The Drude Theory of Metals

Outline

The Drude theory of metals

The model

The metallic state

- A fundamental state of matter
- Most elements are metallic
  - more than $\frac{2}{3}$ of the periodic table
- Possess striking properties:
  - ductile and malleable
  - good conductors of heat and electricity
  - luster of freshly exposed surfaces
- Need of a theory of the metallic state
  - to understand their properties
  - ... and the properties of insulators

The Drude theory of metals

The model

General remarks

- Apply the classical kinetic theory of gases to the conduction electrons
  - viewed as an electron gas
- Works well in some instances, because of cancellation errors
- Can be used for gross estimates of metallic properties

Kinetic theory of gases: overview

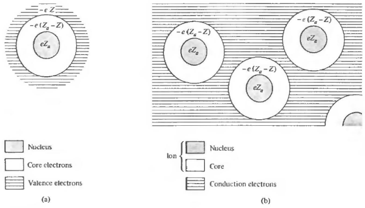
- Atoms/molecules modelled as spheres
- No forces acting between molecules
  - straight motion between collisions
- Collisions are istantaneous and elastic
  - among molecules or with the walls of the container
- Kinetic energy is a measure of T ($\frac{1}{2}mv^2 = \frac{3}{2}k_B T$)

The Drude theory of metals

The model

Valence and core electrons

- Atom of atomic number Z_a with Z valence electrons:
  - involved in the chemical bond
  - periodicity of chemical/physical properties of elements
  - conduction electrons in a metal
- Z_a-Z core electrons:
  - tightly bound to the nucleous, essentially atomic
  - positive background confining the electron gas within the volume of the solid



The Drude theory of metals

The model

Rough estimate of the density of the free electron gas

- Density of the electron gas:

$$n = \frac{N}{V} = 6.022 \times 10^{23} \frac{Z \rho_m}{A}$$

  - n: electron density (number of electrons per cm^3)
  - ρ_m: density of the element ($\frac{g}{cm^3}$)
  - A: atomic mass of the element
- Alternative measure, parameter r_s: radius of a sphere of volume equal to the volume per electron in the electron gas
  - $\frac{4\pi}{3} r_s^3 = \frac{V}{N} = \frac{1}{n}$
  - $r_s = \left(\frac{3}{4\pi n}\right)^{\frac{1}{3}}$
- Electron gas densities are $\sim$ 1000 times higher than those of a classical gas at normal T and p

The Drude theory of metals

The model

Free electron densities

FREE ELECTRON DENSITIES OF SELECTED METALLIC ELEMENTS[a]

ELEMENT	Z	n (10^{22}/cm^3)	r_s(Å)	r_s/a_0
Li (78 K)	1	4.70	1.72	3.25
Na (5 K)	1	2.65	2.08	3.93
K (5 K)	1	1.40	2.57	4.86
Rb (5 K)	1	1.15	2.75	5.20
Cs (5 K)	1	0.91	2.98	5.62
Cu	1	8.47	1.41	2.67
Ag	1	5.86	1.60	3.02
Au	1	5.90	1.59	3.01
Be	2	24.7	0.99	1.87
Mg	2	8.61	1.41	2.66
Ca	2	4.61	1.73	3.27
Sr	2	3.55	1.89	3.57
Ba	2	3.15	1.96	3.71
Nb	1	5.56	1.63	3.07
Fe	2	17.0	1.12	2.12
Mn (α)	2	16.5	1.13	2.14
Zn	2	13.2	1.22	2.30
Cd	2	9.27	1.37	2.59
Hg (78 K)	2	8.65	1.40	2.65
Al	3	18.1	1.10	2.07
Ga	3	15.4	1.16	2.19
In	3	11.5	1.27	2.41
Tl	3	10.5	1.31	2.48
Sn	4	14.8	1.17	2.22
Pb	4	13.2	1.22	2.30
Bi	5	14.1	1.19	2.25
Sb	5	16.5	1.13	2.14

