Exercises on Eigenvalues & Eigenvectors, Matrix Exponential and Characteristic Polynomial of a Matrix

Exercise 1

$$A = \begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix} \quad \Longrightarrow \quad p_A(s) = 0\,,\ s_{1,2} = ? \qquad e^{At} = ?$$

Let's determine the eigenvalues (and later the eigenvectors), and the matrix exponential by exploiting the Symbolic Math Toolbox .

```
syms s

A = sym([ -2,  6; ...
          -2,  5])
```

A =

$$\begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix}$$

```
% the characteristic polynomial
p_A(s) = charpoly(A, s)
```

p_A(s) = $s^2 - 3\,s + 2$

```
% ALTERNATIVE way
ALPp_A(s) = det(s*eye(size(A))-A)
```

ALPp_A(s) = $s^2 - 3\,s + 2$

```
% let's determine the eigenvalues of A
% standard approach:
% using the MATLAB function "eig" [Symbolic Math
% Toolbox]
sA = eig(A)
```

sA =

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

```
s1 = sA(1)
```

s1 = 1

```
s2 = sA(2)
```

s2 = 2

```
% Alternative way: eigenvalues as roots of the characteristic polynomial
ALT_sA = solve(p_A,s)
```

ALT_sA =

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

```
% let's determine the eigenvectors, using the Symbolic Math Toolbox
help sym/eig
```

```
 sym/eig - Eigenvalues and eigenvectors of symbolic matrix
    This MATLAB function returns the eigenvalues of the square symbolic
    matrix A as a symbolic vector.

    Syntax
      lambda = eig(A)
      [V,D] = eig(A)
      [V,D,p] = eig(A)

    Input Arguments
      A - Square matrix
        symbolic matrix

    Output Arguments
      lambda - Eigenvalues (returned as vector)
        symbolic column vector | column vector of symbolic numbers
      V - Right eigenvectors
        square symbolic matrix
      D - Eigenvalues (returned as matrix)
        symbolic diagonal matrix
      p - Vector of indices
        symbolic row vector

    Examples
      Compute Eigenvalues
      Compute Numeric Eigenvalues to High Precision
      Convert Symbolic Eigenvalues to Floating Point
      Compute Eigenvalues and Eigenvectors of Defective Matrix
      Find Linearly Independent Eigenvectors and Their Eigenvalues

    See also charpoly, jordan, svd, vpa

    Introduced in Symbolic Math Toolbox before R2006a
    Documentation for sym/eig
```

```
[V, D, p] = eig(A)
```

V =

$$\begin{bmatrix} 2 & \frac{3}{2} \\ 1 & 1 \end{bmatrix}$$

D =

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

p = 1×2

```
     1     2
```

The length of the row-array p is the total number of linearly independent eigenvectors of A.

Moreover:

- $p(1) = 1$: the first eigenvector (the first column of V, i.e., $V(:, 1)$) corresponds to the first element on the main diagonal of D, i.e., to $D(1, 1)$.
- $p(2) = 2$: the second eigenvector (the second column of V, i.e., $V(:, 2)$) corresponds to the second element on the main diagonal of D, i.e., to $D(2, 2)$.

```
% let's compute the diagonal form of the matrix A
v1 = V(:,1);
v2 = V(:,2);

matT = V; % the full-rank transformation matrix
matT1 = inv(matT); % and the corresponding inverse matrix
```

$\widetilde{A} = T^{-1} A T, \qquad T = [v_1 \mid v_2]$

```
diagA = matT1 * A *matT
```

```
diagA =
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

```
% computation of the matrix exponential
help sym/expm
```

```
 sym/expm - Matrix exponential of symbolic matrices
    This MATLAB function computes the matrix exponential of the square
    symbolic matrix A.

