Lesson 17 - 10/4/2026

Convolution of distributions

goal: give a meaning to $S * T$ where $S \in \mathcal{D}'(\mathbb{R}^n)$, $T \in \mathcal{E}'(\mathbb{R}^n)$

rem. $T \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$

give a meaning to $T * \varphi$

if $T = T_f$ with $f \in L^1(\mathbb{R}^n)$

$$T_f * \varphi(x) \underset{\text{cunt}}{=} f * \varphi(x) = \int_{\mathbb{R}^n} f(y) \cdot \varphi(x-y)\, dy = T_f(\varphi(x - \cdot))$$

parameter

variable "of integration"

$$T_f * \varphi(x) = T_f(\varphi(x-\cdot))$$

def. let $T \in \mathcal{D}'(\mathbb{R}^n)$, let $\varphi \in \mathcal{D}(\mathbb{R}^n)$

we define $(T * \varphi)(x) = T(\varphi(x - \cdot))$ | $T(\varphi_x)$, $\varphi_x(y) = \varphi(x-y)$

Th. let $T \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$

then $T * \varphi \in \mathcal{E}(\mathbb{R}^n) (= \mathcal{C}^\infty(\mathbb{R}^n))$

proof. I start showing that $T * \varphi$ is continuous function.

consider $(x_n)_n$ in $\mathbb{R}^n$, $\bar{x} \in \mathbb{R}^n$ s.t. $x_n \xrightarrow{n} \bar{x}$

I show that $T * \varphi(x_n) \xrightarrow{n} T * \varphi(\bar{x})$

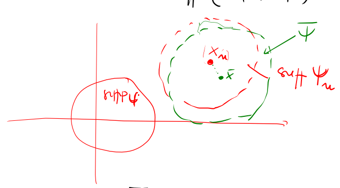
I define $\varphi_n(y) = \varphi(x_n - y)$

$\bar{\varphi}(y) = \varphi(\bar{x} - y)$

I consider $(\varphi_n - \bar{\varphi})_n$ this is a sequence in $\mathcal{D}(\mathbb{R}^n)$

1) $\exists K$ compact s.t. $\forall n$

supp $(\varphi_n - \bar{\varphi}) \subseteq K$



2) $\varphi_n - \bar{\varphi} \to 0$ uniformly

and similarly $D^\alpha(\varphi_n - \bar{\varphi}) \longrightarrow 0$ uniformly $\forall \alpha$

so that $(\varphi_n - \bar{\varphi}) \to 0$ in the sense of $\mathcal{D}(\mathbb{R}^n)$

so that $T(\varphi_n - \bar{\varphi}) \to 0$ in $\mathbb{C}$

$\Rightarrow$ $T * \varphi$ is continuous

How to prove that $T * \varphi$ is differentiable?

Sketch $(n = 1)$

consider $T * \varphi(x + h) = T(\varphi(x + h - \cdot))$

$T * \varphi(x) = T(\varphi(x - \cdot))$

$$\frac{T * \varphi(x+h) - T * \varphi(x)}{h} - T * \varphi'(x)$$

$$T\Big(\underbrace{\frac{\varphi(x+h-\cdot) - \varphi(x-\cdot)}{h} - \varphi'(x - \cdot)}_{\psi_h}\Big) = T(\psi_h)$$

what happens for $\lim_{h \to 0} T(\psi_h)$

$$T\Big(\underbrace{\frac{\varphi(x+e_1-\cdot)-\varphi(x-\cdot)}{e_1}-\varphi'(x-\cdot)}_{\psi_{e_1}}\Big)=T(\psi_{e_1})$$

what happens for $\lim_{e_1\to 0} T(\psi_{e_1})$

as before • $\exists K$ compact s.t. $\forall e_1$, $\operatorname{supp}\psi_{e_1}\subseteq K$

• $\psi_{e_1}\xrightarrow{e_1\to 0} 0$ uniformly

$\psi_{e_1}^{(j)}\longrightarrow 0$ uniformly $\forall j$

So that $\lim_{e_1} T(\psi_{e_1})=0$

$$\lim_{e_1}\frac{T*\varphi(x+e_1)-T*\varphi(x)}{e_1}=T*\varphi'(x)$$

and so on

Corollary $\partial_{x_j}(T*\varphi)=T*\partial_{x_j}\varphi=(\partial_{x_j}T)*\varphi$

proved here above ?