The Drude theory of metals

The model

Assumptions and approximations

- Electron gas treated by the methods of the kinetic theory of diluite gases despite:
    - strong electron-electron repulsion
    - strong electron-ion interactions

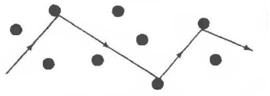
1. Between collisions the interaction of the electron with the other electrons and with the ionic cores is neglected $\longrightarrow$ uniform motion of electrons
    - free electron approximation: neglect electron-ionic cores interaction (serious approx., must be abandoned)
    - independent electron approximation: neglect electron-electron interaction (usually a good approx.)
    - in the absence of external fields (otherwise use Newton's laws)
    - the interaction with ionic cores is implicitly assumed (confinement of electrons) in the volume of the metal

The Drude theory of metals

The model

Assumptions and approximations

2. Collisions are istantaneous
   - change abruptly the electron's velocity
   - Drude assumption: due to collision with the immobile ionic cores (conceptually wrong)
   - electron-electron scattering is not important
   - the detailed scattering mechanism is actually not important for a qualitative understanding of metallic conduction (simply assume that there is a scattering mechanism)



electron's trajectory scattering off the ionic cores (conceptually wrong)

The Drude theory of metals

The model

Assumptions and approximations

3. Probability of collision per unit time: $\frac{1}{\tau}$
    - τ: relaxation time, or mean free time
    - average time interval between successive collisions
    - taken to be independent of electron momentum and position
    - $\frac{dt}{\tau}$: probability of a collision in the time interval dt
4. Thermal equilibrium with the surroundings is reached through collisions
    - local temperature determines the electron velocity after a collision:
        - the velocity is randomly directed
        - not related to the velocity just before the collision

DC electrical conductivity

Predictions from the Drude model

Ohm's law

$$V = RI$$

- V: potential drop
- R: resistance of the wire
  - depends on nature and shape (dimensions) of the material
  - proportionality constant btw V and I
- I: current intensity
  - units of $\frac{C}{s}$
- Drude's model can be used for an estimate of the size of R

DC electrical conductivity

Predictions from the Drude model

Resistivity

- Elimination of the dependence on the shape of the wire:

$$\boldsymbol{E} = \rho \boldsymbol{j}$$

- ρ: resistivity
  - depends only on the nature of the material
  - generally a tensor quantity
- $\boldsymbol{E}$: electric field vector
- $\boldsymbol{j}$: (induced) density current $(\frac{C}{cm^2 s})$
  - direction parallel to the flow of charge
  - amount of charge per unit time crossing an unit area $\perp$ to the flow
- For uniform I flowing in a wire of length L and cross sectional area A: $R = \rho \frac{L}{A}$

DC electrical conductivity

Predictions from the Drude model

Current density

- If all e^- move with constant $\boldsymbol{v}$:

$$dq = -enA(vdt) \rightarrow \boldsymbol{j} = -ne\boldsymbol{v}$$

- when $\boldsymbol{E} = 0 \rightarrow \boldsymbol{v} = 0$
  - $\boldsymbol{v}$: average electronic velocity
  - random orientations, no net flow of charge
- If $\boldsymbol{E} \neq 0$:

$$\boldsymbol{v} = -e\frac{\boldsymbol{E}\bar{t}}{m} = -e\frac{\boldsymbol{E}\tau}{m}$$

  - $\boldsymbol{j} = (\frac{ne^2\tau}{m})\boldsymbol{E} = \sigma\boldsymbol{E}$
  - $\sigma = \frac{1}{\rho} = \frac{ne^2\tau}{m}$: electrical conductivity
- The linear relation between $\boldsymbol{j}$ and $\boldsymbol{E}$ is established

