Introduction to the Geometry of Immersions

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The set-up

Let $\mathcal{I} : M \to \overline{M}$ be a differentiable immersion of a manifold M of dimension n into a Riemannian manifold $\overline{M}$ of dimension $k = n + m$.

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In this manner, the immersion $\mathcal{I}$ becomes an *isometric* immersion.

For each $p \in M$ there exists a neighborhood U of p in M such that $\mathcal{I}(U)$ is a submanifold of $\overline{M}$, therefore $U \simeq \mathcal{I}(U) \subset \overline{M}$ and for each $q \in U$ we can identify $v \in T_q M$ with $d\mathcal{I}_p(v) \in T_{\mathcal{I}(q)}\overline{M} \simeq T_q\overline{M}$.

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For any $p \in M$ the inner product on $T_p\overline{M}$ splits $T_p\overline{M}$ into the direct sum

$$T_p\overline{M} = T_pM \oplus (T_pM)^\perp$$

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In other words the *tangent bundle* $T\overline{M}$ which sits over M decomposes into the direct sum of *tangent bundle* TM and the *normal bundle* $TM^{\perp}$.

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$$\nabla_X Y := (\overline{\nabla}_{\overline{X}} \overline{Y})^T$$

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$$B(X, Y) := \overline{\nabla}_{\overline{X}} \overline{Y} - \nabla_X Y.$$

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- $B(X, Y)$ is a (local) vector field on $\overline{M}$ orthogonal to M $B(X, Y) \in \mathcal{X}(M)^\perp$
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- $B(X, Y)$ does not depend on the extensions $\overline{X}, \overline{Y}$
- $B(X, Y)$ is bilinear and symmetric.

Let $p \in M$ and $\eta \in (T_pM)^\perp$. The mapping

$$H_\eta : T_pM \times T_pM \to \mathbb{R}$$

given by

$$H_\eta(x, y) = \langle B(x, y), \eta \rangle, \quad x, y \in T_pM$$

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is a symmetric bilinear form. Therefore, there exists a linear self-adjoint operator $S_\eta : T_pM \to T_pM$ such that

$$H_\eta(x, y) = \langle B(x, y), \eta \rangle = \langle S_\eta(x), y \rangle$$

Proposition

Let $p \in M$ $x \in T_pM$ and $\eta \in (T_pM)^\perp$. Let N be a local extension of η normal to M. Then

$$S_\eta(x) = -(\overline{\nabla}_x N)^T.$$

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Proof.

If $y \in T_pM$, let $X, Y \in \mathcal{X}(U)$ be local extensions of x, y, where U is a neighborhood of p in M. Let N be a local extension of $\eta \in (T_pM)^\perp$ normal to M.

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$= \langle \overline{\nabla}_X Y - \nabla_X Y, N \rangle_p = \langle \overline{\nabla}_X Y, N \rangle_p = -\langle Y, \overline{\nabla}_X N \rangle_p = \langle -\overline{\nabla}_x N, y \rangle$ for any $y \in T_pM$. □

Definition (Second fundamental form)

The quadratic form

$$H_\eta(x,x) = \langle S_\eta(x), x\rangle =: II_\eta(x)$$

is called the *second fundamental form* of the immersion $\mathcal{I}$ at p along the normal vector η.

If $x, y \in T_pM \subset Tp\overline{M}$ are linearly independent, denote by $K_p(x, y)$ and $\overline{K}_p(x, y)$ the sectional curvatures of M and $\overline{M}$ respectively of the plane spanned by x and y.

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Theorem (Gauss)

Let $p \in M$ and let $x, y \in T_pM$ be orthonormal vectors in T_pM. Then

$$K_p(x, y) - \overline{K}_p(x, y) = \langle B(x, x), B(y, y)\rangle - |B(x, y)|^2$$

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More in general the following identity holds for $X, Y, Z, W \in \mathcal{X}(M)$

$$\mathcal{R}(X, Y, Z, W) - \overline{\mathcal{R}}(X, Y, Z, W) = \langle B(X, Z), B(Y, W)\rangle - \langle B(X, W), B(Y, Z)\rangle$$

(Gauss equation)

