Lesson 17 - 14/4/2026

Exams: June $17^{th}$
July $8^{th}$

rem. consider $\delta_0$, $f \in \mathcal{E}(\mathbb{R}^n)$

$$\delta_0 * f(x) = \delta_0(f(x-\cdot)) = f(x-0) = f(x)$$

$$\delta_0 * f = f$$

consider $a_\alpha \in \mathbb{C}$ and $P(\partial_x) = \sum_{|\alpha| \le m} a_\alpha \partial_x^\alpha$

$P$ is linear partial differential operator with complex constant coefficients

E.g. $\Box = \partial_t^2 - \partial_x^2$ wave operator (d'Alembertian)

$\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + .. + \partial_{x_n}^2$ Laplace operator

$\partial_t - \Delta_x = \partial_t - \sum_{j=1}^n \partial_{x_j}^2$ Heat operator

....

suppose that $T \in \mathcal{D}'(\mathbb{R}^n)$ and $P = \sum_{|\alpha| \le m} a_\alpha \partial_x^\alpha$

suppose $PT = \delta_0$

$T$ is called the <u>fundamental solution</u> to $P$

$\longrightarrow$ $P$ l.p.d.o. w. constant coeff.
$T$ fund. solution
$\varphi \in \mathcal{D}(\mathbb{R}^n)$
then given the equation $Pu = \varphi$ $\quad u = T * \varphi$ is a solution

in fact $PT = \delta_0$ $\quad (PT) * \varphi = P(T * \varphi)$

$$(PT)*\varphi = \delta_0 * \varphi = \varphi, \qquad P(T*\varphi) = Pu$$

$(PT) = \delta_0$

$PT * \varphi = \delta_0 * \varphi = \varphi$

$(PT) * \varphi = P(T * \varphi)$

$P = \sum_{|\alpha| \le m} a_\alpha \partial_x^\alpha$ $\quad P(T*\varphi) = \sum_{|\alpha| \le m} a_\alpha \partial_x^\alpha (T * \varphi)$

$$= \sum_{|\alpha| \le m} a_\alpha (\partial_x^\alpha T) * \varphi$$

$$= (PT) * \varphi$$

<u>Problem</u>. given $P = \sum_{|\alpha| \le m} a_\alpha \partial_x^\alpha$

¿ there exists a fund. solution ?

<u>YES</u>

Th. (Malgrange - Ehrenpreis ~ 1957)
for all $P = \sum_{|\alpha| \le m} a_\alpha \partial_x^\alpha$ there exists a fund. solution.

Rem. <u>The problem of local solvability</u>

Let $P(x,\partial_x) = \sum_{|\alpha|\le m} a_\alpha(x)\partial_x^\alpha$ a linear partial differential operator with $C^\infty$ complex coefficients defined on $\Omega$ open set of $\mathbb{R}^N$. Suppose $x_0 \in \Omega$.

I say that $P$ is locally solvable at $x_0$ if there exists $\Omega'$ open nbh of $x_0$ in $\Omega$ s.t. $\forall \varphi \in C_0^\infty(\Omega')$, $\exists u \in \mathcal{D}'(\Omega')$ s.t. $P(x,\partial_x)u = \varphi$

i.e. given $\varphi \in \mathcal{D}(\Omega')$ $\exists u \in \mathcal{D}'(\Omega)$ solution to $Pu = \varphi$

Corollary to Malgrange - Ehrenpreis
if $P$ has constant coefficients then $P$ is locally solvable.

¿ and if $P$ has not const. coeff. ?

<u>Ex.</u> (H. Lewy, 1957)

consider $L = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + i(x_1 + ix_2)\frac{\partial}{\partial x_3}$ (in $\mathbb{R}^3$)

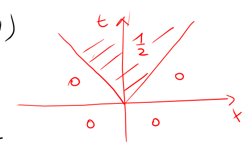
this operator is not locally solvable at any point of $\mathbb{R}^3$

results to this problem by Hörmander
by Nirenberg and Treves
by N. Lerner

<u>Ex.</u> given $\Box = \partial_t^2 - \partial_x^2$

considering $H$ the Heaviside function and

$E(x,t) = \frac{1}{2}H(t-|x|)$




prove that

$\Box E = \delta_0$ in $\mathbb{R}^2$

$$\left( \frac{1}{2}\iint_{t>|x|} (\partial_t^2 - \partial_x^2)\varphi(x,t)\,dx\,dt = \varphi(0,0) \right)$$