DC electrical conductivity

Predictions from the Drude model

Conductivity from Drude model

$$\sigma = \frac{1}{\rho} = \frac{ne^2\tau}{m}$$

- All parameters are known except for τ

Estimate of τ from experimental σ's

$$\tau = \frac{m}{\rho ne^2} = \frac{m}{\rho e^2}\frac{4\pi r_s^3}{3} = (\frac{0.22}{\rho_\mu})(\frac{r_s}{a_0})^3 \times 10^{-14}$$

- ρ_μ (μOhm cm)linear in T at room temperature ($\tau \sim 10^{-14}$–10^{-15}s)
- falls away more steeply at low T

DC electrical conductivity

Predictions from the Drude model

Exp. resistivities (μOhm cm)

ELECTRICAL RESISTIVITIES OF SELECTED ELEMENTS[a]

ELEMENT	77 K	273 K	373 K	$\frac{(\rho/T)_{373\ K}}{(\rho/T)_{273\ K}}$
Li	1.04	8.55	12.4	1.06
Na	0.8	4.2	Melted	
K	1.38	6.1	Melted	
Rb	2.2	11.0	Melted	
Cs	4.5	18.8	Melted	
Cu	0.2	1.56	2.24	1.05
Ag	0.3	1.51	2.13	1.03
Au	0.5	2.04	2.84	1.02
Be		2.8	5.3	1.39
Mg	0.62	3.9	5.6	1.05
Ca		3.43	5.0	1.07
Sr	7	23		
Ba	17	60		
Nb	3.0	15.2	19.2	0.92
Fe	0.66	8.9	14.7	1.21
Zn	1.1	5.5	7.8	1.04
Cd	1.6	6.8		
Hg	5.8	Melted	Melted	
Al	0.3	2.45	3.55	1.06
Ga	2.75	13.6	Melted	
In	1.8	8.0	12.1	1.11
Tl	3.7	15	22.8	1.11
Sn	2.1	10.6	15.8	1.09
Pb	4.7	19.0	27.0	1.04
Bi	35	107	156	1.07
Sb	8	39	59	1.11

DC electrical conductivity

Predictions from the Drude model

Drude relaxation times $\tau = (\frac{0.22}{\rho_\mu})(\frac{r_s}{a_0})^3 \times 10^{-14}$s

DRUDE RELAXATION TIMES IN UNITS OF 10^{-14} SECOND[a]

ELEMENT	77 K	273 K	373 K
Li	7.3	0.88	0.61
Na	17	3.2	
K	18	4.1	
Rb	14	2.8	
Cs	8.6	2.1	
Cu	21	2.7	1.9
Ag	20	4.0	2.8
Au	12	3.0	2.1
Be		0.51	0.27
Mg	6.7	1.1	0.74
Ca		2.2	1.5
Sr	1.4	0.44	
Ba	0.66	0.19	
Nb	2.1	0.42	0.33
Fe	3.2	0.24	0.14
Zn	2.4	0.49	0.34
Cd	2.4	0.56	
Hg	0.71		
Al	6.5	0.80	0.55
Ga	0.84	0.17	
In	1.7	0.38	0.25
Tl	0.91	0.22	0.15
Sn	1.1	0.23	0.15
Pb	0.57	0.14	0.099
Bi	0.072	0.023	0.016
Sb	0.27	0.055	0.036

DC electrical conductivity

Predictions from the Drude model

Relaxation times and electron's velocities

- mean electron's velocity, $\boldsymbol{v}_0$:
  - $\frac{1}{2}mv_0^2 = \frac{3}{2}k_B T$
  - $v_0 \sim 10^7$ cm/s at room T
  - 1 order of magniture too small (and T independent)
- mean free path, l:
  - average distance btw scattering events
  - $l = v_0\tau \sim$1–10 Å (interatomic spacing)
  - τ is actually T dependent
  - $l \sim cm$ can be achieved (low T, low defects)
  - electrons do not simply bump off the ions (contrary to Drude original assumption)
- It is important to find predictions from the Drude's model which are independent of the value of τ:
  - σ's in the present of uniform magnetic fields
  - σ's in the present of uniform time-dependent electric fields