    Syntax
      R = expm(A)

    Input Arguments
      A - Input matrix
        square matrix

    Output Arguments
      R - Resulting matrix
        symbolic matrix

    See also eig, funm, jordan, logm, sqrtm

    Introduced in Symbolic Math Toolbox before R2006a
    Documentation for sym/expm
```

```
expA = expm(A*t)
```

```
expA =
```

$$\begin{bmatrix} 4\,e^{t} - 3\,e^{2t} & 6\,e^{2t} - 6\,e^{t} \\ 2\,e^{t} - 2\,e^{2t} & 4\,e^{2t} - 3\,e^{t} \end{bmatrix}$$

```
% ALTERNATIVE way
help sym/diag
```

```
 sym/diag - Create diagonal matrix or get diagonals from symbolic matrices
    This MATLAB function returns a square diagonal matrix with vector v as
    the main diagonal.

    Syntax
      D = diag(v)
      D = diag(v,k)

      x = diag(A)
      x = diag(A,k)

    Input Arguments
      v - Diagonal elements
        symbolic vector
      A - Input matrix
        symbolic matrix
      k - Diagonal number
        integer

    Examples
      Create Matrix with Diagonal as Vector
      Create Matrix with Subdiagonal as Vector
      Extract Diagonal from Matrix
      Extract Superdiagonal from Matrix

    See also tril, triu

    Introduced in Symbolic Math Toolbox before R2006a
    Documentation for sym/diag
```

```
exp_diagA = diag([exp(s1*t), exp(s2*t)])
```

exp_diagA =

$$\begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$$

```
% let's apply the formula on Part 3 - p.16, p. 20
ALPexpA = matT*exp_diagA*matT1
```

ALPexpA =

$$\begin{bmatrix} 4\,e^t - 3\,e^{2t} & 6\,e^{2t} - 6\,e^t \\ 2\,e^t - 2\,e^{2t} & 4\,e^{2t} - 3\,e^t \end{bmatrix}$$

Exercise 2

$$A = \begin{bmatrix} -2 & 4 & -1 \\ -1 & -2 & 3 \\ 0 & 0 & -5 \end{bmatrix} \implies p_A(s) = 0\,,\ s_{1,2,3} = ? \qquad e^{A\,t} = ?$$

Let's determine the eigenvalues (and later the eigenvectors), and the matrix exponential by exploiting the Symbolic Math Toolbox .

How do you determine the trace of the A matrix?

```
clear variables
syms s t

A = sym([ -2,  4,  -1; ...
         -1,  -2,  3; ...
          0,  0,  -5])
```

A =

$$\begin{bmatrix} -2 & 4 & -1 \\ -1 & -2 & 3 \\ 0 & 0 & -5 \end{bmatrix}$$

```
sA = eig(A)
```

sA =

$$\begin{bmatrix} -5 \\ -2-2\,\mathrm{i} \\ -2+2\,\mathrm{i} \end{bmatrix}$$

```
pA(s) = charpoly(A,s)
```

pA(s) = $s^3 + 9\,s^2 + 28\,s + 40$

```
[V, D, p] = eig(A)
```

V =

$$\begin{bmatrix} \frac{15}{13} & 2\,\mathrm{i} & -2\,\mathrm{i} \\ -\frac{8}{13} & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

D =

$$\begin{bmatrix} -5 & 0 & 0 \\ 0 & -2-2\,\mathrm{i} & 0 \\ 0 & 0 & -2+2\,\mathrm{i} \end{bmatrix}$$

p = 1×3

1	2	3

The length of the row-array p is the total number of linearly independent eigenvectors of A.

Moreover:

- $p(1) = 1$: the first eigenvector (the first column of V, i.e., $V(:,1)$) corresponds to the first element on the main diagonal of D, i.e., to $D(1,1)$.
- $p(2) = 2$: the second eigenvector (the second column of V, i.e., $V(:,2)$) corresponds to the second element on the main diagonal of D, i.e., to $D(2,2)$.
- $p(3) = 3$: the second eigenvector (the second column of V, i.e., $V(:,3)$) corresponds to the third element on the main diagonal of D, i.e., to $D(3,3)$.

