Chapter 6

Covering Spaces

6.1 Local Homeomorphisms and Liftings

Consider two topological spaces X, Y. A map $f\colon X \to Y$ is called a *local homeomorphism* if each point $x \in X$ is contained in an open set U such that $V = f(U)$ is open in Y and the restriction $f|U$ is a homeomorphism from U onto V.

Every (global) homeomorphism is, evidently, a local homeomorphism. A local homeomorphism $f\colon X \to Y$ is a continuous and open map. In particular, the image $f(X)$ is an open set in Y. It follows that if X is compact and Y is a connected Hausdorff space, every local homeomorphism $f\colon X \to Y$ is surjective. When we have a local homeomorphism $f\colon X \to Y$, the space X inherits all of the local topological properties from Y such as, for example, local connectivity, local compactness, and so on. If f is surjective, then Y also inherits the local topological properties of X. Given a local homeomorphism $f\colon X \to Y$ and an open subset $A \subset X$, the restriction $f|A$ is also a local homeomorphism from A onto $f(A)$. A surjective local homeomorphism $f\colon X \to Y$ is a quotient map; that is, $g\colon Y \to Z$ is continuous if, and only if, $g \circ f\colon X \to Z$ is continuous.

If $f\colon X \to Y$ is a local homeomorphism, then f is locally injective; that is, every point $x \in X$ has a neighborhood U such that $f|U$ is injective. But a continuous locally injective map, even when it is surjective, may not be a local homeomorphism. An example is given in Figure 6.1, where X is a segment, Y is a loop, and f is the obvious map from X onto Y. In this case, f is locally injective, we have $f(x_0) = f(x_1) = y_0$, but no neighborhood of x_0 (or of x_1) is transformed homeomorphically by f onto a neighborhood of y_0.

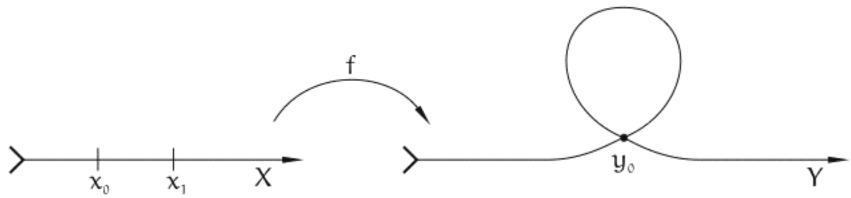


Figure 6.1. A locally injective map that is not a local homeomorphism.

Another example is given by $f\colon [0, 2\pi) \to S^1$, $f(t) = (\cos t, \sin t)$. Here, f is (globally) injective but it does not map any neighborhood of 0 (in the space $[0, 2\pi)$) onto a neighborhood of $f(0)$ in S^1.

A continuous locally injective map is a local homeomorphism if, and only if, it is open.

Example 6.1. The following maps are local homeomorphisms:

a. $\xi\colon \mathbb{R} \to S^1$, $\xi(t) = e^{it} = (\cos t, \sin t)$

b. $\zeta\colon \mathbb{R}^2 \to T = S^1 \times S^1$, $\zeta(s,t) = (e^{is}, e^{it})$

c. $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (e^x \cos y, e^x \sin y)$ or, using complex notation, $f(z) = e^z$

d. $\pi\colon S^n \to P^n$, $\pi(x) = \{x, -x\}$

e. $f\colon S^3 \to \mathrm{SO}(3)$, $f(x)(w) = x \cdot w \cdot x^{-1}$. (See Chapter 4, Section 1.)

With the exception of Case c, in all of the items above, the local homeomorphism is surjective. In Case c, we have $f(\mathbb{R}^2) = \mathbb{R}^2 - \{0\}$.