<u>Ex.</u> consider

$$E(t,x) = \frac{\overset{\text{Heaviside}}{H(t)}}{\sqrt{4\pi t}} \cdot e^{-\frac{x^2}{4t}}$$

$$(\partial_t - \partial_x^2)E = \delta_0$$

<u>Ex</u>

$$E_n(x) = \begin{cases} \log|x| & \text{if } n=2 \\ |x|^{2-n} & \text{if } n \ge 3 \end{cases}$$

then $\Delta E_n = \delta_0$

<u>Fourier transform</u>

def. $f \in L^1(\mathbb{R}^n)$

Fourier transform

def. $f \in L^1(\mathbb{R}^n)$, $\xi \in \mathbb{R}^n$

define $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx$

where $x\cdot\xi = \sum_{j=1}^{n} x_j \xi_j$

$\hat{f}$ is called the Fourier Transf. of $f$

Th. let $f \in L^1(\mathbb{R}^n)$

- $\hat{f} \in L^\infty$ and $\|\hat{f}\|_{L^\infty} \le \|f\|_{L^1}$
- $\hat{f} \in C(\mathbb{R}^n)$
- (Riemann-Lebesgue lemma) $\lim_{|\xi|\to+\infty} \hat{f}(\xi) = 0$

proof. $x \longmapsto e^{-ix\cdot\xi} f(x) \in L^1(\mathbb{R}^n)$

(since $|e^{-ix\cdot\xi}| \le 1$)

$\hat{f}(\xi)$ is defined for all $\xi$ and $|\hat{f}(\xi)| \le \int_{\mathbb{R}^n} |f| = \|f\|_{L^1}$

$\hat{f}$ is continuous (e.g. consider $\xi_n \to \xi$

then $e^{-ix\cdot\xi_n} f(x) \longrightarrow e^{-ix\cdot\xi} f(x)$ pointwise in $x$

and $|e^{-ix\cdot\xi_n} f(x)| \le |f(x)|$

use dominated convergence

$\hat{f}(\xi_n) \longrightarrow \hat{f}(\xi)$ )

To prove Riemann-Lebesgue

step 1 let $\varphi \in C_0^\infty(\mathbb{R}^n)$

consider $\xi_j \hat{\varphi}(\xi) = \int \underbrace{e^{-ix\cdot\xi} \xi_j}_{i\partial_{x_j}(e^{-ix\cdot\xi})} \varphi(x)\,dx$

$= i \int_{\mathbb{R}^n} \partial_{x_j}(e^{-ix\cdot\xi}) \varphi(x)\,dx$

$= i \underbrace{\int \partial_{x_j}(e^{-ix\cdot\xi} \varphi(x))}_{=0} - i \int e^{-ix\cdot\xi} \partial_{x_j}\varphi$

$= -i\, \widehat{\partial_{x_j}\varphi}(\xi)$

$\xi_j \hat{\varphi}(\xi) = -i\, \widehat{\partial_{x_j}\varphi}(\xi) = \widehat{D_{x_j}\varphi}(\xi)$

$\xi_j^2 \hat{\varphi}(\xi) = -\widehat{\partial_{x_j}^2\varphi}$

finally $(1+|\xi|^2)\hat{\varphi}(\xi) = \widehat{(1-\Delta)\varphi}(\xi)$

$\hat{\varphi}(\xi) = \dfrac{\widehat{(1-\Delta)\varphi}(\xi)}{1+|\xi|^2}$ but $(1-\Delta)\varphi \in C_0^\infty$

so $\widehat{(1-\Delta)\varphi} \in L^\infty$

we have $|\hat{\varphi}(\xi)| \le \dfrac{C}{1+|\xi|^2}$ $\left( C = \|\widehat{(1-\Delta)\varphi}\|_{L^\infty} \right)$

we have $|\hat{\varphi}(\xi)| \leq \frac{C}{1+|\xi|^2}$ $\quad \left( C = \|\widehat{(1-\Delta)\varphi}\|_{L^\infty} \right)$

$$\lim_{|\xi|} \hat{\varphi}(\xi) = 0$$

<u>step 2</u> . $C_0^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$

Let $f \in L^1(\mathbb{R}^n)$. take $\varepsilon > 0$

take $\varphi \in C_0^\infty(\mathbb{R}^n)$ s.t. $\|f - \varphi\|_{L^1} \leq \varepsilon/2$

then, we know that $\lim_{|\xi| \to +\infty} |\hat{\varphi}(\xi)| = 0$

so that $\exists R > 0$ s.t. $\forall |\xi| > R$, then $|\hat{\varphi}(\xi)| < \varepsilon/2$

finally $\forall |\xi| > R$

$$|\hat{f}(\xi)| \leq \underbrace{|\hat{f}(\xi) - \hat{\varphi}(\xi)|}_{\leq \|\hat{f} - \hat{\varphi}\|_{L^\infty} \leq \|f - \varphi\|_{L^1} \leq \varepsilon/2} + \overbrace{|\hat{\varphi}(\xi)|}^{< \varepsilon/2} < \varepsilon$$

$$\lim_{|\xi| \to +\infty} |\hat{f}(\xi)| = 0$$

<u>QED</u>

<u>rem</u>. suppose $f \in L^1(\mathbb{R}^n)$, $f$ is diff. w.r.to $x_j$

and suppose that $\partial_{x_j} f \in L^1(\mathbb{R}^n)$

then $\widehat{\partial_{x_j} f}(\xi) = i \xi_j \hat{f}(\xi)$

$\widehat{D_{x_j} f}(\xi) = \xi_j \hat{f}(\xi)$

exercise

$P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ $\quad \left( D_{x_j} = -i \frac{\partial}{\partial x_j} \right)$

