Introduction to Jacobi fields

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Technical Lemma & Gauss' Lemma

Lemma

Let $\sigma : A \to M$ *be a parametrized surface and let* V *be a vector field along* $\sigma(A)$*. If* (t, s) *are the usual coordinates in* $A \subseteq \mathbb{R}^2$*, then, in local coordinates,* $V = \sum_{j=1}^{n} v_j(t,s) X_j$ *with* $X_j = \frac{\partial}{\partial x_j}$ *and we have*

$$\frac{D}{\partial t}\frac{D}{\partial s}V - \frac{D}{\partial s}\frac{D}{\partial t}V = \sum_{j=1}^{n} v_j \left(\frac{D}{\partial t}\frac{D}{\partial s} - \frac{D}{\partial s}\frac{D}{\partial t} \right) X_j = R\left(\frac{\partial \sigma}{\partial s}, \frac{\partial \sigma}{\partial t} \right) V$$

since $\left[\frac{\partial \sigma}{\partial t}, \frac{\partial \sigma}{\partial s}\right] = 0.$

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since $\left[\frac{\partial \sigma}{\partial t}, \frac{\partial \sigma}{\partial s} \right] = 0$.

Lemma (Gauss)

Let $p \in M$ *and let* $v \in T_pM$ *be such that* $\exp_p v$ *is defined. Let* $w \in T_v(T_pM) \simeq T_pM$*, then*

$$\langle (d\exp_p)_v v, (d\exp_p)_v w \rangle = \langle v, w \rangle$$

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For each $s = s_0$ fixed, the curve

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is a geodesic in M such that $\gamma(0) = p$.
Furthermore

$$(d \exp_p)_v w = \frac{\partial \sigma}{\partial s}(1, 0)$$

The problem

We want to study the field

$$J(t) := (d\exp_p)_{tv}(tw) = \frac{\partial \sigma}{\partial s}(t, 0)$$

along the geodesic $\gamma(t) = \exp_p[tv(0)] = \exp_p(tv)$, $0 \leq t \leq 1$ which inuitively is related to the spread of geodesics.

The differential setting

Since $\gamma(t)$ is a geodesic, for all $t \in [0,1]$ for a fixed s such that $-\varepsilon \leq s \leq \varepsilon$, it turns out that

$$\frac{D}{\partial t}\frac{\partial \sigma}{\partial t} = 0.$$

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and

$$\begin{aligned} 0 = \frac{D}{\partial s}\frac{D}{\partial t}\frac{\partial \sigma}{\partial t} \quad &= \quad \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial \sigma}{\partial t} - R\left(\frac{\partial \sigma}{\partial s}, \frac{\partial \sigma}{\partial t}\right)\frac{\partial \sigma}{\partial t} \\ &= \quad \frac{D}{\partial t}\frac{D}{\partial t}\frac{\partial \sigma}{\partial s} + R\left(\frac{\partial \sigma}{\partial t}, \frac{\partial \sigma}{\partial s}\right)\frac{\partial \sigma}{\partial t} \end{aligned}$$

The differential setting

Since $\dfrac{\partial \sigma}{\partial t} = \gamma'(t)$, the previous (differential) condition becomes

$$0 = \frac{D^2}{\partial t^2} J + R(\gamma'(t), J(t))\gamma'(t) \tag{1}$$

with $J(t) = \dfrac{\partial \sigma}{\partial s}$.

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Definition

Given a geodesic $\gamma : [0, t_0] \to M$, a vector field J along γ is said to be a *Jacobi field* if it satisfies the Jacobi equation (1) for all $t \in [0, t_0]$.

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Remark

$t \mapsto \gamma'(t)$ and $t \mapsto t\gamma'(t)$ are examples of Jacobi fields along γ. Notice, furthermore, that the first one never vanishes and the second one vanishes only for $t = 0$. For this reason, we'll often consider Jacobi fields along γ that are orthogonal to γ'.

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Consider an orthonormal *parallel* frame $(e_1(t), \dots, e_n(t))$ along γ and write

$$J(t) = \sum_{j=1}^{n} v_j(t) e_j(t) \quad a_{ij} = \langle R(\gamma'(t), e_i(t))\gamma'(t), e_j(t)\rangle \quad i, j = 1, \dots, n$$

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Then

$$\frac{D^2}{\partial^2 t} J = \sum_{j=1}^{n} v_j''(t) e_j(t)$$

and

$$R(\gamma', J)\gamma' = \sum_{j=1}^{n} \langle R(\gamma', J)\gamma', e_j\rangle e_j = \sum_{i,j=1}^{n} \langle R(\gamma', v_i e_i)\gamma', e_j\rangle e_j = \sum_{i,j=1}^{n} v_i a_{ij} e_j$$

Jacobi fields in local coordinates

Therefore the Jacobi equation in (this) local frame is equivalent to the following linear system of differential equations of second order

$$v_j''(t) + \sum_{i,j=1}^{n} v_i(t) a_{ij}(t) = 0 \tag{2}$$

which has a local solution in $[0, a]$ given the initial conditions $J(0)$ and $\frac{DJ}{\partial t}(0)$.