A more geometric version of Case a above can be obtained by considering an infinite spiral, say $X = \{(1 + e^t)e^{it}; t \in \mathbb{R}\}$, which turns around the unit circle S^1 (see Figure 6.2), and by defining $f\colon X \to S^1$ as the radial projection from the origin; that is, $f(z) = z/|z|$, or, $f((1 + e^t)e^{it}) = e^{it}$. ◁

In Case c, each vertical line passing through the point $(x, 0)$ is transformed by f onto a circle with center at the origin and radius e^x. The horizontal line through the point $(0, y)$ is transformed by f homeomorphically onto an open ray that starts at the origin and makes an angle of y radians with the positive x-axis. The inverse image $f^{-1}(b)$ of each point $b \in \mathbb{R}^2 - \{0\}$ is a countable set of points, all contained in the same vertical line, each one of them at a distance of 2π from the two other closest points. Every open horizontal strip with width 2π is transformed by f homeomorphically onto the complement of a ray ℓ which starts at the origin $0 = (0, 0)$.

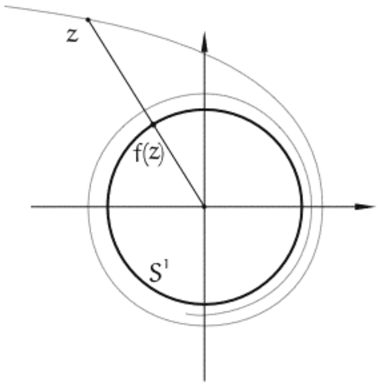


Figure 6.2. A local homeomorphism.

The inverse image $f^{-1}(\mathbb{R}^2 - \ell)$ is the countably infinite disjoint union of open horizontal strips of width 2π. Each one of these strips is transformed homeomorphically by f onto $\mathbb{R}^2 - \ell$.

Example 6.2. Let $U \subset \mathbb{R}^m$ be an open set and $f: U \to \mathbb{R}^m$ be a map of class C^1 whose derivative, $f'(x): \mathbb{R}^m \to \mathbb{R}^m$, is an isomorphism at each point $x \in U$. The Inverse Function theorem guarantees that f is a local homeomorphism. Most examples of local homeomorphisms arise in this context, or in its global version, which can be stated as follows: *Let M^m, $N^m \subset \mathbb{R}^n$ be two differentiable manifolds, and f a map of class C^1 whose derivative $f'(x): T_xM \to T_{f(x)}N$ is an isomorphism at each point $x \in M$. Then f is a local homeomorphism.* The five items in Example 6.1 are special cases of this situation. ◁

Proposition 6.1. *If the map $f: X \to Y$ is continuous and locally injective (in particular, a local homeomorphism), then the inverse image $f^{-1}(y)$ of each point $y \in Y$ is a discrete subset of X.*

Proof. Each point $x \in f^{-1}(y)$ has a neighborhood U, where x is the only point in U such that $f(x) = y$. Then $U \cap f^{-1}(y) = \{x\}$. So, every point $x \in f^{-1}(y)$ is isolated in $f^{-1}(y)$. □

Corollary 6.1. *Let X be a compact space, Y a Hausdorff space, and $f: X \to Y$ a continuous locally injective map. Then $f^{-1}(y)$ is finite for each $y \in Y$.*

Remark. Even if $f^{-1}(y)$ is finite for each $y \in Y$, the continuous map $f: X \to Y$ may not be locally injective. For example, the closed curve X,

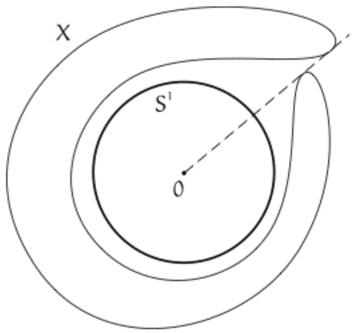
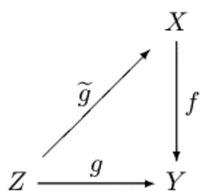


Figure 6.3.

sketched in Figure 6.3, projects radially onto the circle S^1 in such a way that the inverse image of each point of S^1 contains exactly two elements, but the radial projection of X onto S^1 is not locally injective.

Let $f\colon X \to Y$, $g\colon Z \to Y$ be two continuous maps. A *lifting* of g (with respect to f) is a continuous map $\widetilde{g}\colon Z \to X$ such that $f \circ \widetilde{g} = g$. This is illustrated in the diagram below.



One of the basic problems that we study in this chapter is the existence and uniqueness of the lifting, in terms of the properties of the map f. We show now that if f is locally injective then the lifting of g, if it exists, is unique, provided that Z is connected, X is Hausdorff and we fix a value $\widetilde{g}(z_0)$. Note that not every continuous map g has a lifting, even when f is a local homeomorphism. (See Proposition 3.5.)