DC electrical conductivity

Predictions from the Drude model

Equation of motion in an external field, $\boldsymbol{f}$

- If $\boldsymbol{p}(t)$ is the total momentum per electorn
- $\boldsymbol{j} = -\frac{ne\boldsymbol{p}(t)}{m}$
- $\boldsymbol{p}(t+dt) = (1 - \frac{dt}{\tau})[\boldsymbol{p}(t) + \boldsymbol{f}(t)dt]$
- The equation of motion reads

$$\frac{d\boldsymbol{p}(t)}{dt} = -\frac{\boldsymbol{p}(t)}{\tau} + \boldsymbol{f}(t)$$

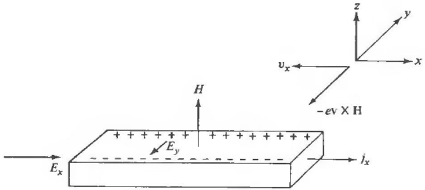
  - correct to order O(dt)
  - $\boldsymbol{f}$: average force per electron
  - Effect of collisions: introduce a frictional damping term in the eom

Hall effect and magnetoresistance

Predictions from the Drude model

Experimental setup

- Uniform $\boldsymbol{E}$ field applied to a wire extending in the x direction
- E_x and j_x: electric field and current density in the wire
- $\boldsymbol{H}$: constant magnetic field in the $\hat{\boldsymbol{z}}$ direction
- Lorentz force: $\boldsymbol{f} = -\frac{e}{c}\boldsymbol{v} \times \boldsymbol{H}$
  - electrons are deflected in the negative $\hat{\boldsymbol{y}}$ direction
  - transverse field E_y builds up to prevent further accumulation
  - at equilibrium E_y balances the Lorentz force

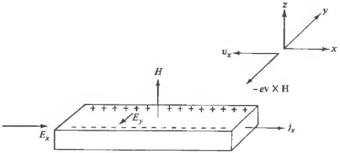


Hall effect and magnetoresistance

Predictions from the Drude model

Magnetoresistance and Hall coefficient

- transverse magnetoresistance: $\rho(H) = \frac{E_x}{j_x}$
  - field-independent in Hall's original exp.
  - longitudinal when $\boldsymbol{H} \parallel \boldsymbol{j}$
- Hall coefficient: $R_H = \frac{E_y}{j_x H}$
  - proportional to E_y
  - must be negative for negative charge carriers ($E_y < 0$)
  - In some metals, $R_H > 0$ (quantum effects)
  - Drude's theory is unable to account for positive R_H



Hall effect and magnetoresistance

Predictions from the Drude model

Magnetoresistance and Hall coefficient

- Consider the equations of motion:

$$\frac{d\boldsymbol{p}(t)}{dt} = -\frac{\boldsymbol{p}(t)}{\tau} + \boldsymbol{f}(t)$$

- with $\boldsymbol{f} = -e(\boldsymbol{E} + \frac{\boldsymbol{p}}{mc} \times \boldsymbol{H})$ (position independent)
- steady-state conditions: $\rightarrow \frac{d\boldsymbol{p}(t)}{dt} = 0$ $(\omega_c = \frac{eH}{mc})$

$$\begin{cases} 0 = & -eE_x - \omega_c p_y - \frac{p_x}{\tau} \\ 0 = & -eE_y + \omega_c p_x - \frac{p_y}{\tau} \end{cases}$$

Hall effect and magnetoresistance

Predictions from the Drude model

Magnetoresistance and Hall coefficient

- Multiply by $-\frac{ne\tau}{m}$ ($\sigma_0 = \frac{ne^2\tau}{m}$, Drude's DC conductivity):

$$\begin{cases} \sigma_0 E_x & = \omega_c \tau j_y + j_x \\ \sigma_0 E_y & = -\omega_c \tau j_x + j_y \end{cases}$$