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Indeed, for any $W \in \mathcal{X}(M)$,

$$\langle R(\gamma', J)\gamma', W\rangle = K[\langle \gamma', \gamma'\rangle\langle J, W\rangle - \langle \gamma', W\rangle\langle J, \gamma'\rangle] = K\langle J, W\rangle$$

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Therefore, the Jacobi equation becomes

$$\frac{D^2}{\partial^2 t} J + KJ = 0 \tag{3}$$

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1) $|w(t)| = 1$;
2) $\langle w(t), \gamma' \rangle = 0$ Then

$$J(t) := \begin{cases} \dfrac{\sin t\sqrt{K}}{\sqrt{K}} w(t) & \text{if} \quad K > 0 \\ \\ tw(t) & \text{if} \quad K = 0 \\ \\ \dfrac{\sinh t\sqrt{-K}}{\sqrt{-K}} w(t) & \text{if} \quad K < 0 \end{cases}$$

is a solution of the Jacobi equation (3) with initial conditions $J(0) = 0$, $J'(0) = w(0)$.

If $\gamma : [0, a] \to M$ is a geodesic with $\gamma(0) = p$, $\gamma'(0) = v$, consider $w \in T_pM$ and a curve $s \to v(s)$ in T_pM such that $v(0) = av$ and $v'(0) = w$.

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Then $J(t) := \frac{\partial}{\partial s} \exp_p(\frac{t}{a} v(s))|_{s=0}$ is the Jacobi field along γ such that $J(0) = 0$ and $\frac{DJ}{\partial t}(0) = w$.

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$$\frac{D}{\partial t}\frac{\partial}{\partial s} \exp_p \left(\frac{t}{a} v(s)\right) |_{s=0} = \frac{D}{\partial t}[(d \exp_p)_{tv(0)}(tv'(0))] = \frac{D}{\partial t}[(d \exp_p)_{tv}(tw)] =$$

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$$= \frac{D}{\partial t}[t(d\exp_p)_{tv}w] = (d\exp_p)_{tv}w + t\frac{D}{\partial t}[(d\exp_p)_{tv}w]$$

hence for $t = 0$

$$\frac{DJ}{\partial t}(0) = (d\exp_p)_0 w = w.$$

Taylor expansion

From the previous considerations, if $\gamma : [0, a] \to M$ is a geodesic with $\gamma(0) = p$, $\gamma'(0) = v$ and if $w \in T_pM$, then

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$$G(0) = |J(0)|^2 = 0$$

and

$$G'(0) = \frac{d}{dt}\langle J(t), J(t)\rangle|_{t=0} = 2\langle \frac{DJ}{\partial t}(t), J(t)\rangle|_{t=0} = 2\langle w, 0\rangle = 0$$

Similarly,

$$
\begin{aligned}
G''(0) &= 2\frac{d}{dt}\langle \frac{DJ}{\partial t}(t), J(t)\rangle|_{t=0} = \\
&= 2\langle \frac{DJ}{\partial t}(t), \frac{DJ}{\partial t}(t)\rangle|_{t=0} + 2\langle \frac{D^2 J}{\partial^2 t}(t), J(t)\rangle|_{t=0} = \\
&= 2\langle w, w\rangle + 2\langle \frac{D^2 J}{\partial t^2}(0), J(0)\rangle = 2
\end{aligned}
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G'''(0) &= 2\frac{d}{dt}\langle \frac{DJ}{\partial t}(t), \frac{DJ}{\partial t}(t)\rangle|_{t=0} + 2\frac{d}{dt}\langle \frac{D^2 J}{\partial t^2}(t), J(t)\rangle|_{t=0} \\
&= 4\langle \frac{D^2 J}{\partial t^2}(t), \frac{DJ}{\partial t}(t)\rangle|_{t=0} + 2\langle \frac{D^2 J}{\partial t^2}(t), \frac{DJ}{\partial t}(t)\rangle|_{t=0} + \\
&+ 2\langle \frac{D^3 J}{\partial t^3}(t), J(t)\rangle|_{t=0} = 6\langle \frac{D^2 J}{\partial t^2}(0), \frac{DJ}{\partial t}(0)\rangle + \\
&+ 2\langle \frac{D^3 J}{\partial t^3}(0), J(0)\rangle = 0
\end{aligned}
$$