Proposition 6.2. *Let X be a Hausdorff space and $f\colon X \to Y$ be a continuous and locally injective map. If Z is connected and $g\colon Z \to Y$ is continuous, then two liftings $\widetilde{g}$, $\hat{g}\colon Z \to X$ of g, which coincide at one point $z \in Z$, are equal.*

Proof. The set $A = \{z \in Z, \widetilde{g}(z) = \hat{g}(z)\}$ is not empty, because $z_0 \in A$. Since X is a Hausdorff space, A is closed in Z. In order to conclude that $\widetilde{g} = \hat{g}$, we just have to prove that A is open in Z. For this, let $a \in A$. There exists a neighborhood V of $\widetilde{g}(a) = \hat{g}(a)$ such that $f|V$ is injective. By the continuity of $\widetilde{g}$ and $\hat{g}$, there exists a neighborhood U of a with $\widetilde{g}(U) \subset V$ and $\hat{g}(U) \subset V$. Hence, for all $z \in U$ we have $f\widetilde{g}(z) = g(z) = f\hat{g}(z)$ and, from the injectivity of f in V, $\widetilde{g}(z) = \hat{g}(z)$. Therefore $U \subset A$. □

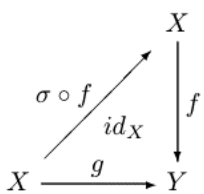
Let $f: X \to Y$ be a continuous map. A *section* of f is a continuous map $\sigma: Y \to X$ such that $f \circ \sigma = id_Y$. In order to provide a section σ we must choose continuously, for each $y \in Y$, a point $\sigma(y)$ belonging to the inverse image (or fiber) $f^{-1}(y)$. This is not always possible. First of all, f must be surjective but this necessary condition is far from being sufficient, as we will see in what follows.

If $\sigma: Y \to X$ is a section of f then the restriction of f to $\sigma(Y)$ is a homeomorphism onto Y.

The following corollaries show some consequences of Proposition 6.2 with respect to the sections of a locally injective map.

Corollary 6.2. *Let X be a connected Hausdorff space. A continuous locally injective map $f: X \to Y$ that admits a section $\sigma: Y \to X$ is a homeomorphism and its inverse is σ.*

In fact, in this case the maps $\sigma \circ f$, $id_X: X \to X$ are liftings of f (relatively to f), as illustrated by the diagram below.



Since $\sigma \circ f$ coincides with id_X in the set $\sigma(Y)$, from Proposition 6.2 we conclude that $\sigma \circ f = id_X$, hence $\sigma = f^{-1}$.

It follows from Corollary 6.2 that a continuous, locally injective and non-injective map, whose domain is Hausdorff connected, does not admit a section. An example of such a map is $f: S^1 \to S^1$, $f(z) = z^2$.

Corollary 6.3. *Let X be a Hausdorff space, Y connected and $f: X \to Y$ a continuous, locally injective map. If $\sigma: Y \to X$ is a section of f then $\sigma(Y)$ is a connected component of X.*

In fact, let C be a connected component of X which contains the connected set $\sigma(Y)$. By Corollary 6.2, $f|C$ is a homeomorphism from C onto Y. Since $f|\sigma(Y)$ is already a homeomorphism onto Y, we have $\sigma(Y) = C$.

Corollary 6.4. *Let A, B be open and connected subsets in the Hausdorff space $X = A \cup B$ and $f\colon X \to Y$ a continuous map such that $f|A$ and $f|B$ are homeomorphisms onto Y. Then $A \cap B = \varnothing$ or $A = B$.*

In fact, f is locally injective, $Y = f(A)$ is connected and $(f|A)^{-1}\colon Y \to X$ is a section. By Corollary 6.3, $A = (f|A)^{-1}(Y)$ is a connected component of X. In a similar way we show that B is also a connected component. It follows that $A = B$ or $A \cap B = \varnothing$.

Remark. Corollary 6.4 would be false without the hypothesis that A and B are open sets. This is shown by the function $f\colon S^1 \to [-1, 1]$, defined by $f(x, y) = x$, and the sets $A = \{(x, y) \in S^1; y \geq 0\}$, $B = \{(x, y) \in S^1; y \leq 0\}$.