- If $j_y = 0 \rightarrow E_y = -(\frac{H}{nec}) j_x \rightarrow R_H = -\frac{1}{nec}$
-
- In Drude's theory, R_H:
  - for a given metal depends only on the density of the charge carriers (n)
  - is independent of H (exp. true for low T and high H)

Hall effect and magnetoresistance

Predictions from the Drude model

Comparison with the experiment

- Experimentally, R_H:
  - depends on T and $\boldsymbol{H}$, conditions of the sample's preparation
  - its limiting value (low T, high H) coincides with Drude's prediction for some metals (alkali metals)
  - can be positive
- Define: $n_0 = -\frac{1}{R_H ec}$, data are in the form of $\frac{n_0}{n}$

Table 1.4
HALL COEFFICIENTS OF SELECTED ELEMENTS IN MODERATE TO HIGH FIELDS[a]

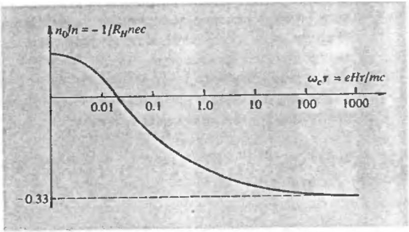
METAL	VALENCE	$-1/R_H nec$
Li	1	0.8
Na	1	1.2
K	1	1.1
Rb	1	1.0
Cs	1	0.9
Cu	1	1.5
Ag	1	1.3
Au	1	1.5
Be	2	−0.2
Mg	2	−0.4
In	3	−0.3
Al	3	−0.3

Hall effect and magnetoresistance

Predictions from the Drude model

Comparison with the experiment

- $n_0 = -\frac{1}{R_H ec}$
- Data are in the form of $\frac{n_0}{n}$
- For Al it predicts one carrier per primitive cell with charge e



Hall effect and magnetoresistance

Predictions from the Drude model

Magnetoresistance and Hall coefficient

- Consider once again the equations:

$$\begin{cases} \sigma_0 E_x & = \omega_c \tau j_y + j_x \\ \sigma_0 E_y & = -\omega_c \tau j_x + j_y \end{cases}$$

- If $\omega_c \tau \to 0 \longrightarrow \boldsymbol{E} \parallel \boldsymbol{j}$
- In general they are at an angle ϕ
  - $\tan \phi = \omega_c \tau$: Hall angle
  - ω_c is the cyclotron frequency: angular frequency of revolution of an electron in a magnetic field
  - $\nu_c(10^9 Hz) = 2.80 \times H(kG)$

AC electrical conductivity

Predictions from the Drude model

Frequency dependent conductivity, $\sigma(\omega)$

- EOM: $\frac{d\boldsymbol{p}(t)}{dt} = -\frac{\boldsymbol{p}}{\tau} - e\boldsymbol{E}$
- $\boldsymbol{E}(t) = \mathrm{Re}(\boldsymbol{E}(\omega)e^{-i\omega t})$, $\boldsymbol{p}(t) = \mathrm{Re}(\boldsymbol{p}(\omega)e^{-i\omega t})$
  - steady-state solution
- $\boldsymbol{p}(\omega)$ satisfies:

$$-i\omega\boldsymbol{p}(\omega) = -\frac{\boldsymbol{p}(\omega)}{\tau} - e\boldsymbol{E}(\omega)$$

- Therefore:

$$\boldsymbol{j}(\omega) = -\frac{ne\boldsymbol{p}(\omega)}{m} = \frac{(\frac{ne^2}{m})\boldsymbol{E}(\omega)}{(\frac{1}{\tau}) - i\omega}$$