since $J(0) = 0$ and $\frac{D^2 J}{\partial^2 t}(0) = -R(\gamma'(0), J(0))\gamma'(0) = -R(v, 0)v = 0$

$$
\begin{aligned}
G^{iv}(0) &= 8\langle \frac{D^3 J}{\partial t^3}(t), \frac{DJ}{\partial t}(t)\rangle|_{t=0} + 6\langle \frac{D^2 J}{\partial t^2}(t), \frac{D^2 J}{\partial t^2}(t)\rangle|_{t=0} + \\
&+ 2\langle \frac{D^4 J}{\partial t^4}(t), J(t)\rangle|_{t=0} = \\
&= 8\langle \frac{D}{\partial t}(-R(\gamma'(t), J(t))\gamma'(t), \frac{DJ}{\partial t}(t)\rangle|_{t=0} + \\
&+ 6\langle R(\gamma'(0), J(0))\gamma'(0), R(\gamma'(0), J(0))\gamma'(0)\rangle + 2\langle \frac{D^4 J}{\partial t^4}(0), J(0)\rangle = \\
&= 8\langle \frac{D}{\partial t}(-R(\gamma'(t), J(t))\gamma'(t), \frac{DJ}{\partial t}(t)\rangle|_{t=0} + 0
\end{aligned}
$$

since $J(0) = 0$.

Now, for any vector field $W(t)$ along γ, we have

$$\begin{aligned}
\langle \frac{D}{\partial t} R(\gamma'(t), J(t))\gamma'(t), W(t)\rangle|_{t=0} &= \frac{d}{dt}\langle R(\gamma'(t), J(t))\gamma'(t), W(t)\rangle|_{t=0} \\
&- \langle R(\gamma'(t), J(t))\gamma'(t), \frac{DW}{\partial t}(t)\rangle|_{t=0}
\end{aligned}$$

$$= \frac{d}{dt}\langle R(\gamma'(t), W(t))\gamma'(t), J(t)\rangle|_{t=0} - \langle R(\gamma'(t), J(t))\gamma'(t), \frac{DW}{\partial t}(t)\rangle|_{t=0}$$

$$= \langle \frac{D}{\partial t} R(\gamma'(t), W(t))\gamma'(t), J(t)\rangle|_{t=0} + \langle R(\gamma'(t), W(t))\gamma'(t), \frac{DJ}{\partial t}(t)\rangle|_{t=0}$$
$$- \langle R(\gamma'(t), J(t))\gamma'(t), \frac{DW}{\partial t}(t)\rangle|_{t=0}$$

Hence, if $W(t) = \frac{DJ}{\partial t}(t)$, since $J(0) = 0$, $\gamma'(0) = v$, $\frac{DJ}{\partial t}(0) = w$

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$$= \langle R(v, w)v, w\rangle = \mathcal{R}(v, w, v, w)$$

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Therefore

$$G^{iv}(0) = -8\mathcal{R}(v, w, v, w)$$

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. Finally, for t in a neighborhood of 0

$$G(t) = |J(t)|^2 = t^2 - \frac{1}{3}\mathcal{R}(v, w, v, w)t^4 + \mathfrak{R}(t)$$

with $\mathfrak{R}(t) = o(t^4)$.

Remark

If - in the above calculations -γ is a geodesic in M parametrized by arc length (hence $|v| = 1$ and w is assumed to be orthogonal to v ($\langle w, v\rangle = 0$), then $\mathcal{R}(v, w, v, w) = K_p(S)$ where S is the plane spanned by v and w.

Remark

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$$|J(t)|^2 = t^2 - \frac{1}{3}K_p(S)t^4 + \mathfrak{R}(t)$$

with $\mathfrak{R}(t) = o(t^4)$ and

$$|J(t)| = t - \frac{1}{6}K_p(S)t^3 + \widehat{\mathfrak{R}}(t)$$

with $\widehat{\mathfrak{R}}(t) = o(t^3)$, since $\sqrt{1+x} = 1 + \frac{1}{2}x + o(x)$.

Conjugate points

Definition

Given a geodesic $\gamma : [0, a] \to M$, the point $\gamma(t_0)$ is said to be *conjugate* to $\gamma(0)$ along γ (with $t_0 \in (0, a]$) if there exists a (not identically zero) Jacobi field J along γ such that $J(0) = 0 = J(t_0)$.