Normal connection

In order to prove this identity, we introduce for $X \in \mathcal{X}(M)$ and η a vector field orthogonal to M

$$\nabla^{\perp}_X \eta := (\overline{\nabla}_X \eta)^N = \overline{\nabla}_X \eta - (\overline{\nabla}_X \eta)^T = \overline{\nabla}_X \eta + S_\eta(X)$$

One can prove that $\nabla^{\perp}$ has all the usual properties of a connection, namely linear in X additive in η and such that if $f \in \mathcal{D}(M)$, then

$$\nabla^{\perp}_X (f\eta) = f\nabla^{\perp}_X \eta + X(f)\nabla^{\perp}_X \eta$$

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As for the tangent bundle, one can then introduce a notion of curvature in the normal bundle, namely

$$R^{\perp}(X,Y)\eta = \nabla^{\perp}_Y \nabla^{\perp}_X Z - \nabla^{\perp}_X \nabla^{\perp}_Y \eta + \nabla^{\perp}_{[X,Y]} \eta.$$

Proof

For $X, Y, Z \in \mathcal{X}(M)$ and $\eta \in \mathcal{X}^{\perp}(M)$

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y) \qquad \overline{\nabla}_X \eta = \nabla^{\perp}_X \eta - S_\eta(X)$$

Proof

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$$\begin{aligned}
\overline{R}(X, Y)Z &= \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_X \overline{\nabla}_Y Z + \overline{\nabla}_{[X,Y]} Z \\
&= \overline{\nabla}_Y (\nabla_X Z + B(X, Z)) - \overline{\nabla}_X (\nabla_Y Z + B(Y, Z)) + \\
&\quad + \nabla_{[X,Y]} Z + B([X, Y], Z) \\
&= \nabla_Y \nabla_X Z + B(Y, \nabla_X Z) + \overline{\nabla}_Y B(X, Z) + \\
&\quad - \nabla_X \nabla_Y Z - B(X, \nabla_Y Z) + \overline{\nabla}_X B(Y, Z) + \\
&\quad + \nabla_{[X,Y]} Z + B([X, Y], Z) \\
&= R(X, Y)Z + B(Y, \nabla_X Z) + \nabla^\perp_Y B(X, Y) - S_{B(X,Z)} Y + \\
&\quad - B(X, \nabla_Y Z) - \nabla^\perp_X B(Y, Z) + S_{B(Y,Z)} X + B([X, Y], Z)
\end{aligned}$$

Proof

Hence, if $W \in \mathcal{X}(M)$

$$\begin{aligned}\overline{\mathcal{R}}(X,Y,Z,W) := &\ \langle \overline{R}(X,Y)Z, W\rangle \\ = &\ \langle R(X,Y)Z, W\rangle + \langle S_{B(Y,Z)}X, W\rangle - \langle S_{B(X,Z)}Y, W\rangle \\ = &\ \mathcal{R}(X,Y,Z,W) + \langle B(X,W), B(Y,Z)\rangle + \\ &\ -\langle B(Y,W), B(X,Z)\rangle\end{aligned}$$

since $\langle S_\eta X, Y\rangle = \langle B(X,Y), \eta\rangle$

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\end{aligned}$$

since $\langle S_\eta X, Y\rangle = \langle B(X,Y), \eta\rangle$

$$\boxed{\mathcal{R}(X,Y,Z,W) - \overline{\mathcal{R}}(X,Y,Z,W) = \langle B(X,Z), B(Y,W)\rangle - \langle B(X,W), B(Y,Z)\rangle}$$

(Gauss equation)

Ricci equation

For $X, Y \in \mathcal{X}(M)$ and $\eta, \xi \in \mathcal{X}^{\perp}(M)$

$$\boxed{\langle \overline{R}(X,Y)\eta, \xi\rangle - \langle R^{\perp}(X,Y)\eta, \xi\rangle = \langle [S_\eta, S_\xi]X, Y\rangle}$$

where $[S_\eta, S_\xi]$ denotes the operator $S_\eta \circ S_\xi - S_\xi, S_\eta$.

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where $[S_\eta, S_\xi]$ denotes the operator $S_\eta \circ S_\xi - S_\xi, S_\eta$.