AC electrical conductivity

Predictions from the Drude model

Frequency dependent conductivity, $\sigma(\omega)$

- $\boldsymbol{j}(\omega) = \sigma(\omega)\boldsymbol{E}(\omega)$
- The AC conductivity is:

$$\sigma(\omega) = \frac{\left(\frac{ne^2\tau}{m}\right)}{1 - i\omega\tau} = \frac{\sigma_0}{1 - i\omega\tau}$$

  - σ_0: DC conductivity
  - reduces to σ_0 for $\omega \to 0$
  - useful for studying EM wave propagation in metals

AC electrical conductivity

Predictions from the Drude model

Electromagnetic wave propagation in metals

- Our derivation considered no magnetic field and spatially uniform $\boldsymbol{E}$ fields
- The effect of $\boldsymbol{H}$ can actually be neglected compared to the electric field:
  - for $v \sim 0.1$ cm/sec (for $i = 1$ amp/mm^2) $\longrightarrow \frac{-e\boldsymbol{p}}{mc} \times \boldsymbol{H} \sim \frac{v}{c} \sim 10^{-10}$
  - negligible compared to $\boldsymbol{E}$
- The field $\boldsymbol{E}$ is assumed constant over distances $\sim l$ (mean free path)
  - good approx. for visible and UV radiation ($\lambda \sim 10^3$–10^4 Å)
  - $\boldsymbol{j}(\boldsymbol{r}, t) = \sigma(\omega)\boldsymbol{E}(\boldsymbol{r}, t)$
- For shorter λ's: $\rightarrow$ non-local theories

AC electrical conductivity

Electromagnetic wave propagation in metals

Maxwell's Equations ($\rho = 0$)

$$\begin{cases} \nabla \cdot \boldsymbol{E} &= 0 \\ \nabla \cdot \boldsymbol{H} &= 0 \\ \nabla \times \boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} \\ \nabla \times \boldsymbol{H} &= \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \end{cases}$$

AC electrical conductivity

Predictions from the Drude model

Electromagnetic wave propagation in metals

- Ansatz: $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{r}, \omega) e^{-i\omega t}$
- Use $\nabla \times \nabla \times \boldsymbol{E} = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E}$
- The wave-equation is obtained

$$-\nabla^2 \boldsymbol{E} = \frac{\omega^2}{c^2} \varepsilon(\omega) \boldsymbol{E}$$

  - $\varepsilon(\omega) = 1 + \frac{4\pi i \sigma}{\omega}$, complex dielectric constant
- If $\omega\tau >> 1 \longrightarrow \varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$
  - $\omega_p^2 = \frac{4\pi n e^2}{m}$: plasma frequency

AC electrical conductivity

Predictions from the Drude model

Electromagnetic wave propagation in metals

- Wave propagation can only occur for $\omega > \omega_p$
  - provided $\omega\tau >> 1$ for $\omega \sim \omega_p$
- $\omega_p \tau = 1.6 \times 10^2 (\frac{r_s}{a_0})^{3/2} (\frac{1}{\rho_\mu})$
  - $\frac{r_s}{a_0}$ ~2–6; $\rho_\mu \sim \mu\Omega$cm
- $\lambda_p = \frac{2\pi c}{\omega_p} = 0.26 \times (\frac{r_s}{a_0})^{3/2} \times 10^3$Å

Table 1.5
OBSERVED AND THEORETICAL WAVELENGTHS BELOW WHICH THE ALKALI METALS BECOME TRANSPARENT

ELEMENT	THEORETICAL[a] λ (10^3 Å)	OBSERVED λ (10^3 Å)
Li	1.5	2.0
Na	2.0	2.1
K	2.8	3.1
Rb	3.1	3.6
Cs	3.5	4.4

AC electrical conductivity

Predictions from the Drude model

Charge density oscillations (plasmons)