6.2 Covering Maps

The Inverse Function Theorem is usually employed to prove that a certain map $f\colon X \to Y$ is a local homeomorphism, but a natural question remains open: Is f a (global) homeomorphism from X onto $f(X)$? Since f is already an open map, this is equivalent to ask if the map f is injective. This is a global question, of topological nature, whose answer cannot be given by Differential Calculus theorems, which are essentially local. We will discuss this problem here.

A local homeomorphism $f\colon X \to Y$ can be interpreted from the following viewpoint: given $a \in X$ and $b = f(a) \in Y$, the equation $f(x) = y$ has, for each y sufficiently close to b, a unique solution x, close to a, which depends continuously on y. It remains to be known the conditions under which this locally unique solution is globally unique in X.

The classical (and the most adequate) instrument to investigate if a given local homeomorphism is global and, more generally, to obtain regions where the homeomorphism is injective, is the *method of analytic continuation*, which we briefly describe now. Given a local homeomorphism $f\colon X \to Y$, let $y \in f(X)$. For each $x \in X$ with $f(x) = y$, there exist neighborhoods $U \ni x$ and $V \ni y$ such that $f\colon U \to V$ is a homeomorphism; $g = (f|U)^{-1}\colon V \to U$ is a local inverse of f, known classically as a "branch of f^{-1}." The problem consists of extending this branch g to a region larger than V. Given another point $y' \in f(X)$, we connect y' to y by a path a and we try to extend g along this path. This is possible provided that the path a has a lifting; that is, there exists a path $\widetilde{a}$ in X, with initial point x, such that $f(\widetilde{a}(s)) = a(s)$ for all $s \in I$. Then, since $y' = a(1)$, we define $g(y') = \widetilde{a}(1)$. The existence of the lifting $\widetilde{a}$ cannot be guaranteed, as the example below shows. The concept of *covering*, which we introduce below,

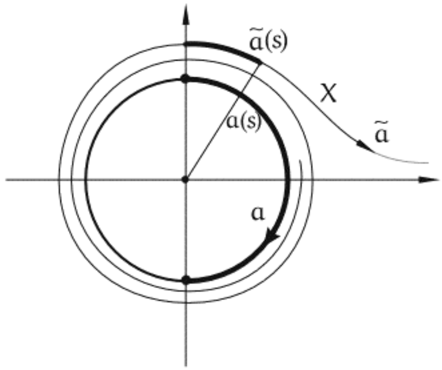


Figure 6.4.

provides additional conditions that will enable the use of the method of analytical continuation.

Example 6.3. Let $f\colon (0, 3\pi) \to S^1$ be the surjective local homeomorphism defined by $f(t) = (\cos t, \sin t)$. It is easy to obtain, in the circle S^1, paths that cannot be lifted relatively to f. We just have to take, for example, a closed path in S^1 whose degree is ≥ 2. For another example, consider the set $X = \{(1+t)e^{i/t}; 0 < t < +\infty\}$ and define $g\colon X \to S^1$ as the radial projection, $g(z) = z/|z|$. By taking $t = 2/\pi$, we see that the point $\left(0, 1 + \frac{2}{\pi}\right)$ belongs to X. Let $a\colon I \to S^1$ be a path with origin $(0,1)$, which describes homeomorphically the semicircle $x > 0$, and ends at the point $(0,-1)$ (see Figure 6.4). There is no path $\widetilde{a}\colon I \to X$ with $\widetilde{a}(0) = \left(0, 1 + \frac{2}{\pi}\right) \in X$ such that $g \circ \widetilde{a} = a$, even though we have $g(\widetilde{a}(0)) = a(0)$. ◁

A map $p\colon \widetilde{X} \to X$ is called a *covering map* (or, simply, a *covering*) when each point $x \in X$ belongs to an open set $V \subset X$ such that

$$p^{-1}(V) = \bigcup_{\alpha} U_\alpha$$

is a union of pairwise disjoint open sets U_α such that, for each α, the restriction $p|U_\alpha\colon U_\alpha \to V$ is a homeomorphism. The open set V satisfying the above condition is called a *distinguished neighborhood*. The space $\widetilde{X}$ is called a *covering space* of X and, for each $x \in X$, the set $p^{-1}(x)$ is called a *fiber* over x. Sometimes, X is called the *base* of the covering.