Proof.

$$\begin{aligned}\overline{R}(X, Y)\eta &= \overline{\nabla}_Y\overline{\nabla}_X\eta - \overline{\nabla}_X\overline{\nabla}_Y\eta + \overline{\nabla}_{[X,Y]}\eta \\ &= \overline{\nabla}_Y(\nabla^\perp_X\eta - S_\eta(X)) - \overline{\nabla}_X(\nabla^\perp_Y\eta - S_\eta(Y)) + \\ &\quad + \nabla^\perp_{[X,Y]}\eta - S_\eta([X, Y]))\end{aligned}$$

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$$\begin{aligned}
\overline{R}(X,Y)\eta &= \overline{\nabla}_Y \overline{\nabla}_X \eta - \overline{\nabla}_X \overline{\nabla}_Y \eta + \overline{\nabla}_{[X,Y]}\eta \\
&= \overline{\nabla}_Y(\nabla^{\perp}_X \eta - S_\eta(X)) - \overline{\nabla}_X(\nabla^{\perp}_Y \eta - S_\eta(Y)) + \\
&\quad + \nabla^{\perp}_{[X,Y]}\eta - S_\eta([X,Y])) \\
&= \nabla^{\perp}_Y \nabla^{\perp}_X \eta - S_{\nabla^{\perp}_X}(Y) - \overline{\nabla}_Y(S_\eta(X))) + \\
&\quad - \nabla^{\perp}_X \nabla^{\perp}_Y \eta + S_{\nabla^{\perp}_Y}(X) + \overline{\nabla}_X(S_\eta(Y))) + \\
&\quad + \nabla^{\perp}_{[X,Y]}\eta - S_\eta([X,Y]))
\end{aligned}$$

□

Ricci equation

Proof.

$$
\begin{aligned}
= \quad & \nabla^\perp_Y \nabla^\perp_X \eta - S_{\nabla^\perp_X}(Y) - \overline{\nabla}_Y(S_\eta(X))) + \\
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\end{aligned}
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= \quad & R^\perp(X, Y)\eta - S_{\nabla^\perp_Y}(X) + S_{\nabla^\perp_X}(Y) - S_\eta([X, Y])) + \\
& + \nabla_X(S_\eta(Y)) + B(X, S_\eta Y) - \nabla_Y(S_\eta(X)) - B(S_\eta X, Y)
\end{aligned}
$$

Ricci equation

Proof.

$$\begin{aligned}
&= \nabla^{\perp}_{Y}\nabla^{\perp}_{X}\eta - S_{\nabla^{\perp}_{X}}(Y) - \overline{\nabla}_{Y}(S_{\eta}(X)))+ \\
&\quad -\nabla^{\perp}_{X}\nabla^{\perp}_{Y}\eta + S_{\nabla^{\perp}_{Y}}(X) + \overline{\nabla}_{X}(S_{\eta}(Y)))+ \\
&\quad +\nabla^{\perp}_{[X,Y]}\eta - S_{\eta}([X,Y]))
\end{aligned}$$

$$\begin{aligned}
&= R^{\perp}(X,Y)\eta - S_{\nabla^{\perp}_{Y}}(X) + S_{\nabla^{\perp}_{X}}(Y) - S_{\eta}([X,Y]))+ \\
&\quad +\nabla_{X}(S_{\eta}(Y)) + B(X,S_{\eta}Y) - \nabla_{Y}(S_{\eta}(X)) - B(S_{\eta}X,Y)
\end{aligned}$$

Therefore

$$\langle \overline{R}(X,Y)\eta,\xi\rangle = \langle R^{\perp}(X,Y)\eta,\xi\rangle + \langle B(X,S_{\eta}Y),\xi\rangle - \langle B(S_{\eta}X,Y),\xi\rangle$$