$$(T*\partial_{x_j}\varphi)(x)=T\big((\partial_{x_j}\varphi)(x-\cdot)\big)=T\big((-\partial_{y_j}\varphi)(x-\cdot)\big)$$
$$=(\partial_{x_j}T*\varphi)(x)$$

Th. Let $T\in\mathcal{D}'(\mathbb{R}^n)$

consider $T*:\mathcal{D}(\mathbb{R}^n)\to\mathcal{E}(\mathbb{R}^n)$

this operator is linear and continuous

proof linearity: easy

continuity ?

it is sufficient to prove that

if $(\varphi_n)_n$ is a sequence in $\mathcal{D}(\mathbb{R}^n)$ which goes to 0 (in the sense $\nabla\alpha$)

then $\tilde{p}_j(T*\varphi_n)\xrightarrow{n}0$

where $\tilde{p}_j$ is a seminorm of $\mathcal{E}(\mathbb{R}^n)$

$$\Big(\tilde{p}_j(\psi)=\sum_{|\alpha|\le j}\sup_{\Omega_j}|D^\alpha\psi|\Big)$$

Ω_j s.t. ...

$(\bigcup_j\Omega_j=\Omega$ as usual$)$

take (φ_n) s.t. $\forall n$ $\operatorname{supp}\varphi_n\subseteq K$ fixed

and $D^\beta\varphi_n\to 0$ uniformly for all β

idea

$$\tilde{p}_j(T*\varphi_n)\xrightarrow{n}0$$

$$\tilde{p}_j(T*\varphi_n)=\sum_{|\alpha|\le j}\sup_{\Omega_j}|D^\alpha(T*\varphi_n)(x)|$$
$$=\sum_{|\alpha|\le j}\sup_{\Omega_j}|(T*D^\alpha\varphi_n)(x)|$$
$$\sup_{y\in\Omega_j}|T((D^\alpha\varphi_n)(x-\cdot)|$$

$$\tilde{p}_j(T*\varphi_n) = \sum_{|\alpha|\le j} \sup_{\overline{\Omega}_j} |D^\alpha (T*\varphi_n)(x)|$$

$$= \sum_{|\alpha|\le j} \sup_{\overline{\Omega}_j} |(T*D^\alpha \varphi_n)(x)|$$

$$\sup_{x\in\overline{\Omega}_j} |T(\underbrace{(D^\alpha\varphi_n)(x-\cdot)}_{\psi_n(y)})|$$

all the ψ_n have suppt in a same compact (fixed $x\in\overline{\Omega}_j$, $y\in K$

and $D^\alpha\varphi_n(x-\cdot)\longrightarrow 0$ unifmly

so $\tilde{p}_j(\psi_n)\longrightarrow 0$ .....

<u>Th.</u> let $T\in\mathcal{D}'(\mathbb{R}^n)$ and $\varphi,\psi\in\mathcal{D}(\mathbb{R}^n)$

then $$(\underbrace{T*\varphi}_{\in C^\infty})*\underbrace{\psi}_{\in C_0^\infty} = T*(\underbrace{\varphi*\psi}_{\in C_0^\infty})$$

proof (idea)

<u>rem.</u> $T\in\mathcal{D}'(\mathbb{R}^n)$, $\varphi\in\mathcal{D}(\mathbb{R}^n)$

connection between $T*\varphi$ and $T(\varphi)$?

$$T(\varphi) = (T*\check{\varphi})(0) \quad \text{where } \check{\varphi}(y)=\varphi(-y)$$

or $$T(\check{\varphi}) = (T*\varphi)(0)$$

$$T*\varphi(x) = T(\varphi(x-\cdot)) \implies T*\varphi(0) = T(\varphi(0-\cdot)) = T(\check{\varphi})$$

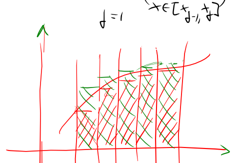
<u>rem.</u> let $f\in C_0(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} f(y)\,dy = \lim_{\varepsilon\to0^+} \varepsilon^n \sum_{\nu\in\mathbb{Z}^n} f(\nu\varepsilon)$$

in calculus 1
$$\int_a^b f(t)\,dt = \sup_\Delta s(f,\Delta) = \inf_\Delta S(f,\Delta)$$

$$s(f,\Delta) = \sum_{j=1}^n \Big(\inf_{x\in[t_{j-1},t_j]} f\Big)(t_j-t_{j-1})$$

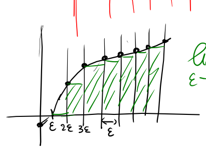
$$S(f,\Delta) = \sum_{t_j} \Big(\sup_{x\in[t_{j-1},t_j]} f\Big)(t_j-t_{j-1})$$



