Lesson 16 - 9/4/2026

Ex. let $f: [0,1]\times\mathbb{R} \to \mathbb{R}$ a $\mathcal{C}^1$ function

define $g(x) = \int_0^1 f(t,x)\,dt$

prove that $g \in \mathcal{C}^1(\mathbb{R})$ and $g'(x) = \int_0^1 \frac{\partial}{\partial x} f(t,x)\,dt$

solution

1) to see that g is well defined consider that, for x fixed, the function $t \longmapsto f(t,x)$ is continuous w.r.t. t so that it is integrable in $t \in [0,1]$

2) to see that g is continuous w.r.t. x

I use the dominated convergence

let $(x_n)_n$ sequence, $x_n \to \bar{x}$

$$g(x_n) \xrightarrow{n} g(\bar{x})$$

because $f(t,x_n) \xrightarrow{n} f(t,\bar{x})$ pointwise

and $|f(t,x_n)| \le \max_{\substack{t\in[0,1]\\ x\in[\bar{x}-1,\bar{x}+1]}} |f(t,x)|$

$$\underbrace{\int_0^1 f(t,x_n)\,dt}_{= g(x_n)} \xrightarrow{n} \underbrace{\int_0^1 f(t,\bar{x})\,dt}_{= g(\bar{x})} \quad \left(\forall \quad \underline{x_n \in [\bar{x}-1, \bar{x}+1]}\right)$$

3) to prove that g is differentiable at $\bar{x}$

consider $\frac{g(\bar{x}+y) - g(\bar{x})}{y} = \int_0^1 \frac{f(t,\bar{x}+y) - f(t,\bar{x})}{y}\,dt$

$\to \frac{\partial f}{\partial x}(t,\bar{x})$

also here use the dominated convergence

Ex. Let $\varphi \in \mathcal{C}^\infty(\mathbb{R})$

consider $\psi(x) = \begin{cases} \frac{\varphi(x)-\varphi(0)}{x} & \text{if } x \neq 0 \\ \varphi'(0) & \text{if } x = 0 \end{cases}$

prove that $\psi \in \mathcal{C}^\infty(\mathbb{R})$

solution

$\varphi(x) - \varphi(0) = \int_0^x \varphi'(t)\,dt \qquad t = sx$

$= \int_0^1 x\varphi'(sx)\,ds \qquad dt = x\,ds$

$$= \begin{cases} \frac{\varphi(x)-\varphi(0)}{x} & x \neq 0 \\ \varphi'(0) & x = 0 \end{cases} = \underbrace{\int_0^1 \varphi'(sx)\,ds}_{= \psi(x)} \left(= \int_0^1 f(s,x)\,ds\right)$$

$\psi'(x) = \int_0^1 s\varphi''(sx)\,ds$

use recursively ex. 1

$\psi^{(n)}(x) = \int_0^1 s^n \varphi^{(n+1)}(sx)\,ds$

$\Rightarrow \psi \in \mathcal{C}^\infty$

Ex. Find all the distributions $T \in \mathcal{D}'(\mathbb{R})$ s.t. $xT = 0$

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s.t. $\quad xT = 0$

(rem. find $f: \mathbb{R} \to \mathbb{R}$ (e.g., $f \in L^1_{loc}(\mathbb{R})$)

s.t. $\quad x f(x) = 0$

$f = 0$ a.e. $\quad$ so only $f = 0$ is OK)

solution.

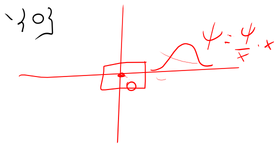
$T \in \mathcal{D}'(\mathbb{R})$ and

$$\boxed{\text{for all } \varphi \in \mathcal{D}(\mathbb{R}), \quad T(x \cdot \varphi(x)) = 0}$$

we want to find $T(\psi)$ for all ψ

suppose ψ has support in $\mathbb{R} \setminus \{0\}$

then $\psi(x) = \underbrace{\frac{\psi(x)}{x}}_{\in\, \mathcal{D}(\mathbb{R})} \cdot x$



so $T(\psi) = T\left(\left(\frac{\psi}{x}\right) \cdot x\right) = 0$

$\Rightarrow$ supp $T \subseteq \{0\}$

$\psi = \chi\psi + (1-\chi)\psi \quad$ where $\chi \in C_0^\infty(\mathbb{R})$, $\chi = 1$ in a nbh of 0