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&\quad +\nabla^\perp_{[X,Y]}\eta - S_\eta([X,Y]))\\
&= R^\perp(X,Y)\eta - S_{\nabla^\perp_Y}(X) + S_{\nabla^\perp_X}(Y) - S_\eta([X,Y]))+\\
&\quad +\nabla_X(S_\eta(Y)) + B(X, S_\eta Y) - \nabla_Y(S_\eta(X)) - B(S_\eta X, Y)
\end{aligned}
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\langle \overline{R}(X,Y)\eta, \xi\rangle &= \langle R^\perp(X,Y)\eta,\xi\rangle + \langle B(X,S_\eta Y),\xi\rangle - \langle B(S_\eta X, Y),\xi\rangle\\
&= \langle R^\perp(X,Y)\eta,\xi\rangle + \langle S_\xi X, S_\eta Y\rangle - \langle S_\xi(S_\eta X), Y\rangle
\end{aligned}
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= \quad & \nabla^\perp_Y \nabla^\perp_X \eta - S_{\nabla^\perp_X}(Y) - \overline{\nabla}_Y(S_\eta(X)))+ \\
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& +\nabla^\perp_{[X,Y]}\eta - S_\eta([X,Y]))
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\end{aligned}
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&= \quad \langle R^\perp(X,Y)\eta,\xi\rangle + \langle S_\eta(S_\xi X), Y\rangle - \langle S_\xi(S_\eta X), Y\rangle \\
&= \quad \langle R^\perp(X,Y)\eta,\xi\rangle + \langle (S_\eta \circ S_\xi X - S_\xi \circ S_\eta)XY\rangle
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&= \langle R^\perp(X,Y)\eta,\xi\rangle + \langle [S_\eta, S_\xi]X, Y\rangle
\end{aligned}
$$

Codazzi equation

If $X, Y, Z \in \mathcal{X}(M)$ and $\eta \in \mathcal{X}^{\perp}(M)$, consider the tensor $\mathcal{B} : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}^{\perp}(M) \to \mathbb{R}$ defined as follows

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Its covariant derivative with respect to Z is

$$(\overline{\nabla}_Z \mathcal{B})(X, Y, \eta) = Z(\mathcal{B}(X, Y, \eta)) - \mathcal{B}(\nabla_Z X, Y, \eta) - \mathcal{B}(X, \nabla_Z Y, \eta) - \mathcal{B}(X, Y, \nabla^\perp_Z \eta)$$

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$$\boxed{\langle \overline{R}(X, Y)Z, \eta \rangle = (\overline{\nabla}_Y \mathcal{B})(X, Z, \eta) - (\overline{\nabla}_X \mathcal{B})(Y, Z, \eta)}$$

(Codazzi equation)

Codazzi equation

Proof.

$$
\begin{aligned}
(\overline{\nabla}_X \mathcal{B})(Y, Z, \eta) &= X\langle B(Y,Z), \eta\rangle - \langle B(\nabla_X Y, Z), \eta\rangle - \langle B(Y, \nabla_X Z), \eta\rangle + \\
&\quad -\langle B(Y,Z), \nabla^{\perp}_X \eta\rangle = \\
&= \langle \nabla^{\perp}_X (B(Y,Z)), \eta\rangle - \langle B(\nabla_X Y, Z), \eta\rangle - \langle B(Y, \nabla_X Z), \eta\rangle
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Similarly

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(\overline{\nabla}_Y \mathcal{B})(X, Z, \eta) = \langle \nabla^{\perp}_Y (B(X,Z)), \eta\rangle - \langle B(\nabla_Y X, Z), \eta\rangle - \langle B(X, \nabla_Y Z), \eta\rangle
$$

Codazzi equation

Proof.

$$(\overline{\nabla}_X\mathcal{B})(Y,Z,\eta) = X\langle B(Y,Z),\eta\rangle - \langle B(\nabla_X Y,Z),\eta\rangle - \langle B(Y,\nabla_X Z),\eta\rangle + -\langle B(Y,Z),\nabla^{\perp}_X\eta\rangle =$$

$$= \langle\nabla^{\perp}_X(B(Y,Z)),\eta\rangle - \langle B(\nabla_X Y,Z),\eta\rangle - \langle B(Y,\nabla_X Z),\eta\rangle$$

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On the other hand,

$$\overline{R}(X,Y)Z = R(X,Y)Z + B(Y,\nabla_X Z) + \nabla^{\perp}_Y B(X,Z) - S_{B(X,Z)}Y + -B(X,\nabla_Y Z) - \nabla^{\perp}_X B(Y,Z) + S_{B(Y,Z)}X + B([X,Y],Z)$$

$$= R(X,Y)Z + B(Y,\nabla_X Z) + \nabla^{\perp}_Y B(X,Z) - S_{B(X,Z)}Y + -B(X,\nabla_Y Z) - \nabla^{\perp}_X B(Y,Z) + S_{B(Y,Z)}X + B(\nabla_X Y - \nabla_Y X,Z)$$

Codazzi equation

Proof.