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Given a geodesic $\gamma : [0, a] \to M$, the point $\gamma(t_0)$ is said to be *conjugate* to $\gamma(0)$ along γ (with $t_0 \in (0, a]$) if there exists a (not identically zero) Jacobi field J along γ such that $J(0) = 0 = J(t_0)$.

The maximum number of linearly independent Jacobi fields along γ such that $J(0) = 0 = J(t_0)$ is called the *multiplicity* of the conjugate point $\gamma(t_0)$.

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The set of (first) conjugate points to $p = \gamma(0)$ is called the *conjugate locus* of p and denoted by $C(p)$.

Conjugate points: a characterization

Proposition

Let $\gamma : [0, a] \to M$ *be a geodesic with* $\gamma(0) = p$. *Then the point* $q = \gamma(t_0)$ *(with* $t_0 \in (0, a]$*) is conjugate to* $p = \gamma(0)$ *along* γ *if and only if* $v_0 := t_0\gamma'(t_0)$ *is a critical point of* $\exp_p$.

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Furthermore the multiplicity of $q = \gamma(t_0)$ *as a conjugate point of* $p = \gamma(0)$ *is equal to the dimension of* $\ker(d\exp_p)_{v_0}$.

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$$J(t) = (d\exp_p)_{tv}(tw)$$

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Then J is a Jacobi field such that $J(0) = 0$ and $w = \frac{DJ}{\partial t}(0)$; observe that J is a non-zero vector field along γ if and only if $w \neq 0$.

Hence $q = \gamma(t_0)$ is conjugate to $p = \gamma(0)$ (when $w \neq 0$) if and only if

$$0 = J(t_0) = (d\exp_p)_{t_0 v}(t_0 w)$$

or $v_0 = t_0\gamma'(t_0)$ is a critical point of $\exp_p$. □

Additional Lemmas on Jacobi fields

Lemma

Let J be a Jacobi field along the geodesic $\gamma : [0, a] \to M$. Then, for any $t \in [0, a]$

$$\langle J(t), \gamma'(t) \rangle = \langle \frac{DJ}{\partial t}(0), \gamma'(0) \rangle t + \langle J(0), \gamma'(0) \rangle$$

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Corollary

Let J be a Jacobi field along the geodesic $\gamma : [0, a] \to M$. If for some $t_1, t_2 \in [0, a]$ $(t_1 \neq t_2)$

$$\langle J(t_1), \gamma'(t_1) \rangle = \langle J(t_2), \gamma'(t_2) \rangle$$

then $\langle J(t), \gamma'(t) \rangle$ does not depend on t. In particular if $J(0) = J(a) = 0$, then $J(t)$ is orthogonal to $\gamma'(t)$ for any $t \in [0, a]$ (along γ).

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Corollary

Let J be a Jacobi field along the geodesic $\gamma : [0, a] \to M$ such that $J(0) = 0$. Then

$$\langle \frac{DJ}{\partial t}(0), \gamma'(0) \rangle = 0$$

if and only if

$$\langle J(t), \gamma'(t) \rangle \equiv 0.$$

In particular the space of Jacobi fields with $J(0) = 0$ and $\langle J(t), \gamma'(t) \rangle \equiv 0$ has dimension $n - 1$.

Additional Lemmas on Jacobi fields

Proposition

Let $\gamma : [0, a] \to M$ *be a geodesic. Take* $v_1 \in T_{\gamma(0)}M$ *and* $v_2 \in T_{\gamma(a)}M$*. If* $\gamma(a)$ *is not conjugate to* $\gamma(0)$*, then there exists a unique Jacobi field* J *along* γ *such that* $J(0) = v_1$ *and* $J(a) = v_2$*.*

Proposition

Let $\gamma : [0, a] \to M$ be a geodesic in a Riemannian manifold M and let $p = \gamma(0)$, $q = \gamma(b)$.

If there is no conjugate point of p along γ, then there exists $\varepsilon > 0$ such that for any piecewise differentiable curve $\alpha : [0, a] \to M$ from p to q such that $g(\gamma(t), \alpha(t)) < \varepsilon$ we have $L(\alpha) \geq L(\gamma)$ with equality if and only if α is a reparameterization of γ.

Jacobi

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If there exists $t_0 \in (0, a]$ *such that* $q_0 = \gamma(t_0)$ *is a conjugate of* p, *then there is a proper variation of* γ *so that* $L(\gamma_s) \leq L(\gamma)$ *for* $-\varepsilon < s < \varepsilon$.