Lezione 19 - 17/04/2026

Th $f \in \mathcal{S}(\mathbb{R}^N_x) \Rightarrow \hat{f} \in \mathcal{S}(\mathbb{R}^N_\xi)$ $\qquad \left(\hat{f}' = \widehat{-ix f(x)}\right)$

proof. $(N=1)$

we know that $f(x) \in L^1(\mathbb{R})$, $x f(x) \in L^1(\mathbb{R}) \Rightarrow \hat{f} \in C^1$

$\|\hat{f}'\|_{L^\infty} \le \|x f(x)\|_{L^1}$

so that $f(x), x f(x), x^2 f(x), \dots, x^k f(x) \in L^1(\mathbb{R})$

$\Rightarrow \hat{f} \in C^k$

Consequently $f \in \mathcal{S} \Rightarrow x^k f(x) \in L^1 \ \forall k \Rightarrow \hat{f} \in C^\infty$ *

$\|\hat{f}^{(k)}\|_{L^\infty} \le \|x^k f(x)\|_{L^1}$

$(\Rightarrow \hat{f}^{(k)}$ is bounded for all $k)$

similarly we know that $f \in L^1$, f diff. and $f' \in L^1 \Rightarrow \|\xi \hat{f}(\xi)\|_{L^\infty} \le \|f'\|_{L^1}$

$\left(\xi \hat{f}(\xi) = -i \widehat{f'}(\xi)\right)$

so that

$f \in L^1, f' \in L^1, f'' \in L^1, \dots f^{(k)} \in L^1 \Rightarrow \|\xi^k \hat{f}(\xi)\|_{L^\infty} \le C$

consequently $f \in \mathcal{S}(\mathbb{R}) \Rightarrow \|\xi^k \hat{f}(\xi)\|_{L^\infty} \le C \ \forall k$ **

now mixing

* and ** we have that $\hat{f} \in \mathcal{S}$

i.e. $\forall k, \forall h$ $\quad \|\xi^k \hat{f}^{(h)}(\xi)\|_{L^\infty} \le C$

is the L^1 norm of $(x^h f(x))^{(k)}$

It remains to prove that

$\|(x^h f(x))^{(k)}\|_{L^1} \le C \, \tilde{q}_j(f)$ $\leftarrow \exists j$

$\tilde{q}_j(f) = \|(1+|x|)^j \sum_{|\alpha| \le j} |\partial^\alpha f(x)|\|_{L^\infty}$

Exercise

example $\quad f \in \mathcal{S} \quad$ $\|f\|_{L^1} = \int |f|$

$= \int (1+|x|)^2 |f(x)| \cdot \frac{1}{(1+|x|)^2} dx$

$\le \underbrace{\sup (1+|x|^2)|f(x)|}_{\le \tilde{q}_2(f)} \cdot \underbrace{\int \frac{1}{1+|x|^2} dx}_{C}$

Th (inversion formula for Fourier transform)

Let $f \in \mathcal{S}(\mathbb{R}^N)$

then $\forall x \in \mathbb{R}^N$, $\quad f(x) = \frac{1}{(2\pi)^N} \underbrace{\int_{\mathbb{R}^N_\xi} e^{i x \cdot \xi} \hat{f}(\xi) \, d\xi}$

$\frac{1}{(2\pi)^N} \hat{\hat{f}}(-x)$

Corollary

$\mathcal{F} : \mathcal{S}(\mathbb{R}^N_x) \to \mathcal{S}(\mathbb{R}^N_\xi)$ is bijective continuous with continuous inverse.

proof. idea

compute $\int_{\mathbb{R}^N_\xi} e^{i x \xi} \hat{f}(\xi) \, d\xi$

proof. idea

compute $\int_{\mathbb{R}^N_\xi} e^{ix\xi} \hat{f}(\xi)\, d\xi$

$$= \int_{\mathbb{R}^N_\xi} e^{ix\xi} \left(\int_{\mathbb{R}^N_y} e^{-iy\xi} f(y)\, dy \right) d\xi$$

exchange the order of integration?