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z + S_{B(Y,Z)}X - S_{B(X,Z)}Y + \\ &+ B(Y,\nabla_X Z) - B(X,\nabla_Y Z) - \nabla^{\perp}_X B(Y,Z) + \nabla^{\perp}_Y B(X,Z) + \\ &+ B(\nabla_X Y, Z) - B(\nabla_Y X, Z)\end{aligned}$$

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Proof.

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Hence

$$\langle \overline{R}(X,Y)Z,\eta\rangle = (\overline{\nabla}_Y \mathcal{B})(X,Z,\eta) - (\overline{\nabla}_X \mathcal{B})(Y,Z,\eta)$$

since

$$(\overline{\nabla}_Y \mathcal{B})(X,Z,\eta) = \langle \nabla^{\perp}_Y (B(X,Z)),\eta\rangle - \langle B(\nabla_Y X, Z),\eta\rangle - \langle B(X,\nabla_Y Z),\eta\rangle$$

and

$$(\overline{\nabla}_X \mathcal{B})(Y,Z,\eta) = \langle \nabla^{\perp}_X (B(Y,Z)),\eta\rangle - \langle B(\nabla_X Y, Z),\eta\rangle - \langle B(Y,\nabla_X Z),\eta\rangle$$

□

Geodesic immersion

Definition

An immersion $\mathcal{I} : M \to \overline{M}$ is *geodesic* at $p \in M$ if and only if any geodesic γ of M starting from p is a geodesic in $\overline{M}$ at p.

Proposition

An immersion $\mathcal{I} : M \to \overline{M}$ is geodesic at $p \in M$ if and only if for every $\eta \in (T_p M)^\perp$ the second bilinear form H_η is identically zero at p.

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$$\begin{aligned} H_\eta(x,x) &= \langle S_\eta(x), y\rangle = -\langle \overline{\nabla}_X N, X\rangle_p = \\ &= -X\langle N, X\rangle_p + \langle N, \overline{\nabla}_X X\rangle_p = \langle N, \overline{\nabla}_X X\rangle_p. \end{aligned}$$

□

Minimal immersion

Definition

An immersion $\mathcal{I} : M \to \overline{M}$ is *minimal* at $p \in M$ if and only if for every $\eta \in (T_pM)^\perp$ the trace of S_η is identically zero at p.

Complete Manifolds

Definition

A Riemannian manifold M is (geodesically) *complete* if for all $p \in M$ the exponential map $\exp_p$ is defined for all $v \in T_pM$, i.e. any geodesic $\gamma(t)$ starting at p is defined for all $t \in \mathbb{R}$.

Riemannian Manifolds as metric spaces

Proposition

If M is a Riemannian manifold and $p, q \in M$, then define $d_M(p, q)$ as the infimum of the lengths of all piece-wise differentiable curves in M joining p to q. Then (M, d_M) is a metric space.

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Corollary

The (metric) topology induced by d_M on M coincides with the original topology on M.

Hopf Rinow Thm

Theorem (Hopf-Rinow part I)

Let M be a Riemannian manifold. Then M is geodesically complete if and only if the metric space (M, d_M) is a complete metric space.

