Autocorrelation Function of an AR(1) Model

When an AR(1) process $X_t = \phi_0 + \phi_1 X_{t-1} + a_t$ is weakly stationary, which implies that $|\phi_1| < 1$, then

$$\begin{aligned} \mathrm{E}(X_t) =& \mu \quad \forall t & (1) \\ \mathrm{Var}(X_t) =& \gamma_0 = \frac{\sigma_a^2}{1-\phi_1^2} & (2) \\ \mathrm{Cov}(X_t, X_{t+k}) =& \gamma_k = \phi_1^k \frac{\sigma_a^2}{1-\phi_1^2} \quad \forall k > 0 & (3) \end{aligned}$$

It follows that the model ACF is

$$\rho_0 = 1; \quad \rho_k = \frac{\gamma_k}{\gamma_0} \phi_1^k \quad \forall k > 0$$

and satisfies the recursion

$$\rho_k = \phi_1 \rho_{k-1} \quad k = 1, 2, \ldots$$

Autocovariance Function of an AR(1) Model

The AR(1) is a linear process of the form

$$X_t = \mu + \sum_{i=0}^{\infty} \phi_1^i \, a_{t-i}$$

Hence, for $k > 0$, Eq.(3) is obtained as

$$\text{Cov}(X_t, X_{t+k}) = \text{Cov}\left(\sum_{i=0}^{\infty} \phi_1^i \, a_{t-i}, \sum_{j=0}^{\infty} \phi_1^j \, a_{t+k-j} \right)$$

Since $\text{Cov}(a_l, a_m) = 0$ when $l \neq m$, setting $t - i = t + k - j$ (that is $j = k + i$) we obtain

$$\text{Cov}\left(\sum_{i=0}^{\infty} \phi_1^i \, a_{t-i}, \sum_{j=0}^{\infty} \phi_1^j \, a_{t+k-j} \right) = \sum_{i=0}^{\infty} \phi_1^i \phi_1^{k+i} \text{Cov}\left(a_{t-i}, a_{t-i}\right)$$

Autocovariance Function of an AR(1) Model (cont)

The latter expression becomes (using Var$(a_t) = \sigma_a^2, \forall t$)

$$
\begin{aligned}
\sum_{i=0}^{\infty} \phi_1^i \phi_1^{k+i} \text{Cov}\,(a_{t-i}, a_{t-i}) &= \sum_{i=0}^{\infty} \phi_1^i \phi_1^{k+i} \text{Var}(a_{t-i}) \\
&= \sigma_a^2 \sum_{i=0}^{\infty} \phi_1^i \phi_1^{k+i} = \sigma_a^2 \phi_1^k \sum_{i=0}^{\infty} \phi_1^i \phi_1^i \\
&= \sigma_a^2 \phi_1^k \sum_{i=0}^{\infty} (\phi_1^2)^i \\
&= \sigma_a^2 \phi_1^k \frac{1}{1-\phi_1^2} \quad \forall k > 0, -1 < \phi_1 < 1,
\end{aligned}
$$

(using the convergence of the geometric series $\sum_{i=0}^{\infty} r^i = 1/(1-r), |r| < 1$)