$(\xi, y) \longmapsto e^{-i(y-x)\cdot\xi} f(y)$ is not in $L^1(\mathbb{R}^N_\xi \times \mathbb{R}^N_y)$

Consider $g \in \mathcal{S}(\mathbb{R}^N)$

consider $\int e^{ix\xi} \hat{f}(\xi) g(\xi)\, d\xi$

$$= \int e^{ix\xi} \int e^{-iy\xi} f(y)\, dy\, g(\xi)\, d\xi$$

$$= \int e^{-i(y-x)\xi} f(y) g(\xi)\, dy\, d\xi$$

$$(y, \xi) \longmapsto \underbrace{e^{-i(y-x)\cdot\xi}}_{\in L^\infty} \underset{L^1(\mathbb{R}^N_y)}{\underset{\cap}{f(y)}} \underset{L^1(\mathbb{R}^N_\xi)}{\underset{\cap}{g(\xi)}} \in L^1(\mathbb{R}^N_y \times \mathbb{R}^N_\xi)$$

exchange the order

$$= \int f(y) \cdot \underbrace{\int e^{-i(y-x)\xi} g(\xi)\, d\xi}_{\hat{g}(y-x)}\, dy$$

$$\boxed{\int_{\mathbb{R}^N_\xi} e^{ix\xi} \hat{f}(\xi) g(\xi)\, d\xi = \int_{\mathbb{R}^N_y} f(y)\, \hat{g}(y-x)\, dy} \quad \boxed{f, g \in \mathcal{S}}$$

Consider $\boxed{g_k(\xi) = e^{-\xi^2/k}} \longleftrightarrow \boxed{\hat{g}_k(x) = (\sqrt{\pi k})^N e^{-\frac{x^2 k}{4}}}$

$g(\xi) = e^{-\xi^2} \Rightarrow \hat{g}(x) = (\sqrt{\pi})^N e^{-x^2/4}$

$\hat{g}(x) = \int e^{-ix\xi} g(\xi)\, d\xi$

$\widehat{g\left(\frac{\xi}{\sqrt{k}}\right)}(x) = \int e^{-ix\xi} g\left(\frac{\xi}{\sqrt{k}}\right) d\xi$ $\qquad$ $\frac{\xi}{\sqrt{k}} = \eta$, $\quad d\xi = (\sqrt{k})^N d\eta$

$= \int e^{-ix\sqrt{k}\eta} g(\eta) \sqrt{k}^N d\eta$

$= (\sqrt{k})^N \hat{g}(x\sqrt{k})$

$$\boxed{\int_{\mathbb{R}^N} e^{ix\xi} \hat{f}(\xi)\, e^{-\xi^2/k}\, d\xi = \int f(y) (\sqrt{\pi k})^N e^{-\frac{|y-x|^2 k}{4}}\, dy} \xrightarrow{k}$$

$k \to +\infty$

$e^{ix\xi} \hat{f}(\xi) e^{-\xi^2/k} \xrightarrow{k} e^{ix\xi} \hat{f}(\xi)$ pointwise

$|e^{ix\xi} \hat{f}(\xi) e^{-\xi^2/k}| \le |\hat{f}|$

dom. convergence

$k \to +\infty$

$\int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi)\, d\xi$

$$= \int f(y) (\sqrt{\pi k})^N e^{-\frac{|y-x|^2 k}{4}} dy \qquad \frac{\sqrt{k}(y-x)}{2} = z$$

$$y - x = \frac{2z}{\sqrt{k}}$$

$$y = x + \frac{2z}{\sqrt{k}}$$

$$= \int f\left(x+\frac{2z}{\sqrt{k}}\right) (\sqrt{\pi k})^N \cdot e^{-z^2} \frac{2^N}{(\sqrt{k})^N} dz \qquad dy = \left(\frac{2}{\sqrt{k}}\right)^N dz$$

$$= \int f\left(x+\frac{2z}{\sqrt{k}}\right) (\sqrt{\pi})^N \cdot 2^N e^{-z^2} dz$$