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Then, since any Cauchy sequence in a metric space is bounded, its closure is compact. Thus from any Cauchy sequence in the metric space (M, d_M), one can extract a convergent subsequence. Hence, the Cauchy sequence is convergent. □

Theorem (Hopf-Rinow part II)

Let M be a Riemannian manifold. Then M is complete if and only if there exists an exhaustion *by compact sets of M (i.e. an (ascending) chain of compacta in M, that is a sequence of compact sets in M $\{K_n\}_{n\in\mathbb{N}}$ such that $K_n \subset\subset K_{n+1}$, whose union is M) such that, if one picks $q_n \in M \setminus K_n$ for any $n \in \mathbb{N}$, then, for the sequence of points $\{q_n\}_{n\in\mathbb{N}}$ in M it turns out that for any $p \in M$ $d_M(p, q_n) \to +\infty$ as $n \to +\infty$.*

Hopf Rinow - consequences

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Corollary

If M is a complete manifold, then for any $p \in M$ there exists a geodesic γ joining p to q such that $L(\gamma) = d_M(p, q)$.

Ambrose Thm

Proposition

Let (M, g) be a complete Riemannian manifold, and $\pi : \tilde{M} \to M$ be a smooth covering map. Then $\tilde{M}$ (with the metric induced by g, i.e. the pull-back of g via π) is a complete Riemannian manifold.

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Theorem (Ambrose)

Let (M, g_M) and (N, g_N) be two (connected) Riemannian manifolds. If $\phi : M \to N$ is a local isometry and M is complete, then ϕ is a smooth covering map and (N, g_N) is a complete manifold.

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Corollary

Let M be a (connected) Riemannian manifold. If for $p \in M$ $\exp_p : T_pM \to M$ is a local diffeomorphism, then $\exp_p$ is a covering map.

Hadamard Thm

Theorem (Hadamard)

Let M be a complete Riemannian manifold of dimension n, simply connected and with sectional curvature $K_p(\sigma) \leq 0$ for all $p \in M$ and for all two-dimensional subspace of T_pM. Then $\exp_p : T_pM \to M$ is a diffeomorphism, or – equivalently – M is diffeomorphic to $\mathbb{R}^n$.

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Lemma

Let M be a complete Riemannian manifold and let $\psi : M \to N$ be a local diffeomorphism onto a Riemannian manifold N. If for all $p \in M$ and for all $v \in T_pM$ it turns out that $d\psi_p(v)| \geq |v|$, then ψ is a covering map.

Cartan Thm

Let M and $\tilde{M}$ be two Riemmanian manifolds of dimension n. Consider $p \in M$ and $\tilde{p} \in \tilde{M}$.

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Let M and $\tilde{M}$ be two Riemmanian manifolds of dimension n. Consider $p \in M$ and $\tilde{p} \in \tilde{M}$. Take $\varphi : T_pM \to T_p\tilde{M}$ a linear isometry and let $p \ni V \subset M$ a normal neighborhood of p such that $\exp_{\tilde{p}}$ is defined at $\varphi \circ \exp_p^{-1}(V)$.

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$$\Psi := \exp_{\tilde{p}} \circ \varphi \circ \exp_p^{-1} .$$

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For any $q \in V$ there exists a unique (normalized) geodesic γ joining p to q.

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For any $q \in V$ there exists a unique (normalized) geodesic γ joining p to q. Consider the (normalized) geodesic $\tilde{\gamma}$ in $\tilde{M}$ uniquely determined by $\tilde{\gamma}(0) = \tilde{p}$ and $\tilde{\gamma}'(0) = \varphi(\gamma'(0))$.

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Let P_t and $\tilde{P}_t$ be the parallel transports along γ and $\tilde{\gamma}$ respectively and define

$$\psi_t(v) := \tilde{P}_t \circ \varphi \circ P_t^{-1}(v)$$

for $v \in T_qM$.

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$$\psi_t : T_qM \to T_{\Psi(q)}\tilde{M}$$

Cartan Thm

Theorem (Cartan)

Denote by $\mathcal{R}$ *and* $\tilde{\mathcal{R}}$ *the curvatures in* M *and* $\tilde{M}$ *respectively; if for all* $q \in V$ *and all* $x, y, z, w \in T_q M$ *we have*

$$\mathcal{R}(x, y, z, w) = \tilde{\mathcal{R}}(x, y, z, w)$$

then $\Psi : V \subseteq M \to \Psi(V) \subseteq \tilde{M}$ *is a local isometry and* $d\Psi_p = \varphi$*.*