(t) = A \sin(\omega t)$

$$y(t) = A \sin[\omega(t-\tau)] = 1 \cdot A \sin(\omega t - \omega\tau)$$

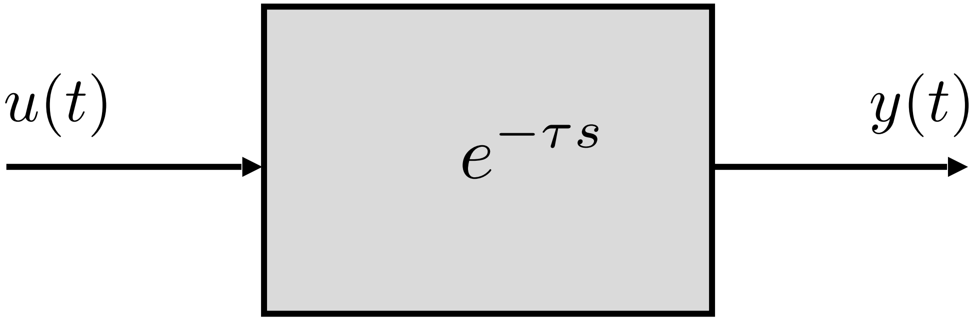
$|G(j\omega)|$ $\quad$ $\arg G(j\omega)$

The frequency response result holds!

Time-Delay and Frequency Response (contd.)

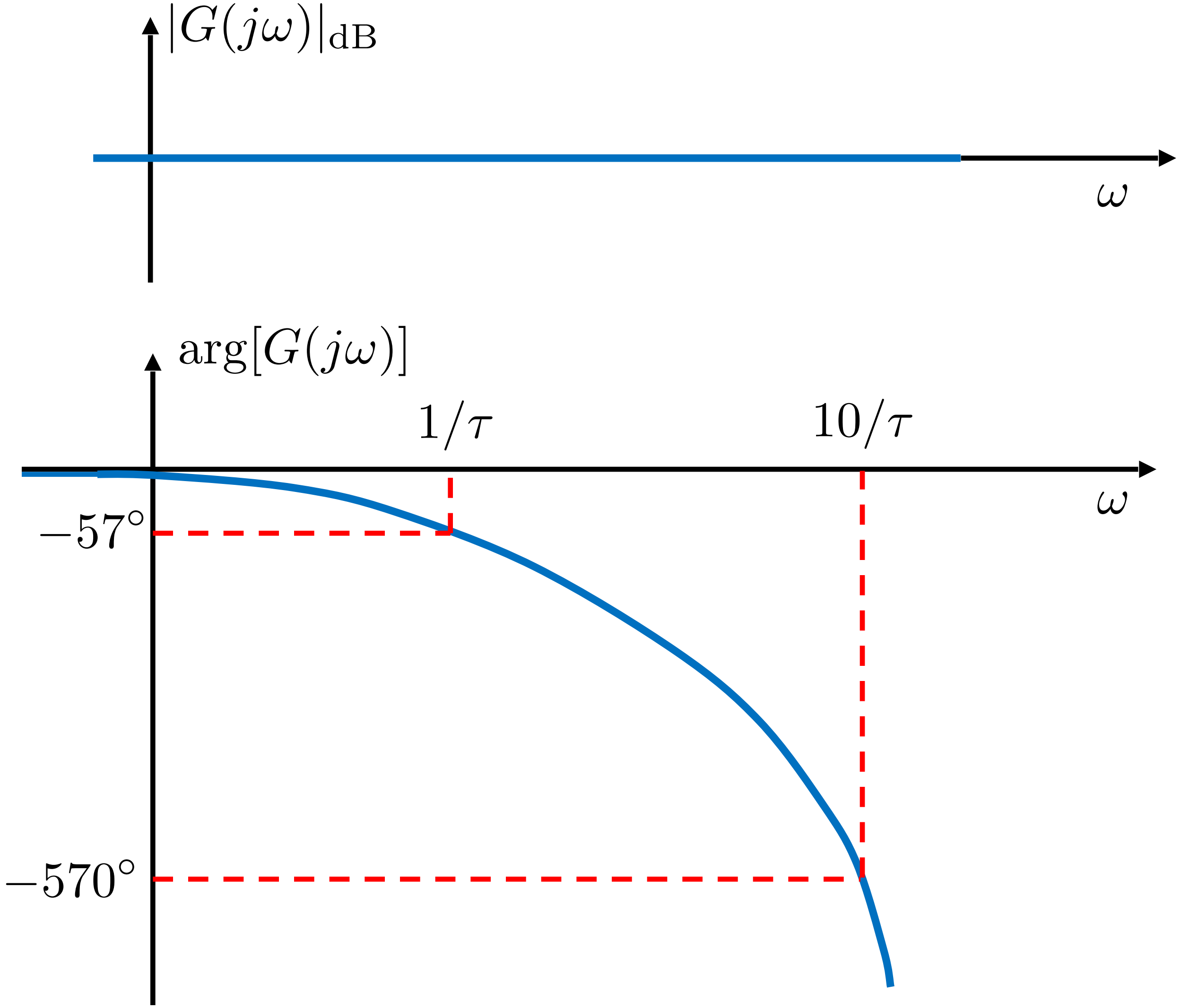
Bode diagram of the time-delay block:

$$G(s) := e^{-\tau s}$$

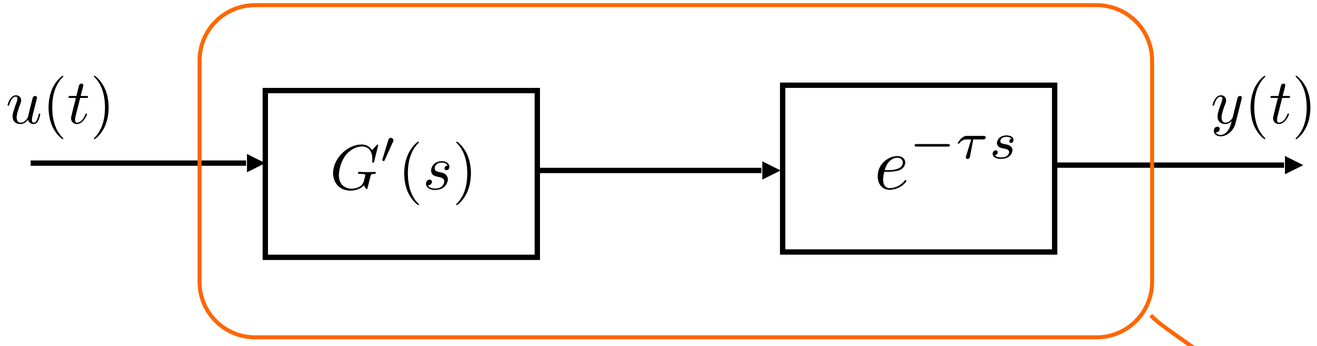


$$|G(j\omega)|_{\text{dB}} = 0$$

$$\arg G(j\omega) = -\omega\tau\frac{180}{\pi}$$



Systems with Time-Delay Blocks



$$G(s) = G'(s) \cdot e^{-\tau s}$$

$$|G(j\omega)| = |G'(j\omega)| \cdot \underbrace{|e^{j\omega\tau}|}_{1} = |G'(j\omega)|$$

$$\arg G(j\omega) = \arg G'(j\omega) - \omega\tau\frac{180}{\pi}$$