$= s(f,\Delta)$

$= S(f,\Delta)$

f <u>continuous</u>



$$\lim_{\varepsilon\to0}\Big(\sum_{j=0}^{n_\varepsilon} \varepsilon\cdot f(\varepsilon j)\Big) = \int_a^b f(t)\,dt$$

I have to prove $T*(\varphi*\psi) = (T*\varphi)*\psi$

$$\varphi*\psi(x) = \int_{\mathbb{R}^n}\varphi(x-y)\psi(y)\,dy = \lim_{\varepsilon\to0} \underbrace{\varepsilon^n\sum_{\nu\in\mathbb{Z}^n}\varphi(x-\varepsilon\nu)\psi(\varepsilon\nu)}_{f_\varepsilon(x)\,\in\, C_0^\infty}$$

$$T*f_\varepsilon(x) = T*\Big(\varepsilon^n\sum_{\nu\in\mathbb{Z}^n}\varphi(x-\varepsilon\nu)\psi(\varepsilon\nu)\Big)$$

$$\underbrace{\lim_{\varepsilon \to 0} \varepsilon^n \sum_{\nu \in \mathbb{Z}^n} \varphi(x - \varepsilon\nu)\psi(\varepsilon\nu)}_{f_\varepsilon(x) \in C_0^\infty}$$

¿ $f_\varepsilon \to \varphi * \psi$ in the sense of $\mathcal{D}$?

I hope yes....

conclusion

$$T * f_\varepsilon(x) = T * \left(\varepsilon^n \sum_{\nu\in\mathbb{Z}^n} \varphi(x-\varepsilon\nu)\psi(\varepsilon\nu)\right)$$

$$= \varepsilon^n \sum_{\nu\in\mathbb{Z}^n} \left(T * \varphi(x-\varepsilon\nu)\right)\cdot\psi(\varepsilon\nu)$$

$$\left(T * \tau_{\varepsilon\nu}\varphi\right)(x)\cdot\psi(\varepsilon\nu)$$

$$\left(\tau_h\varphi = \varphi(\cdot-h)\right)$$

$$= \varepsilon^n \sum_{\nu\in\mathbb{Z}^n} \psi(\varepsilon\nu)\,(T*\varphi)(x-\varepsilon\nu)$$

$$\lim_{\varepsilon\to0^+} (T*f_\varepsilon)(x) = \underbrace{\lim_{\varepsilon\to0^+} \varepsilon^n \sum_{\nu\in\mathbb{Z}_n} (T*\varphi)(x-\varepsilon\nu)\psi(\varepsilon\nu)}_{\int (T*\varphi)(x-y)\,\psi(y)\,dy}$$

$$= \left((T*\varphi)*\psi\right)(x)$$

it remain to prove that $\lim_{\varepsilon\to0^+} (T*f_\varepsilon)(x) = T*(\varphi*\psi)$

idea $(f_\varepsilon)_{\varepsilon>0}$ goes to $\varphi*\psi$ in the sense of $\mathcal{D}$

conclusion

$$T*(\varphi*\psi) = (T*\varphi)*\psi$$

Last property

Th. suppose $T\in\mathcal{D}'(\mathbb{R}^n)$ suppose $\varphi\in\mathcal{D}(\mathbb{R}^n)$ and $h\in\mathbb{R}^n$

then $(T*\tau_h\varphi)(x) = \tau_h(T*\varphi)(x)$

(so $T*$ commutes with τ_h)

proof.

$$(T*\tau_h\varphi)(x) = T\left((\tau_h\varphi)(x-\cdot)\right)$$

$$= T\left(\varphi(x-\cdot-h)\right)$$

$$= T\left(\varphi((x-h)-\cdot)\right)$$

$$= (T*\varphi)(x-h)$$

$$= \tau_h(T*\varphi)(x)$$

remark. let $T\in\mathcal{D}'(\mathbb{R}^n)$

consider
$$T*: \mathcal{D}(\mathbb{R}^n)\to\mathcal{E}(\mathbb{R}^n)$$
$$\varphi\longmapsto T*\varphi$$