A covering map $p\colon \widetilde{X} \to X$ is a local homeomorphism from $\widetilde{X}$ onto X. Example 6.4 shows that not every local homeomorphism is a covering map.

The local homeomorphisms in Example 6.1 are covering maps. When the space Y is discrete, the projection $p\colon X \times Y \to X$ is a covering map.

Every open subset of a distinguished neighborhood is itself a distinguished neighborhood. Thus, when X is locally connected, locally compact, etc., we may choose the distinguished neighborhoods in such a way to be connected, with compact support, and so on.

If X is a locally connected and locally Hausdorff space, each distinguished neighborhood V can be chosen to be connected and Hausdorff. Thus we do not need to suppose that, in the decomposition

$$p^{-1}(V) = \bigcup_{\alpha} U_\alpha,$$

where $p|U_\alpha$ is, for each α, a homeomorphism onto V, the open sets U_α be pairwise disjoint. (See Corollary 6.4.) In this case, the sets U_α are the connected components of the inverse image $p^{-1}(V)$.

When $p\colon \widetilde{X} \to X$ is a covering, the condition that $\widetilde{X}$ be a Hausdorff space can be omitted from Proposition 6.2. We have the following proposition.

Proposition 6.3. *Let $p\colon \widetilde{X} \to X$ be a covering map and Z a connected space. If $\widetilde{g}$, $\hat{g}\colon Z \to \widetilde{X}$ satisfy $p \circ \widetilde{g} = p \circ \hat{g} = g$, then either $\widetilde{g}(z) \neq \hat{g}(z)$ for all $z \in Z$ or $\widetilde{g} = \hat{g}$.*

Proof. Since p is locally injective, we use the proof of Proposition 6.2 in order to see that the set $A = \{z \in Z; \widetilde{g}(z) = \hat{g}(z)\}$ is open. In order to show that A is closed, without using that $\widetilde{X}$ is a Hausdorff space, choose $z \in Z$ such that $\widetilde{g}(z) \neq \hat{g}(z)$. The image of these two points by the map p is the same point $g(z) \in X$. Let V be a distinguished neighborhood of $g(z)$. Then

$$p^{-1}(V) = \bigcup_{\alpha} U_\alpha,$$

the disjoint union of open set which are mapped homeomorphically by p onto V. Therefore, there exists $\alpha \neq \beta$ such that $\widetilde{g}(z) \in U_\alpha$ and $\hat{g}(z) \in U_\beta$. By taking an open neighborhood $W \ni z$ in Z such that $\widetilde{g}(W) \subset U_\alpha$ and $\hat{g}(W) \subset U_\beta$, we see that $\widetilde{g}(w) \neq \hat{g}(w)$ for all $w \in W$. Hence, $z \notin A \Rightarrow z \in W$ with $W \cap A = \varnothing$. Thus, A is closed. □

Proposition 6.4. *If the base X of a covering $p\colon \widetilde{X} \to X$ is connected, then each fiber $p^{-1}(x)$ $x \in X$, has the same cardinal number, which is called the* number of leaves *of the covering.*

Proof. For every point x of a distinguished neighborhood V, the cardinal number of the fiber $p^{-1}(x)$ is the same. Hence, the set of the points $x \in X$ such that $p^{-1}(x)$ has a prescribed cardinal number is open. This determines a decomposition of X as the union of disjoint open sets, where in each of them the cardinal number of $p^{-1}(x)$ is constant. Since X is connected, this family of disjoint open sets has only one set. □

Remarks. 1. By Corollary 6.1, when $\widetilde{X}$ is compact and X is a connected Hausdorff space, every covering map $p\colon \widetilde{X} \to X$ has a finite number of leaves. In this case, $X = p(\widetilde{X})$ is necessarily compact.

2. A covering map $p\colon \widetilde{X} \to X$ whose base X is connected is a locally trivial fibration whose typical fiber F is discrete (and whose cardinality is equal to the number of leaves of p).