- Solutions of the type $\rho(\boldsymbol{r}, t) = \rho(\boldsymbol{r}, \omega) e^{-i\omega t}$, from:
  - charge conservation: $\frac{dq}{dt} + \int_S \boldsymbol{j} \cdot \hat{\boldsymbol{n}} dS = 0$
  - Gauss theorem

$$\begin{cases} \nabla \cdot \boldsymbol{j} & = -\frac{\partial \rho}{\partial t} \\ \nabla \cdot \boldsymbol{E} & = 4\pi\rho \end{cases}$$

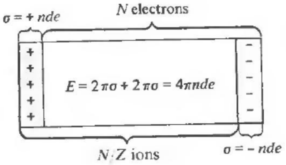
- Non trivial solutions for: $1 + \frac{i4\pi\sigma(\omega)}{\omega} = 0 \Longrightarrow \omega = \omega_p$

AC electrical conductivity

Predictions from the Drude model

Charge density oscillations (plasmons)

- Displace the electron charge by a distance d from the positive uniform background
- Charge separation, distributed on two opposite surfaces: $\sigma = \frac{-enAd}{A} = -nde$ is the charge density
- Equation of motion: $Nm\ddot{d} = -4\pi ne^2 Nd$
  - Harmonic motion with $\omega^2 = \frac{4\pi ne^2}{m} = \omega_p^2$



simple model of plasma oscillation

Thermal conductivity

Predictions from the Drude model

Explanation of the Wiedemar-Franz law (1853)

- For many metals $\frac{\kappa}{\sigma} \propto T$
  - σ: electrical conductivity
  - κ: thermal conductivity (positive quantity)
- $\frac{\kappa}{\sigma T}$ roughly the same for all metals (Lorentz number)

EXPERIMENTAL THERMAL CONDUCTIVITIES AND LORENZ NUMBERS OF SELECTED METALS

	273 K		373 K	
ELEMENT	κ (watt/cm-K)	$\kappa/\sigma T$ (watt-ohm/K^2)	κ (watt/cm-K)	$\kappa/\sigma T$ (watt-ohm/K^2)
Li	0.71	2.22×10^{-8}	0.73	2.43×10^{-8}
Na	1.38	2.12		
K	1.0	2.23		
Rb	0.6	2.42		
Cu	3.85	2.20	3.82	2.29
Ag	4.18	2.31	4.17	2.38
Au	3.1	2.32	3.1	2.36
Be	2.3	2.36	1.7	2.42
Mg	1.5	2.14	1.5	2.25
Nb	0.52	2.90	0.54	2.78
Fe	0.80	2.61	0.73	2.88
Zn	1.13	2.28	1.1	2.30
Cd	1.0	2.49	1.0	
Al	2.38	2.14	2.30	2.19
In	0.88	2.58	0.80	2.60
Tl	0.5	2.75	0.45	2.75
Sn	0.64	2.48	0.60	2.54
Pb	0.38	2.64	0.35	2.53
Bi	0.09	3.53	0.08	3.35
Sb	0.18	2.57	0.17	2.69

Thermal conductivity

Predictions from the Drude model

Explanation of the Wiedemar-Franz law

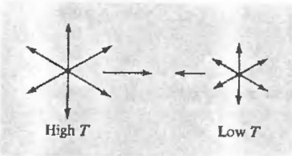
- The thermal current is carried by the conduction electrons
  - insulators do not conduct heat well
  - thermal conduction by the ions is less important but not zero (elastic waves propagating through the network of ions)
- Consider a metal bar along which T varies slowly. Supply heat to the hot end to compensate the flow of heat opposite to the gradient of T: a steady state can be reached
- Fourier's Law: $\boldsymbol{j}^q = -\kappa \nabla T$
  - small temperature gradient (∇T)
  - $\boldsymbol{j}^q$: thermal current density
  - κ: thermal conductivity ($\kappa > 0$)
- $\boldsymbol{j}^q \parallel$ to the heat flow