let $k \to +\infty$ $\qquad f\left(x+\frac{2z}{\sqrt{k}}\right) (\sqrt{\pi})^N 2^N e^{-z^2} \longrightarrow f(x) (\sqrt{\pi})^N 2^N e^{-z^2}$

$$\left| f\left(x+\frac{2z}{\sqrt{k}}\right) (\sqrt{\pi})^N 2^N e^{-z^2} \right| \le \underbrace{\|f\|_{L^\infty} (\sqrt{\pi})^N 2^N \cdot e^{-z^2}}_{\in L^1}$$

dom. conv.

$$\xrightarrow{k} \int f(x) (\sqrt{\pi})^N 2^N e^{-z^2} dz = (\sqrt{\pi})^N 2^N \underbrace{\int e^{-z^2} dz}_{(\sqrt{\pi})^N} \cdot f(x)$$

$$= (2\pi)^N f(x)$$

conclusion

$$\int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) d\xi = (2\pi)^N f(x)$$

QED

rem. what happens if $f \notin \mathcal{S}$ but $f \in L^1(\mathbb{R}^N)$ such that $\hat{f} \in L^1(\mathbb{R}^N)$?

in this case which is the value of $\int e^{ix\xi} \hat{f}(\xi) d\xi$?

def. Let $T \in \mathcal{S}'(\mathbb{R}^N)$ (T is a tempered distribution)

we define $\hat{T}(f) = T(\hat{f}) \quad \forall f \in \mathcal{S}$.

Th. $\hat{T} \in \mathcal{S}'(\mathbb{R}^N)$

proof $\hat{T}$ is linear (because $\mathcal{F}$ is linear on $\mathcal{S}$)

$$\left(\hat{T}(f_1+f_2) = T(\widehat{f_1+f_2}) = T(\hat{f_1}+\hat{f_2}) = T(\hat{f_1}) + T(\hat{f_2}) = \hat{T}(f_1) + \hat{T}(f_2) \right)$$

moreover $\quad |\hat{T}(f)| = |T(\hat{f})| \le C\, q_j(\hat{f}) \le C \cdot \tilde{C}\, \tilde{q}_k(f)$

($\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is continuous)

Ex. $\delta_0 \in \mathcal{S}'(\mathbb{R}^N)$ ($\mathcal{E}' \subseteq \mathcal{S}'$ dist. with compact support) $\quad (\mathcal{E}' \subseteq \mathcal{S}' \subseteq \mathcal{D}')$

$$\hat{\delta}_0(f) = \delta_0(\hat{f}) = \hat{f}(0) = \int e^{-i0\cdot x} f(x) dx = \int f(x) dx = T_1(f)$$

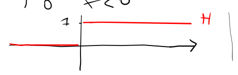
$$\boxed{\hat{\delta}_0 = T_1} \qquad (\hat{\delta}_0 = 1)$$

$$\hat{T}_1(f) = T_1(\hat{f}) = \int_{\mathbb{R}^N} \hat{f}(\xi) d\xi = \underbrace{\int_{\mathbb{R}^N} e^{i0\cdot\xi} \hat{f}(\xi) d\xi}_{= (2\pi)^N f(0) = (2\pi)^N \delta_0(f)}$$

$$\widehat{T_1}(f) = T_1(\hat f) = \int_{\mathbb{R}^N} \hat f(\xi)\,d\xi = \underbrace{\int_{\mathbb{R}^N} e^{i\,0\cdot\xi}\,\hat f(\xi)\,d\xi}_{} = (2\pi)^N f(0) = (2\pi)^N \delta_0(f)$$

$$\boxed{\widehat{T_1} = (2\pi)^N \delta_0} \qquad \left(\hat 1 = (2\pi)^N \delta_0\right)$$

Ex. compute $\hat H$ $\qquad H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$