$\widehat{P(D) f}(\xi) = P(\xi) \hat{f}(\xi)$

similarly let $f \in L^1(\mathbb{R}^n)$ suppose that

also $x_j f(x) \in L^1(\mathbb{R}^n)$

then $\hat{f}$ is differentiable w.r.t. $\xi_j$

and $\widehat{x_j f(x)}(\xi) = i \frac{\partial}{\partial \xi_j} \hat{f}(\xi)$

<u>idea</u> (in 1 d)

$f(x) \in L^1(\mathbb{R})$, $x f(x) \in L^1(\mathbb{R})$

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx$$

$$\hat{f}(\xi + \eta) = \int_{\mathbb{R}} e^{-ix(\xi+\eta)} f(x)\,dx$$

$$\frac{\hat{f}(\xi+\eta) - \hat{f}(\xi)}{\eta} = \int_{\mathbb{R}} \left( \frac{e^{-ix(\xi+\eta)} - e^{ix\xi}}{\eta} \right) f(x)\,dx$$

$$= \int_{\mathbb{R}} \left( \frac{e^{-ix\eta} - 1}{-ix\eta} \right) -ix e^{-ix\xi} f(x)\,dx$$

$$\frac{\hat f(\xi+y)-\hat f(\xi)}{y} = \int_{\mathbb R}\left(\frac{e^{-ixy}-1}{-ixy}\right)-ix\,e^{-ix\xi}f(x)\,dx$$

$$= \int \underbrace{\left(\frac{e^{-ixy}-1}{-ixy}\right)}_{\xrightarrow{y\to 0}\ 1} -ix\,e^{-ix\xi}f(x)\,dx$$

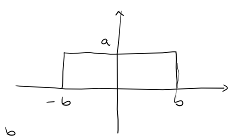
pass to the limit under the integral

$$\hat f'(\xi) = \int -i\,e^{-ix\xi}\,x f(x)\,dx$$

$$\boxed{i\,\hat f'(\xi) = \widehat{x f(x)}(\xi)}$$

Example of Fourier transform

1) $f(x) = \begin{cases} a & \text{if } |x|\le b \\ 0 & \text{if } |x|>b\end{cases}$




$$\hat f(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}f(x)\,dx = a\int_{-b}^{b} e^{-ix\xi}\,dx$$

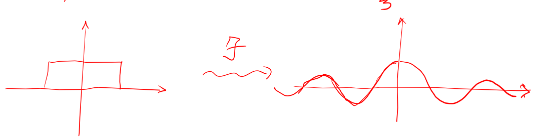
$\hat f(0) = a\int_{-b}^{b} 1\,dx = 2ab$ $\qquad e^{-ix\xi} = \cos x\xi - i\sin x\xi$

$\xi\neq 0$, $\hat f(\xi) = a\int_{-b}^{b}(\cos x\xi - i\sin x\xi)\,dx$ $\qquad \int_{-b}^{b}\sin x\xi\,dx = 0$

$$= a\int_{-b}^{b}\cos x\xi\,dx$$

$$= a\,\frac{\sin x\xi}{\xi}\Big|_{-b}^{b} = 2a\,\frac{\sin b\xi}{\xi} = 2ab\cdot\frac{\sin b\xi}{b\xi}$$

( $a=\frac12$, $b=1$ $\qquad \hat f(\xi) = \begin{cases} 1 & \xi = 0 \\ \frac{\sin\xi}{\xi} & \xi\neq 0\end{cases}$