Thermal conductivity

Predictions from the Drude model

Explanation of the Wiedemar-Franz law

- Consider a 1D model metal bar (uniform T drop along x)
- $j^q = -\kappa \frac{dT}{dx}$
  - $\frac{n}{2}$ e^- move $\rightarrow$, $\frac{n}{2}$ e^- move $\leftarrow$
  - T is a function of x ($T = T[x]$)
- Thermal energy per e^-: $\epsilon = \epsilon(T[x])$ (assumption 4 of the model)



net flow of heat to the low T side

Thermal conductivity

Predictions from the Drude model

Explanation of the Wiedemar-Franz law

- At point x, $j^q = \frac{1}{2}nv\{\epsilon(T[x - v\tau] - \epsilon(T[x + v\tau])\}$
- Therefore $j^q = nv^2\tau\frac{d\epsilon}{dT}(-\frac{dT}{dx})$
  - Taylor exp. around x, small variations of T
- In 3D: $\boldsymbol{j}^q = \frac{1}{3}v^2\tau c_v(-\nabla T)$
- $\kappa = \frac{1}{3}v^2\tau c_v = \frac{1}{3}lvc_v$
  - $\boldsymbol{v}$ distribution is isotropic: $\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2} = \frac{1}{3}\overline{v^2}$
  - $\overline{v^2}$: electron's mean square speed
  - $c_v = \frac{N}{V}\frac{d\epsilon}{dT}$: specific heat capacity
  - Assume $\overline{v^2}$ is T independent

Thermal conductivity

Predictions from the Drude model

Explanation of the Wiedemar-Franz law

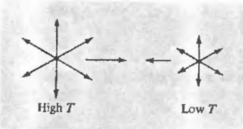
- $\frac{\kappa}{\sigma} = \frac{\frac{1}{3} c_v m v^2}{n e^2}$
- Using ideal gas law: $\frac{\kappa}{\sigma} = \frac{3}{2}(\frac{k_B}{e})^2 T$
  - $c_v = \frac{3}{2} n k_B$
  - $\frac{1}{2} m v^2 = \frac{3}{2} k_B T$
- $\frac{\kappa}{\sigma T} = \frac{3}{2}(\frac{k_B}{e})^2$
  - $\frac{\kappa}{\sigma T} = 1.11 \times 10^{-8}$ watt·ohm/K^2 (half the correct value)
- Two errors compensating each other:
  - c_v is about 100 times smaller than the classical prediction
  - v^2 is about 100 times larger than the classical prediction
- Need quantum statistical mechanics (Fermi-Dirac distribution)

Thermoelectric effects

Predictions from the Drude model

The Seebeck effect

- A temperature gradient is accompained by $\boldsymbol{E}$ opposite to ∇T
- $\boldsymbol{E} = Q\nabla T$
  - Q: thermopower
- It cancels the effect of ∇T on v^2



net flow of heat to the low T side

Thermoelectric effects

Predictions from the Drude model

Estimation of Q

- $\boldsymbol{v}_Q$: mean $\boldsymbol{v}$ due to ∇T
- $\boldsymbol{v}_E$: mean $\boldsymbol{v}$ due to $\boldsymbol{E}$
  - $\boldsymbol{v}_E = -\frac{e\boldsymbol{E}\tau}{m}$
  - $\boldsymbol{v}_Q = -\frac{\tau}{6}\frac{dv^2}{dT}(\nabla T)$
- Put $\boldsymbol{v}_E + \boldsymbol{v}_Q = 0 \Longrightarrow Q = -\frac{c_v}{3ne}$
- $Q = -\left(\frac{1}{3e}\right)\frac{d}{dT}\frac{mv^2}{2} = -\frac{c_v}{3ne}$
- If $c_v = \frac{3nk_B}{2} \rightarrow Q = -\frac{k_B}{2e} = -0.43 \times 10^{-4}\mathrm{V/K}$
  - Exp. Q values are $\sim \mu\mathrm{V/K}$
  - due to non-compensating error on c_v
- classical statistical mechanics is inadequate