(rem. $f \in L^p(\mathbb{R}^N)$ for some $p \in [1, +\infty]$ then $T_f \in \mathcal{S}'(\mathbb{R}^N)$

we have to show that $\left| \int_{\mathbb{R}^N} f\varphi \right| \leq C\, \tilde q_j(\varphi)$ for some j $\forall \varphi \in \mathcal{S}$

$$\left|\int f\varphi\right| \leq \|f\|_{L^p} \|\varphi\|_{L^{p'}}$$

$$\left(\int |\varphi|^{p'}\right)^{\frac{1}{p'}} \leq \left(\int \left|(1+|x|)^{N+1}\varphi\right|^{p'} \cdot \left|(1+|x|)^{-(N+1)}\right|^{p'} dx\right)^{\frac{1}{p'}}$$

$$\leq \underbrace{\left(\sup (1+|x|)^{N+1}|\varphi|\right)}_{\leq \tilde q_j(\varphi)} \cdot \underbrace{\left(\int (1+|x|)^{-(N+1)p'}\right)^{\frac{1}{p'}}}_{C_{p'}}$$

$\hat H$?

$$x\hat H(f) = \hat H(x f(x)) = H(\widehat{x f(x)}) = H(i \hat f') = i H(\hat f') = i \int_0^{+\infty} \hat f'(\xi)\,d\xi = -i \hat f(0) = -i \int f(x)\,dx = -i\, T_1(f)$$

conclusion $\quad i x \hat H = T_1 \qquad \boxed{x(i\hat H) = 1}$

we know that $x T = 1 \Leftrightarrow T = PV\frac{1}{x} + c\delta_0$

$$i\hat H = PV\frac{1}{x} + c\delta_0$$

$$\hat H = -i\, PV\frac{1}{x} + c\delta_0 \qquad c \text{ constant}$$

$$\hat H(e^{-x^2}) = H(\sqrt{\pi} e^{-\xi^2/4}) = \sqrt{\pi}\int_0^{+\infty} e^{-\xi^2/4}\,d\xi = 2\sqrt{\pi}\int_0^{+\infty} e^{-y^2}\,dy = \pi \qquad \xi/2 = y, \quad d\xi = 2\,dy$$

$$\hat H(e^{-x^2}) = \pi$$

$$\sqrt{\pi} = \left(-i\,PV\frac{1}{x} + c\delta_0\right)(e^{-x^2}) = -i \underbrace{PV\frac{1}{x}(e^{-x^2})}_{=0} + \underbrace{c\delta_0(e^{-x^2})}_{=c}$$

conclusion $\quad \boxed{\hat H = -i\, PV\frac{1}{x} + \pi\delta_0}$

Fourier transform in $L^2(\mathbb{R}^N)$

$f \in L^2(\mathbb{R}^N) \Rightarrow T_f \in \mathcal{S}'(\mathbb{R}^N)$

it makes sense to compute $\widehat{T_f}$

(we will write $\hat{f}$)

Lemma let $f, g \in \mathcal{S}(\mathbb{R}^N)$ we have

1) $\int_{\mathbb{R}^N} f(x)\hat{g}(x)\,dx = \int_{\mathbb{R}^N} \hat{f}(\xi) g(\xi)\,d\xi$

2) $\int_{\mathbb{R}^N} f(x)\overline{g(x)}\,dx = \frac{1}{(2\pi)^N}\int \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi$ (Parseval's formula)

3) $\widehat{fg}(\xi) = \frac{1}{(2\pi)^N}(\hat{f} * \hat{g})(\xi)$

4) $\widehat{f*g}(x) = \hat{f}(x)\,\hat{g}(x)$

proof

1) $\int f(x)\hat{g}(x)\,dx = \int f(x)\left(\int e^{-ix\xi} g(\xi)\,d\xi\right)dx$

