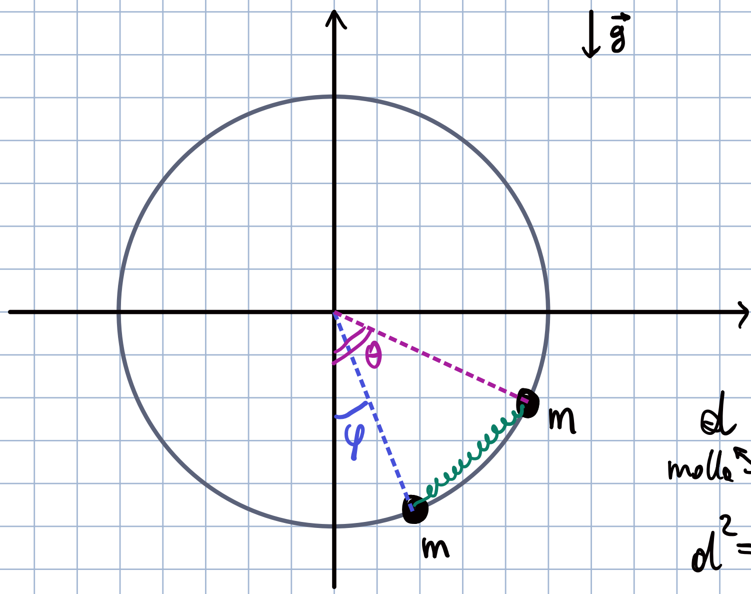


$x_\varphi = R \,\text{sen}\, \varphi$
$y_\varphi = -R\cos\varphi$

$x_\theta = R \,\text{sen}\, \theta$
$y_\theta = -R\cos\theta$

$d = 2R \,\text{sen}\left(\frac{\theta-\varphi}{2}\right)$
molla

$\Downarrow$

$d^2 = 4R^2 \,\text{sen}^2\left(\frac{\theta-\varphi}{2}\right) = 2R^2\left(1-\cos(\theta-\varphi)\right)$

1)
$$T = \frac{1}{2} m R^2 \left(\dot\theta^2 + \dot\varphi^2\right)$$

cost. irrilevante

$$V = mg\left(y_\theta + y_\varphi\right) + \frac{1}{2} K d^2 = -mgR\left(\cos\theta + \cos\varphi\right) + KR^2\left(\cancel{1} - \cos(\theta-\varphi)\right)$$
$$\downarrow = -mgR\left(\cos\theta + \cos\varphi\right) - KR^2 \cos(\theta-\varphi)$$

$$L = \frac{1}{2} m R^2 \left(\dot\theta^2 + \dot\varphi^2\right) + mgR\left(\cos\theta + \cos\varphi\right) + KR^2 \cos(\theta-\varphi)$$

2)
$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = mR^2\ddot\theta \qquad \frac{\partial L}{\partial \theta} = -mgR \,\text{sen}\,\theta - KR^2 \,\text{sen}(\theta-\varphi)$$

$$\ddot\theta = -\frac{g}{R}\,\text{sen}\,\theta - \frac{K}{m}\,\text{sen}(\theta-\varphi)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} = mR^2\ddot\varphi \qquad \frac{\partial L}{\partial \varphi} = -mgR \,\text{sen}\,\varphi + KR^2 \,\text{sen}(\theta-\varphi)$$

$$\ddot\varphi = -\frac{g}{R}\,\text{sen}\,\varphi + \frac{K}{m}\,\text{sen}(\theta-\varphi)$$

3)
$$V = -mgR\left(\cos\theta + \cos\varphi\right) - KR^2 \cos(\theta-\varphi)$$

$$\begin{cases} \partial_\theta V = mgR \,\text{sen}\,\theta + KR^2 \,\text{sen}(\theta-\varphi) = 0 \\ \partial_\varphi V = mgR \,\text{sen}\,\varphi - KR^2 \,\text{sen}(\theta-\varphi) = 0 \end{cases}$$

$\downarrow$

sommando le due eq.

$$\begin{cases} \sin\theta + \sin\varphi = 0 \\ mgR\sin\theta + kR^2\sin(\theta-\varphi) = 0 \end{cases} \Rightarrow \quad \varphi = -\theta, \quad \theta+\pi$$

$$\varphi = -\theta \rightarrow 0 = mgR\sin\theta + kR^2\sin(2\theta) = mgR\sin\theta + 2kR^2\sin\theta\cos\theta$$

$$= 2kR^2\sin\theta\left(\cos\theta + \frac{mg}{2kR}\right)$$

$\rightarrow \theta = 0, \pi$ $\qquad$ $\cos\theta^* = -\frac{mg}{2kR}$ $\rightarrow$ 2 pti, esistono solo se $\underline{mg \leq 2kR}$

$$\rightarrow (\theta,\varphi) = (0,0), (\pi,\pi), (\theta^*_{1,2}, -\theta^*_{1,2})$$

$$\varphi = \theta+\pi \rightarrow 0 = mgR\sin\theta + \cancel{kR^2\sin\pi}_{=0}$$