```
matT = V; % the full-rank transformation matrix
matT1 = inv(matT); % and the corresponding inverse matrix

diagA = matT1 * A *matT
```

diagA =

$$\begin{bmatrix} -5 & 0 & 0 \\ 0 & -2-2\,\mathrm{i} & 0 \\ 0 & 0 & -2+2\,\mathrm{i} \end{bmatrix}$$

```
expA = simplify(expm(A*t),500)
```

expA =

$$\begin{bmatrix} \sigma_1 & 2\sin(2\,t)\,\mathrm{e}^{-2\,t} & \frac{15\,\mathrm{e}^{-5\,t}}{13} - \frac{15\cos(2\,t)\,\mathrm{e}^{-2\,t}}{13} + \frac{16\sin(2\,t)\,\mathrm{e}^{-2\,t}}{13} \\ -\frac{\sin(2\,t)\,\mathrm{e}^{-2\,t}}{2} & \sigma_1 & \frac{8\cos(2\,t)\,\mathrm{e}^{-2\,t}}{13} - \frac{8\,\mathrm{e}^{-5\,t}}{13} + \frac{15\sin(2\,t)\,\mathrm{e}^{-2\,t}}{26} \\ 0 & 0 & \mathrm{e}^{-5\,t} \end{bmatrix}$$

where

$$\sigma_1 = \cos(2\,t)\,\mathrm{e}^{-2\,t}$$

```
diagA_exp = expm(diagA*t)
```

diagA_exp =

$$\begin{bmatrix} \mathrm{e}^{-5\,t} & 0 & 0 \\ 0 & \mathrm{e}^{t\,(-2-2\,\mathrm{i})} & 0 \\ 0 & 0 & \mathrm{e}^{t\,(-2+2\,\mathrm{i})} \end{bmatrix}$$

What about the trace of A?

```
trA = trace(A)
```

trA = -9

```
sum(sA)
```

ans = -9

Exercise 3

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & -10 \end{bmatrix} \implies p_A(s) = 0\,,\ s_{1,2,3,4,5} = ? \qquad e^{A\,t} = ?$$

Let's determine the eigenvalues, the eigenvectors, and the matrix exponential by exploiting the Symbolic Math Toolbox .

Does this matrix admit the diagonal form?

```
clear variables
syms t

A = sym([-2,  1,  0,  0,  0; ...
         0, -2,  0,  0,  0; ...
         0,  0, -5,  1,  0; ...
         0,  0,  0, -5,  0; ...
         0,  0,  0,  0, -10])
```

A =

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & -10 \end{bmatrix}$$

```
[eigV, eigVALDiag, p ] = eig(A)
```

eigV =

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

eigVALDiag =

$$\begin{bmatrix} -10 & 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

p = 1×3

```
     1     2     4
```

The length of the row-array p is the total number of linearly independent eigenvectors of A. This time we obtained only 3 independent eigenvectors: thus, there is no way to obtain the diagonal form of the matrix A, because we can not assembly a full-rank, square transformation matrix T such that $\widetilde{A} = T^{-1} \cdot A \cdot T$ is the diagonal form of A.

Moreover:

- $p(1) = 1$: the first eigenvector (the first column of V, i.e., $V(:, 1)$) corresponds to the first element on the main diagonal of D, i.e., to $D(1, 1)$.
- $p(2) = 2$: the second eigenvector (the second column of V, i.e., $V(:, 2)$) corresponds to the second element on the main diagonal of D, i.e., to $D(2, 2)$.
- $p(3) = 4$: the second eigenvector (the third column of V, i.e., $V(:, 3)$) corresponds to the fourth element on the main diagonal of D, i.e., to $D(4, 4)$.

Exploiting the MATLAB function expm, we are able to determine the matrix exponential, as follows:

```
expAnoDiag = expm(A*t)
```

```
expAnoDiag =
```

$$\begin{bmatrix} e^{-2t} & t\,e^{-2t} & 0 & 0 & 0 \\ 0 & e^{-2t} & 0 & 0 & 0 \\ 0 & 0 & e^{-5t} & t\,e^{-5t} & 0 \\ 0 & 0 & 0 & e^{-5t} & 0 \\ 0 & 0 & 0 & 0 & e^{-10t} \end{bmatrix}$$