Example 6.4. Let $f\colon X \to Y$ be a local homeomorphism, where Y is connected and each inverse image $f^{-1}(y)$, $y \in Y$, is finite. Given any $x_0 \in X$, the restriction $f_0 = f|(X - \{x_0\})$ is still a local homeomorphism. But at least one of the maps, f or f_0, is not a covering map. In fact, either the number of elements of $f^{-1}(y)$ is not constant or the number of elements of $f_0^{-1}(y)$ is not constant. ◁

It is important to recognize when a local homeomorphism $f\colon X \to Y$ is a covering map. Now we characterize the coverings with a finite number of leaves. Other sufficient conditions will be studied later on.

A map $f\colon X \to Y$ is called *closed* when the image $f(F)$ of every closed subset $F \subset X$ is a closed subset of Y.

In order that a map $f\colon X \to Y$ be closed, it is necessary and sufficient that, for every $y \in Y$ and every open set $U \supset f^{-1}(y)$ in X, there exists an open set $V \ni y$ in Y such that $f^{-1}(V) \subset U$. (See the Appendix at the end of the book.) Note that this condition suggests something like the continuity of the correspondence $y \mapsto f^{-1}(y)$, which is not a function.

A continuous map $f\colon X \to Y$ is called *proper* when it is closed and, for every $y \in Y$, the inverse image $f^{-1}(y)$ is compact. In the Appendix, we prove some basic properties of proper maps.

Proposition 6.5. *Let X be a Hausdorff space and $f\colon X \to Y$ a local homeomorphism. Each of the following statements implies the next one:*

1. *There exists $n \in \mathbb{N}$ such that each inverse image $f^{-1}(y)$, $y \in Y$, has n elements.*
2. *f is proper and surjective.*

3. f is a covering map whose fibers $f^{-1}(y)$ are finite.

If Y is connected, then the three statements are equivalent.

Proof. $1 \Rightarrow 2$. We just need to prove that f is closed. Let $y \in Y$ and $A \supset f^{-1}(y)$ be an open set. Since $f^{-1}(y) = \{x_1, \dots, x_n\}$ is finite and X is Hausdorff, there exist pairwise disjoint open sets $W_1 \ni x_1, \dots, W_n \ni x_n$, such that $W_1 \cup \ldots \cup W_n \subset A$. Hence,

$$V = \bigcap_{i=1}^{n} f(W_i)$$

is an open neighborhood of y. For each $i = 1, \dots, n$, $U_i = W_i \cap f^{-1}(V)$ is open and, by setting $U = \cup U_i$, we have $f^{-1}(V) \supset U$. We claim that we must have $f^{-1}(V) = U$. In fact, if $w \in f^{-1}(V)$—that is, $f(w) = v \in V = \cap f(W_i)$—then there exist $w_1 \in W_1, \dots, w_n \in W_n$ such that $f(w_i) = v$ for every i. Since $f^{-1}(v)$ has n elements and the sets W_i are pairwise disjoint, we must have $w = w_i$ for some i, thus $w \in U_i = f^{-1}(V) \cap W_i$, that is, $w \in U$. Hence, $f^{-1}(V) = U \subset \cup W_i \subset A$, which proves that f is closed, because of the criterion about closed maps mentioned above and proved in the Appendix.

$2 \Rightarrow 3$. Given an arbitrary point $y \in Y$, its inverse image is a compact discrete set, therefore, it is finite: $f^{-1}(y) = \{x_1, \dots, x_n\}$. Let $W_1 \ni x_1, \ldots W_n \ni x_n$ be pairwise disjoint open sets in X, which are mapped homeomorphically by f onto open sets of Y. Then $f(W_1) \cap \ldots \cap f(W_n)$ is an open neighborhood of y and, since f is closed, we can obtain an open set V with $y \in V \subset \cap f(W_i)$ and such that $f^{-1}(V) \subset \cup W_i$. For every $i = 1, \dots, n$, we take $U_i = f^{-1}(V) \cap W_i$. Then $f^{-1}(V) = (\cup W_i) \cap f^{-1}(V) = \cup (f^{-1}(V) \cap W_i) = \cup U_i$ and, since $V \subset f(W_i)$, f maps each one of the open sets U_i homeomorphically onto V.