$\hookrightarrow \theta = 0, \pi$

$$\longrightarrow (\theta,\varphi) = (0,\pi), (\pi,0)$$

$\partial_\theta V = mgR\sin\theta + kR^2\sin(\theta-\varphi)$

$\partial_\varphi V = mgR\sin\varphi - kR^2\sin(\theta-\varphi)$

$$\partial^2 V = \begin{pmatrix} mgR\cos\theta + kR^2\cos(\theta-\varphi) & -kR^2\cos(\theta-\varphi) \\ -kR^2\cos(\theta-\varphi) & mgR\cos\varphi + kR^2\cos(\theta-\varphi) \end{pmatrix}$$

$$\partial^2 V_{(0,0)} = \begin{pmatrix} \underbrace{mgR + kR^2}_{>0} & -kR^2 \\ -kR^2 & \underbrace{mgR+kR^2}_{>0} \end{pmatrix} \rightarrow \det = mgR(mgR + 2kR^2) > 0$$

$\Rightarrow$ $\underline{\text{STABILE}}$

$$\partial^2 V_{(\pi,\pi)} = \begin{pmatrix} -mgR + kR^2 & -kR^2 \\ -kR^2 & -mgR+kR^2 \end{pmatrix} \rightarrow \det = mgR(mgR - 2kR^2)$$

$\det \geq 0$ per $mgR \geq 2kR^2$, ma in quel caso elem. sulla diagonale sono negativi $\Rightarrow$ INSTABILE

(si può dimostrare che entrambi gli autoval. sono negativi)

$$\partial^2 V_{\substack{(0,\pi)\\ \pi,0}} = \begin{pmatrix} \pm mgR - kR^2 & kR^2 \\ kR^2 & \mp mgR - kR^2 \end{pmatrix}$$ INSTABILI

(uno degli elem. diag. è sempre negativo)

$mg \leq 2kR$

$$\partial^2 V_{(\theta^*, -\theta^*)} = \begin{pmatrix} mgR\cos\theta^* + kR^2\cos 2\theta^* & -kR^2\cos 2\theta^* \\ -kR^2\cos 2\theta^* & mgR\cos\theta^* + kR^2\cos 2\theta^* \end{pmatrix}$$

$$\cos\theta^* = -\frac{mg}{2kR} \qquad \cos 2\theta^* = 2\cos^2\theta^* - 1 = \frac{(mg)^2}{2(kR)^2} - 1$$

$$mgR\cos\theta^* + kR^2\cos 2\theta^* = -\frac{(mg)^2}{2k} + \frac{2(kR)^2}{2k}\left(\frac{(mg)^2}{2(kR)^2} - 1\right) =$$

$$= \frac{1}{2k}\left(-(mg)^2 + (mg)^2 - 2(kR)^2\right) = -kR^2 < 0$$

$\Rightarrow$ INSTABILI

4) $$B = \partial^2 V_{(0,0)} = \begin{pmatrix} mgR + kR^2 & -kR^2 \\ -kR^2 & mgR + kR^2 \end{pmatrix}$$

$$A = a_{(0,0)} = \begin{pmatrix} mR^2 & \\ & mR^2 \end{pmatrix}$$

$$\det(B - \lambda A) = \det\ mR^2 \begin{pmatrix} \frac{g}{R} + \frac{k}{m} - \lambda & -\frac{k}{m} \\ -\frac{k}{m} & \frac{g}{R} + \frac{k}{m} - \lambda \end{pmatrix} =$$

$$= (mR^2)^2 \left(\lambda^2 - 2\left(\frac{g}{R} + \frac{k}{m}\right)\lambda + \frac{g}{R}\left(\frac{g}{R} + \frac{2k}{m}\right)\right) = 0$$

$$\lambda_{1,2} = \left(\frac{g}{R} + \frac{k}{m}\right) \pm \sqrt{\left(\frac{g}{R} + \frac{k}{m}\right)^2 - \frac{g}{R}\left(\frac{g}{R} + \frac{2k}{m}\right)} =$$