$T*$ is
- linear
- continuous
- commutes with τ_h

Th let $\Phi: \mathcal{D}(\mathbb{R}^n)\to\mathcal{E}(\mathbb{R}^n)$

suppose Φ is linear, continuous, it commutes with τ_h

Th let $\Phi : \mathcal{D}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n)$

suppose Φ is linear, continuous, it commutes with τ_a

Then $\exists\ T \in \mathcal{D}'(\mathbb{R}^n)$ s.t.

$$\Phi(\varphi) = T * \varphi$$

$T*\varphi(0) = T(\check{\varphi})$

$(T*\check{\varphi})(0) = T(\varphi)$

$\check{\varphi}(x) = \varphi(-x)$

proof. we have Φ.

define $T(\varphi) = \Phi(\check{\varphi})(0)$

with this definition

$T : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{C}$

T is linear $\Big(T(\varphi+\psi) = \Phi((\varphi+\psi)^\vee)(0) = \Phi(\check{\varphi}+\check{\psi})(0) = \Phi(\check{\varphi})(0) + \Phi(\check{\psi})(0) = T(\varphi) + T(\psi) \Big)$

T is continuous

($\varphi \to \check{\varphi}$ is cont.

Φ is cont.

evaluation in 0 is cont.)

it remains to verify that $T*\varphi = \Phi(\varphi)$, $\forall \varphi$.

$$\begin{aligned} T*\varphi(x) &= T(\varphi(x-\cdot)) \\ &= \Phi\big((\varphi(x-\cdot))^\vee\big)(0) \\ &= \Phi(\varphi(\cdot+x))(0) \\ &= \Phi(\tau_{-x}\varphi)(0) \\ &= (\tau_{-x}\Phi(\varphi))(0) = \Phi(\varphi)(x) \end{aligned}$$

$$T*\varphi(x) = \Phi(\varphi)(x)$$

QED

Conclusion.

def let $T \in \mathcal{D}'(\mathbb{R}^n)$, $S \in \mathcal{E}'(\mathbb{R}^n)$

distribution with compact support

Consider $\Phi : \mathcal{D}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n)$

$$\varphi \longmapsto \underbrace{T*(\underbrace{S*\varphi}_{\in \mathcal{E}_0^\infty})}_{\in \mathcal{E}^\infty}$$

Φ is
- linear
- continuous
- it commutes with τ_a

so that Φ is the convolution with a distribution U

$\in \mathcal{E}^\infty$

Φ is
- linear
- continuous
- it commutes with τ_{e_1}

so that Φ is the convolution with a distribution U

I define $T * S = U$

It is possible to prove that

$$S * T = T * S$$

$$T * (S_1 * S_2) = (T * S_1) * S_2$$

Fourier transform.

Fourier transform of $L^1(\mathbb{R}^n)$

Fourier transform of distributions (tempered)

def. $f \in L^1(\mathbb{R}^n)$

$\xi \in \mathbb{R}^n$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i x \cdot \xi} f(x)\, dx$$

$$x \cdot \xi = \sum_{j=1}^{n} x_j \xi_j$$

Th $\hat{f} \in L^\infty(\mathbb{R}^n_\xi)$

$\hat{f} \in \mathcal{C}(\mathbb{R}^n_\xi)$

$\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$

$\lim_{|\xi| \to +\infty} \hat{f}(\xi) = 0$ (so called Riemann - Lebesgue lemma)

proof. • $|\hat{f}(\xi)| \leq \int_{\mathbb{R}^n} \underbrace{|e^{-ix\xi} f(x)|}_{= |f(x)|}\, dx \leq \|f\|_{L^1}$

$\Rightarrow \hat{f} \in L^\infty$ and $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$

• $\hat{f}$ is continuous

let $(\xi_n)_n$ in $\mathbb{R}^n$ s.t. $\xi_n \to \bar{\xi}$

then $e^{-i \xi_n \cdot x} f(x) \longrightarrow e^{-i \bar{\xi} \cdot x} f(x)$ pointwise

$|e^{-i \xi_n \cdot x} f(x)| \leq |f(x)|$

so that $\hat{f}(\xi_n) = \int_{\mathbb{R}^n} e^{-i \xi_n x} f(x)\, dx \longrightarrow \int_{\mathbb{R}} e^{-i \bar{\xi} \cdot x} f(x)\, dx$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \quad \hat{f}(\bar{\xi})$

it remains R-L.