Finally, when Y is connected, $3 \Rightarrow 1$ by Proposition 6.4. □

Corollary 6.5. *If X is a Hausdorff compact space and Y is Hausdorff, then every surjective local homeomorphism $f \colon X \to Y$ is a covering.*

In fact, f is proper.

Corollary 6.6. *Let X, Y be Hausdorff spaces. If $A \subset X$ has compact closure and the local homeomorphism $f \colon A \to Y$ extends continuously to a map $\bar{f} \colon \overline{A} \to Y$ such that $\bar{f}(\partial A) \subset \partial \bar{f}(A)$ (that is, $\bar{f}$ maps the boundary of A into the boundary of $\bar{f}(A)$), then the restriction $f|A \colon A \to f(A)$ is a covering.*

In fact, under these conditions, f is a proper map from A onto $f(A)$.

Remarks. 1. Let $f\colon X \to Y$ be a covering map. If Y is a Hausdorff compact space and each fiber $f^{-1}(y)$ is finite, then X is compact. The proof is easy (even in the general case where f is a locally trivial fibration, with compact base and fiber). By supposing that X is Hausdorff and Y is connected, it follows as a corollary of Proposition 6.5, because f is proper and $X = f^{-1}(Y)$.

2. In order to prove that $1 \Rightarrow 2$ we used only the fact that f is open and continuous. Therefore, we can state that if $f\colon X \to Y$ is a continuous open map, and there exists $n \in \mathbb{N}$ such that all of the inverse images $f^{-1}(y)$, $y \in Y$ have n elements, then f is a covering map. About the necessity that f be an open map, see Figure 6.3.

3. In the Appendix, Proposition A.4, we show that if X and Y are metric spaces without isolated points, a local homeomorphism $f\colon X \to Y$ which is also a closed map is necessarily a proper map.

Example 6.5. The maps in the items a), b), and c) in Example 6.1 are not proper maps; the maps in items d) and e) are proper. The map $f\colon S^1 \to S^1$, defined by $f(z) = z^n$, is a local homeomorphism (by the Inverse Function theorem). Since S^1 is a Hausdorff compact space and f is surjective, we see that it is a covering map with n leaves. ◁

Example 6.6. Let $p\colon \mathbb{C} \to \mathbb{C}$ be a non constant complex polynomial and $F \subset \mathbb{C}$ the finite set whose elements are the roots of $p'(z)$. By setting $X = \mathbb{C} - p^{-1}(p(F))$ and $Y = \mathbb{C} - p(F)$, we see that the restriction $p|X : X \to Y$ is a local homeomorphism and a proper map; $p|X$ is surjective because Y is connected. Hence, $p|X$ is a covering with n leaves, where n is the degree of p. In particular, the map $p\colon \mathbb{C} - \{0\} \to \mathbb{C} - \{0\}$, $p(z) = z^n$, is a covering. ◁

6.3 Properly Discontinuous Groups

Important examples of covering maps are obtained when we consider properly discontinuous groups of homeomorphisms, which we study now.

The set of homeomorphisms of a topological space X is a group with the operation of composition. A subgroup G of this group is called a *group of homeomorphisms* of X. Therefore, we must have:

1. $id_X \in G$;
2. $g, h \in G \Rightarrow gh \in G$;
3. $g^{-1} \in G$.

(Here, gh is the composition of g with h.)

For the sake of simplicity in the notation, the image of the point x by the homeomorphism $g\colon X \to X$ is denoted by gx.

The *orbit* of a point $x \in X$ relative to a group of homeomorphisms G is the set $G \cdot x = \{gx; g \in G\}$. The relation "there exists $g \in G$ such that $gx = y$" is an equivalence relation on the set X. The equivalence class of a point $x \in X$ according to this relation is the orbit of the point $G \cdot x$. Therefore, given x, $y \in X$, either $G \cdot x = G \cdot y$ or $G \cdot x \cap G \cdot y = \varnothing$.

A group G of homeomorphisms of a space X is said to be *properly discontinuous* when every point $x \in X$ has a neighborhood V such that, for every $g \in G$ different from the identity, we have $g \cdot V \cap V = \varnothing$. Equivalently: If $g \neq h$ in G, then $g \cdot V \cap h \cdot V = \varnothing$. We say that V is a *convenient neighborhood* of the point x.