$= \iint f(x) e^{-ix\xi} g(\xi)\,d\xi\,dx$

exchange

$= \int\left(\int e^{-ix\xi} f(x)\,dx\right) g(\xi)\,d\xi$

$= \int \hat{f}(\xi) g(\xi)\,d\xi$

2) $\int f(x)\overline{g(x)}\,dx \overset{\uparrow}{=} \int f(x)\cdot\overline{\frac{1}{(2\pi)^N}\int e^{ix\xi}\hat{g}(\xi)\,d\xi}\,dx$

inversion formula

$= \frac{1}{(2\pi)^N}\iint f(x) e^{-ix\xi}\,\overline{\hat{g}(\xi)}\,d\xi\,dx$

exchange

$= \frac{1}{(2\pi)^N}\int \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi$

3) $\int e^{-ix\xi} f(x) g(x)\,dx \overset{\uparrow}{=} \int e^{-ix\xi} f(x)\frac{1}{(2\pi)^N}\int e^{ixy}\hat{g}(y)\,dy\,dx$

inversion formula

exchange

$= \iint e^{-ix(\xi-y)} f(x)\frac{1}{(2\pi)^N}\hat{g}(y)\,dx\,dy$

$= \frac{1}{(2\pi)^N}\int \hat{f}(\xi-y)\hat{g}(y)\,dy$

$= \frac{1}{(2\pi)^N}\hat{f} * \hat{g}(\xi)$

Th (Plancherel)

let $f \in L^2(\mathbb{R}^N)$. There exists a unique $g \in L^2(\mathbb{R}^N)$

s.t. $\widehat{T_f} = T_g$ we will write $\hat{f} = g$

moreover $\|\hat{f}\|_{L^2} = (2\pi)^{-N/2}\|f\|_{L^2}$

Th (Plancherel)

let $f \in L^2(\mathbb{R}^N)$. There exists a unique $g \in L^2(\mathbb{R}^N)$ s.t. $\widehat{T_f} = T_g$ we will write $\hat{f} = g$

moreover $\|\hat{f}\|_{L^2} = (2\pi)^{-N/2} \|f\|_{L^2}$

proof. take $f \in L^2(\mathbb{R}^N)$

we know that $\mathcal{S}(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$

($\mathcal{D}(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$)

so $\exists (f_n)_n$ in $\mathcal{S}$ s.t. $\|f_n - f\|_{L^2} \xrightarrow{n} 0$

$(f_n)_n$ is a Cauchy sequence in L^2

from Parseval $(\hat{f}_n)_n$ is a Cauchy sequence in L^2

($f_n \in \mathcal{S} \Rightarrow \hat{f}_n \in \mathcal{S} \Rightarrow \hat{f}_n \in L^2$

$\|f_n - f_m\|_{L^2}^2 = (2\pi)^{-N} \|\hat{f}_n - \hat{f}_m\|_{L^2}^2$ $\quad \int f\bar{g} = (2\pi)^{-N} \int \hat{f}\bar{\hat{g}}$

$\|f\|_{L^2}^2 = \int f\bar{f} = (2\pi)^{-N} \|\hat{f}\|_{L^2}$)

so that $\exists g \in L^2$ s.t. $\lim_n \hat{f}_n = g$ in L^2

$T_f(\varphi) = \int f\varphi = \lim \int f_n \varphi$ $\quad$ ($f_n \to f$ strong, $f_n \to f$ weakly)

$\widehat{T_f}(\varphi) = T_f(\hat{\varphi}) = \lim_n \int f_n \hat{\varphi} = \lim_n \int \hat{f}_n \varphi = \int g\varphi$

$\widehat{T_f}(\varphi) = T_g(\varphi) \Rightarrow \widehat{T_f} = T_g$

rem. g does not depend on $(f_n)_n$ but only on f

and g is unique

finally $\|f\|_{L^2} = \lim_n \|f_n\|_{L^2}$ $\leftarrow$ Parseval

$= \lim_n (2\pi)^{-N/2} \|\hat{f}_n\|_{L^2}$

$= (2\pi)^{-N/2} \|g\|_{L^2}$

QED

rem. it is possible to prove that

$f \in L^2(\mathbb{R}^N)$ $\quad \hat{f}(\xi) = \lim_{R \to +\infty} \int_{|x| \le R} e^{-ix\xi} f(x)\,dx$

(exercise ?)