$\psi = \chi\left(\psi(x) - \psi(0)\right) + \chi \cdot \psi(0) + (1-\chi)\psi$

$$\psi = \chi \cdot \left(\frac{\psi(x) - \psi(0)}{x}\right) \cdot x + \chi \cdot \psi(0) + (1-\chi)\psi$$

$$T(\psi) = \underbrace{T\Big(\underbrace{\chi \cdot \Big(\frac{\psi(x) - \psi(0)}{x}\Big)}_{C_0^\infty \ (\text{ex. } 2)} \cdot \underbrace{x}_{\text{mult. by } x}\Big)}_{=0} + T(\chi \cdot \psi(0)) + \underbrace{T((1-\chi)\psi)}_{=0}$$

T has support in 0 and $(1-\chi)\psi$ is 0 in a nbh of 0

$$T(\psi) = \psi(0) \cdot T(\chi) = c\,\psi(0) = c\,\delta_0(\psi)$$

$T(\chi) = c$

conclusion $\quad T = c\,\delta_0 \qquad c = \underbrace{T(\chi)}_{\text{does not depend on } \chi \text{ but only on } T}$

χ: 1 in a nbh of 0

Ex find all $T \in \mathcal{D}'(\mathbb{R})$

s.t. $\quad xT = 1 \ (= T_1)$

Ex find all $T \in \mathcal{D}'(\mathbb{R})$
s.t. $x T = 1 \ (= T_1)$

rem. in the sense of functions

$$x \cdot f(x) = 1 \iff f(x) = \frac{1}{x}$$

hint $T = PV_{\frac{1}{x}}$

solution We verify that $PV_{\frac{1}{x}}$ is a good T
s.t. $x \cdot T = T_1$

$$x\, PV_{\frac{1}{x}}(\varphi) = PV_{\frac{1}{x}}(x\varphi(x)) = \lim_{\varepsilon \to 0^+} \left(\int_{-\infty}^{-\varepsilon} \cancel{x} \cdot \frac{\varphi(x)}{\cancel{x}}\, dx + \int_{\varepsilon}^{+\infty} \cancel{x} \cdot \frac{\varphi(x)}{\cancel{x}}\, dx \right)$$

$$= \lim_{\varepsilon \to 0^+} \int_{-\infty}^{-\varepsilon} \varphi(x)\, dx + \int_{\varepsilon}^{+\infty} \varphi(x)\, dx$$

$$= \int_{-\infty}^{+\infty} \varphi(x)\, dx = \int_{-\infty}^{+\infty} 1 \cdot \varphi(x)\, dx = T_1(\varphi)$$

$$x\, PV_{\frac{1}{x}} = T_1$$

suppose that $x \cdot T = T_1$

we have $x\left(PV_{\frac{1}{x}} - T\right) = T_1 - T_1 = 0$

if $xT = T_1$ then $x\left(PV_{\frac{1}{x}} - T\right) = 0$

$$\Downarrow$$

$$PV_{\frac{1}{x}} - T = c\,\delta_0 \quad (\text{previous ex.})$$

finally $\boxed{T = PV_{\frac{1}{x}} + c\,\delta_0} \iff [xT = T_1]$

Ex. Show that, given $\varphi \in \mathcal{D}(\mathbb{R})$

$$T(\varphi) = \lim_{m} \left(\sum_{j=1}^{m} \varphi\left(\frac{1}{j}\right) - m\,\varphi(0) - \log m\, \varphi'(0) \right)$$

is a distribution.
find the order and the support

Ex. suppose $\varphi \in \mathcal{D}(\mathbb{R})$, $T \in \mathcal{D}'(\mathbb{R})$
suppose $\varphi = 0$ on supp T
¿ $T(\varphi) = 0$?