Lesson 19 - 16/4/2026

Fourier transform $\quad f \in L^1(\mathbb{R}^N)$, $\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\,dx$.

def. $\mathcal{S}(\mathbb{R}^N) = \{ f \in \mathcal{C}^\infty(\mathbb{R}^N) : \forall \alpha, \beta \in \mathbb{N}^n, \ \sup_{x\in\mathbb{R}^N} |x^\alpha D^\beta f(x)| < +\infty \}$

$\uparrow$

Schwartz space or space of rapidly decreasing (at infinity) functions

$= \{ f \in \mathcal{C}^\infty(\mathbb{R}^N) : \forall j \in \mathbb{N} \quad \sup_{x\in\mathbb{R}^N} | (1+|x|)^j \sum_{|\alpha|\le j} |D^\alpha f(x)| | < +\infty \}$

rem. $f \in \mathcal{S}(\mathbb{R}^N)$ then $|D^\alpha f(x)| \le \dfrac{C_{\alpha,j}}{(1+|x|)^j}$

$\forall \alpha \in \mathbb{N}^n, \forall j \in \mathbb{N} \ \exists C_{\alpha,j}$ s.t

ex. $e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^N)$

rem. consider $\forall \alpha, \beta \in \mathbb{N}^n$ $\quad q_{\alpha,\beta}(f) = \sup_{x\in\mathbb{R}^N} |x^\alpha D^\beta f(x)|$

the family $(q_{\alpha,\beta})_{\alpha,\beta}$ gives a Fréchet topology on $\mathcal{S}(\mathbb{R}^N)$

equivalently $\tilde{q}_j(f) = \sup_{x\in\mathbb{R}^N} \left((1+|x|)^j \sum_{|\alpha|\le j} |D^\alpha f(x)|\right)$

rem. $\quad \mathcal{D}(\mathbb{R}^N) \subseteq \mathcal{S}(\mathbb{R}^N) \subseteq \mathcal{E}(\mathbb{R}^N)$

with continuous immersion

Ex. take $(\varphi_n)_n$ in $\mathcal{D}(\mathbb{R}^N)$ s.t. $\varphi_n \xrightarrow{n} 0$ in the sense of $\mathcal{D}$

Then $\forall j$, $\tilde{q}_j(\varphi_n) \xrightarrow{n} 0$

$\left(\mathcal{D}(\mathbb{R}^N) \hookrightarrow \mathcal{S}(\mathbb{R}^N) \text{ is continuous} \right)$

$\tilde{q}_j(\varphi_n) = \sup_{x\in\mathbb{R}^N} \left((1+|x|)^j \sum_{|\alpha|\le j} |D^\alpha \varphi_n(x)|\right)$

fixed

we know that $\forall \varphi_n$, $\operatorname{supp} \varphi_n \subseteq B(0,R)$

then on the support of φ_n

$(1+|x|)^j \le (1+R)^j$

then $\sup_{x\in\mathbb{R}^N} (1+|x|)^j \sum_{|\alpha|\le j} |D^\alpha \varphi_n(x)| \le (1+R)^j \underbrace{\sup_{x\in\mathbb{R}^N} \sum_{|\alpha|\le j} |D^\alpha \varphi_n(x)|}_{\downarrow \ 0}$

Ex. take $(f_n)_n$ in $\mathcal{S}(\mathbb{R}^N)$ s.t.

$\tilde{q}_j(f_n) \xrightarrow{n} 0 \quad \forall j$

then $\forall \tilde{p}_k$ we have $\tilde{p}_k(f_n) \xrightarrow{n} 0$

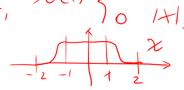
$\tilde{p}_k(f_n) = \sup_{x\in\Omega_k} \sum_{|\alpha|\le k} |D^\alpha f_n(x)|$ $\quad \left(\mathbb{R}^N = \bigcup_k \Omega_k, \ \overline{\Omega}_k \text{ compact}, \ \Omega_k \text{ open}, \ \overline{\Omega}_k \subseteq \Omega_{k+1} \right)$

rem. $\mathcal{D}(\mathbb{R}^N)$ is dense in $\mathcal{S}(\mathbb{R}^N)$

$\mathcal{S}(\mathbb{R}^N)$ is dense in $\mathcal{E}(\mathbb{R}^N)$

in fact. take $f \in \mathcal{S}(\mathbb{R}^N)$

take $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ s.t. $\chi(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| \ge 2 \end{cases}$



$\chi_n(x) = \chi\left(\frac{x}{n}\right)$

take $\varphi_n(x) = f(x) \cdot \chi_n(x)$

show that $\forall j$, $\tilde{q}_j(f - \varphi_n) = \tilde{q}_j(f(1-\chi_n)) \xrightarrow{n} 0 \quad \forall j$

take $\varphi_n(x) = f(x)\cdot\chi_n(x)$

show that $\forall r$, $\quad \tilde q_j(f-\varphi_n) = \tilde q_j(f(1-\chi_n)) \xrightarrow{n} 0 \quad \forall j$

$$\tilde q_j(f(1-\chi_n)) = \sup_{x\in\mathbb{R}^N}\Big((1+|x|)^j \sum_{|\alpha|\le j} |D^\alpha(f(1-\chi_n))|\Big)$$