$$= \left(\frac{g}{R} + \frac{k}{m}\right) \pm \sqrt{\left(\frac{k}{m}\right)^2} = \begin{cases} \frac{g}{R} \\ \frac{g}{R} + \frac{2k}{m} \end{cases}$$

$$\underset{\text{anche}}{\overset{\text{*}}{=}} \left(\frac{g}{R} + \frac{k}{m} - \lambda\right)^2 - \left(\frac{k}{m}\right)^2 = \left(\frac{g}{R} + \frac{k}{m} - \lambda + \frac{k}{m}\right)\left(\frac{g}{R} + \frac{k}{m} - \lambda - \frac{k}{m}\right) =$$

$$= \left(\frac{g}{R} + \frac{2k}{m} - \lambda\right)\left(\frac{g}{R} - \lambda\right) = 0 \to \lambda_{1/2} = \begin{cases} \frac{g}{R} + \frac{2k}{m} \\ \frac{g}{R} \end{cases}$$

4bis) Modi normali di oscillaz.

$$A = \begin{pmatrix} mR^2 & \\ & mR^2 \end{pmatrix}$$

$$\lambda_1 = \frac{g}{R} + \frac{2k}{m}$$

$$(B - \lambda_1 A) = \begin{pmatrix} mgR + kR^2 - mgR - 2kR^2 & -kR^2 \\ -kR^2 & mgR + kR^2 - mgR - 2kR^2 \end{pmatrix}$$

$$= \begin{pmatrix} -kR^2 & -kR^2 \\ -kR^2 & -kR^2 \end{pmatrix} \to \bar{u}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = \frac{g}{R}$$

$$(B - \lambda_2 A) = \begin{pmatrix} mgR + kR^2 - mgR & -kR^2 \\ -kR^2 & mgR + kR^2 - mgR \end{pmatrix} =$$

$$= \begin{pmatrix} kR^2 & -kR^2 \\ -kR^2 & kR^2 \end{pmatrix} \to \bar{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Soluz. $\begin{pmatrix} \theta(t) \\ \varphi(t) \end{pmatrix} = A_1 \cos(\sqrt{\lambda_1}\, t + \varphi_1)\bar{u}_1 + A_2 \cos(\sqrt{\lambda_2}\, t + \varphi_2)\bar{u}_2$

5) $L = \frac{1}{2} m R^2 (\dot\theta^2 + \dot\varphi^2) + K R^2 \cos(\theta - \varphi)$

$$\psi \equiv \theta - \varphi \qquad \phi = \theta + \varphi \quad \rightarrow \quad \theta = \frac{1}{2}(\psi + \phi) \qquad \varphi = \frac{1}{2}(\phi - \psi) \quad \rightarrow \quad \dot\theta = \frac{1}{2}(\dot\psi + \dot\phi) \qquad \dot\varphi = \frac{1}{2}(\dot\phi - \dot\psi)$$

$$L = \frac{1}{2} m R^2 \left( \frac{1}{4}(\dot\psi + \dot\phi)^2 + \frac{1}{4}(\dot\psi - \dot\phi)^2 \right) + k R^2 \cos\psi$$

$$= \frac{1}{4} m R^2 (\dot\psi^2 + \dot\phi^2) + k R^2 \cos\psi$$

6) Coord ciclica $\phi$

costi del moto : $\frac{\partial L}{\partial \dot\phi} = \frac{1}{2} m R^2 \dot\phi$ → momento angolare totale in direzione perpendicolare al foglio.

Sist. è inv. per rotazioni attorno asse $\perp$ foglio.

7) $\dot{\phi} = \dfrac{2p_\phi}{mR^2}$ $\quad L^* = \dfrac{1}{4} mR^2 \left( \dot{\psi}^2 + \dfrac{4p_\phi^2}{(mR^2)^2} \right) + kR^2 \cos\psi$

$$- \frac{2p_\phi^2}{mR^2}$$

$$\rightarrow L^* = \frac{1}{4} mR^2 \dot{\psi}^2 + kR^2 \cos\psi - \cancel{\frac{p_\phi^2}{mR^2}}$$ cost. irrilevante

$$V_{eff} = -kR^2 \cos\psi \rightarrow \text{min in } \psi = 0$$

cost. $\qquad$ potenze più alte

$$L^*_{lin} = \frac{1}{4} mR^2 \dot{\psi}^2 + kR^2 \left( \cancel{1} - \frac{\psi^2}{2} \cancel{+ \dots} \right)$$

$$= \frac{1}{4} mR^2 \dot{\psi}^2 - \frac{kR^2}{2} \psi^2$$