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2) $f(x) = e^{-|x|} = \begin{cases} e^{-x} & x\ge 0 \\ e^{x} & x<0\end{cases}$

$$\hat f(\xi) = \int_{-\infty}^{0} e^{-ix\xi+x}\,dx + \int_{0}^{+\infty} e^{-ix\xi-x}\,dx$$

$$= \frac{e^{-ix\xi+x}}{1-i\xi}\Big|_{-\infty}^{0} + \frac{e^{-ix\xi-x}}{-1-i\xi}\Big|_{0}^{+\infty}$$

$$= \frac{1}{1-i\xi} + \frac{1}{1+i\xi} = \frac{2}{1+\xi^2}$$

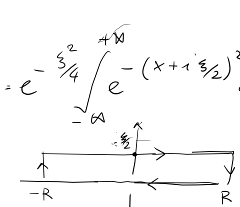
3) let $f(x) = e^{-x^2}$

$$\hat f(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi-x^2}\,dx$$

$-(+i\xi/2)^2$

$$= e^{-\xi^2/4}\cdot e^{\xi^2/4}\int_{-\infty}^{+\infty} e^{-ix\xi-x^2}\,dx = e^{-\xi^2/4}\int_{-\infty}^{+\infty} e^{-(x+i\xi/2)^2}\,dx$$

consider $\int_\gamma e^{-z^2}\,dz$



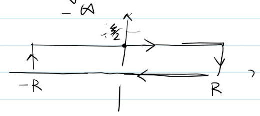
,

$$\hat{f}(\xi) = \int_{-\infty} e^{-ix\xi - x^2} dx$$

$$-\left(+i\tfrac{\xi}{2}\right)^2$$

$$= e^{-\xi^2/4} \cdot e^{\xi^2/4} \int_{-\infty}^{+\infty} e^{-ix\xi - x^2} dx = e^{-\xi^2/4} \int_{-\infty}^{+\infty} e^{-\left(x + i\xi/2\right)^2} dx$$

consider $\int_\gamma e^{-z^2} dz$

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$$\int_{-\infty}^{+\infty} e^{-\left(x + i\xi/2\right)^2} dx = \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

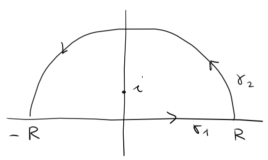
$$\widehat{e^{-x^2}}(\xi) = e^{-\xi^2/4} \cdot \sqrt{\pi}$$

4) $f(x) = \frac{1}{1+x^2}$

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} \frac{1}{1+x^2} dx$$

if $\xi < 0$ consider $z \longrightarrow \frac{e^{-iz\xi}}{1+z^2} = \frac{e^{-iz\xi}}{(z+i)(z-i)}$

$$\mathrm{Res}(f, i) = \frac{e^{\xi}}{2i}$$



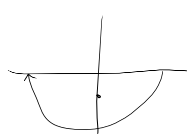

$$\int_{\gamma_1} + \int_{\gamma_2} = 2\pi i \cdot \mathrm{Res}(i) = \pi e^{\xi} \qquad \xi < 0$$

$$\lim_{R\to+\infty} \int_{\gamma_1} = \int_{-\infty}^{+\infty} \frac{e^{-ix\xi}}{1+x^2} dx$$

$\underline{\underline{\xi < 0}}$

$$\lim_{R\to+\infty} \int_{\gamma_2} = 0 = \lim_R \int_0^{\pi} \frac{e^{R\xi e^{i\theta}}}{1+R^2 e^{2i\theta}} iRe^{i\theta} d\theta$$

$\xi > 0$



$$\hat{f}(\xi) = \pi e^{-\xi}$$

$$f(x) = e^{-|x|} \rightsquigarrow \hat{f}(\xi) = \frac{2}{1+\xi^2}$$

$$f(x) = \frac{1}{1+x^2} \rightsquigarrow \hat{f}(\xi) = \pi e^{-|x|}$$

<u>rem.</u> $f(x) = e^{-x^2}$

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} e^{-x^2 - ix\xi} dx$$

$$i \hat{f}'(\xi) = \widehat{xf(x)}(\xi) = \int_{-\infty}^{+\infty} x e^{-x^2 - ix\xi} dx$$

$$\hat{f}'(\xi) = -i \int_{-\infty}^{+\infty} x e^{-x^2 - ix\xi} dx$$

$$\xi \hat{f}(\xi) = \int \underbrace{\xi e^{-ix\xi}} e^{-x^2} dx$$

$$i\partial_x(e^{-ix\xi}) e^{-x^2} = -i e^{-ix\xi} \partial_x(e^{-x^2})$$

$$= +2i x e^{-x^2 - ix\xi}$$

$$\xi \hat{f}(\xi) = 2i \int_{-\infty}^{+\infty} x e^{-ix\xi - x^2} dx = -2 \hat{f}'(\xi)$$

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$$\xi \hat{f}(\xi) = -2 \hat{f}'(\xi)$$

$$\hat{f}(0) = \int e^{-x^2} dx \overset{!}{=} \sqrt{\pi}$$

$$\boxed{\begin{cases} \hat{f}'(\xi) + \frac{1}{2}\xi \hat{f}(\xi) = 0 \\ \hat{f}(0) = \sqrt{\pi} \end{cases}}$$

$$e^{-\xi^2/4}$$