$1-\chi_n = 0$ if $|x|\le n$

$$\le \sup_{|x|\ge n}\Big((1+|x|)^{j+1}\cdot\frac{1}{1+|x|}\cdot\sum_{|\alpha|\le j+1}|D^\alpha(f(1-\chi_n))|\Big)$$

$$\le \underbrace{\sup_{x\in\mathbb{R}^N}\Big((1+|x|)^{j+1}\sum_{|\alpha|\le j+1}|D^\alpha f(1-\chi_n)|\Big)}_{C\,\sup_{x\in\mathbb{R}^N}(1+|x|)^{j+1}\sum_{|\alpha|\le j+1}|D^\alpha f|}\cdot\underbrace{\sup_{|x|\ge n}\frac{1}{1+|x|}}_{\frac{1}{1+n}}$$

C depending on χ but not on n

conclusion

$$\tilde q_j(f-\varphi_n) \le C\cdot \tilde q_{j+1}(f)\cdot\frac{1}{1+n} \xrightarrow{n} 0$$

C does not dep on n

Ex density of $\mathcal{S}(\mathbb{R}^N)$ in $\mathcal{E}(\mathbb{R}^N)$ exercise

conclusion

$$\mathcal{D}(\mathbb{R}^N) \subseteq \mathcal{S}(\mathbb{R}^N) \subseteq \mathcal{E}(\mathbb{R}^N)$$

with continuous and dense inclusions.

then

$$\mathcal{E}'(\mathbb{R}^N) \subseteq \mathcal{S}'(\mathbb{R}^N) \subseteq \mathcal{D}'(\mathbb{R}^N)$$

distribution with compact support — def. tempered distributions — distributions

$S \in \mathcal{S}'(\mathbb{R}^N)$ $\quad S : \mathcal{S}(\mathbb{R}^N) \to \mathbb{C}$ linear and

$\forall \tilde q_j \ \exists C_j$ s.t. $|S(f)| \le C_j \tilde q_j(f) \quad \forall f\in\mathcal{S}$

$\forall j \ \exists C_j > 0$ s.t. $\forall f\in\mathcal{S}(\mathbb{R}^N)$

$$|S(f)| \le C_j \sup_{\mathbb{R}^N}(1+|x|)^j\sum_{|\alpha|\le j}|D^\alpha f(x)|$$

Ex. if $T\in\mathcal{D}'(\mathbb{R}^N)$ and supp $T\subseteq K$ compact

then $|T(f)| \le C \sup_{\mathbb{R}^N}(1+|x|)^j\sum_{|\alpha|\le j}|D^\alpha f(x)|$

$\forall j, \exists C_j$ $\quad \forall f\in\mathcal{S}$.

Ex. if $f\in\mathcal{S}(\mathbb{R}^N)$ then $f\in L^p(\mathbb{R}^N)$ $\forall p\in[1,+\infty]$

if $f\in L^p(\mathbb{R}^N)$ for some $p\in[1,+\infty]$ then $T_f\in\mathcal{S}'(\mathbb{R}^N)$

$\int |fg| \le C\tilde q_j(g)$ $\quad \forall j \ \forall g\in\mathcal{S}$

Ex. $\mathcal{D}'(\mathbb{R}^N) \supsetneq \mathcal{S}'(\mathbb{R}^N)$

$N=1$, $\quad f(x) = e^{x^2}$ $\quad \exists g\in\mathcal{S}(\mathbb{R})$ s.t. $fg\notin L^1(\mathbb{R})$

$e^{-\sqrt{1+x^2}}$ $\quad$ so that $T_f\notin\mathcal{S}'(\mathbb{R})$

Th. take $f\in\mathcal{S}(\mathbb{R}^N)$

we know that $f\in L^1(\mathbb{R}^N)$ so that $\hat f$ is well defined

Th. take $f \in \mathcal{S}(\mathbb{R}^N)$

we know that $f \in L^1(\mathbb{R}^N)$. So that $\hat{f}$ is well defined

we have that $\hat{f} \in \mathcal{S}(\mathbb{R}^N)$

and $\mathcal{F} : \mathcal{S}(\mathbb{R}^N_x) \longrightarrow \mathcal{S}(\mathbb{R}^N_\xi)$

$f \longmapsto \hat{f}$ is linear and continuous

__idea__

$$|\xi \hat{f}(\xi)| = |\widehat{f'(x)}(\xi)|$$

$$|\xi^k \hat{f}(\xi)| = |\widehat{f^{(k)}(x)}(\xi)|$$

if $f^{(k)} \in L^1 \; \forall k$ then $\xi^k \cdot \hat{f}(\xi) \in L^\infty$

if $\underbrace{x^k f^{(h)}(x)}_{f \in \mathcal{S}} \in L^1$ then $\underset{\hat{f} \in \mathcal{S}}{\left(\xi^\alpha \hat{f}(\xi)\right)^{(h)}} \in L^\infty$

Th. $\mathcal{F} : \mathcal{S}(\mathbb{R}^N_x) \longrightarrow \mathcal{S}(\mathbb{R}^N_\xi)$ is bijective

continuous with continuous inverse