If G is a properly discontinuous group of homeomorphisms of a topological space X, then for every $g \neq id_X$ in G and every $x \in X$, we have $gx \neq x$. That is, with the exception of the identity, the homeomorphisms that belong to G do not have fixed points. This is equivalent to stating that $g \neq h$ in $G \Rightarrow gx \neq hx$ for all $x \in X$. We also say, in this case, that G operates *freely* in X.

Given a properly discontinuous group G of homeomorphisms of a space X, the orbit $G \cdot x$ of each point of X is a discrete set. In fact, if V is a convenient neighborhood of the point x then each set $g \cdot V$, $g \in G$ is a neighborhood of gx which contains only this point of the orbit $G \cdot x$.

A neighborhood V is convenient with respect to a properly discontinuous group G if, and only if, it contains at most one element of any orbit of G. In fact, let V be a convenient neighborhood. If there exist y, $gy \in V$ then $gy \in V \cap gV$; hence, $g = id_X$ and from this, $gy = y$. Conversely, if V does not contain two distinct elements of any orbit of G then, for all $y \in V$ and $g \neq id_X$ in G, we have $gy \notin V$; that is, $V \cap g \cdot V = \varnothing$.

If the points of the space X are closed sets (for example, if X is a Hausdorff, or locally Hausdorff, space), then the orbits $G \cdot x$ relative to a properly discontinuous group G of homeomorphisms of X are closed subsets of X. In fact, if $y \notin G \cdot x$, then a convenient neighborhood of y contains, at most, one point gx of the orbit $G \cdot x$. Since $\{gx\}$ is a closed set, a smaller neighborhood of y will be disjoint of $G \cdot x$.

In particular, when X is a compact Hausdorff space, every properly discontinuous group G of homeomorphisms of X is finite. In fact, by fixing $x_0 \in X$, the map $g \mapsto gx_0$ is a bijection from G onto the orbit $G \cdot x_0$ which, being a discrete and closed subset of the compact space X, is finite.

The condition that the points of X be closed is essential in order that the orbits of a properly discontinuous group of homeomorphisms of X be closed. This can be easily seen by considering the group $G = \{id_X\}$.

Example 6.7. For each $m \in \mathbb{Z}$, let $T_m : \mathbb{R} \to \mathbb{R}$ be the translation $T_m(x) = x + m$. The set $G = \{T_m; m \in \mathbb{Z}\}$ is a properly discontinuous group of homeomorphisms of $\mathbb{R}$. More generally, let $\mathbb{Z}^n \subset \mathbb{R}^n$ be the additive subgroup that consists of the vectors whose coordinates are integers. For each $v \in \mathbb{Z}^n$, let $T_v : \mathbb{R}^n \to \mathbb{R}^n$ be the translation $T_v(x) = x + v$. The set $G = \{T_v; v \in \mathbb{Z}^n\}$ is a properly discontinuous group of homeomorphisms of $\mathbb{R}^n$. Any neighborhood whose diameter is smaller than 1 is a convenient neighborhood for this group. ◁

Example 6.8. Let $\alpha : S^m \to S^m$ be the antipodal map. The set $G = \{id, \alpha\}$ is a group of homeomorphisms of S^m since $\alpha \circ \alpha = id$. G is properly discontinuous because if V is an open set contained in a hemisphere, then $\alpha \cdot V \cap V = \varnothing$. ◁

Example 6.9. Let G be a finite group of homeomorphisms of a Hausdorff space X such that, with the exception of the identity, no element $g \in G$ has fixed points. Then G is properly discontinuous. In fact, given $x \in X$, if $g \neq h$ in G, we have $gx \neq hx$. By Hausdorff axiom it is possible to obtain, for each $g \in G$, an open set V_g containing gx, such that $g \neq h$ implies that $V_g \cap V_h = \varnothing$. By the continuity of the homeomorphisms $g \in G$ and the fact that G is finite, we can take a neighborhood $V = V_{id}$ of x so small that $g \cdot V \subset V_g$ for every $g \in G$. Then $g \cdot V \cap V = \varnothing$ for every $g \in G$. Note that Example 6.8 is a particular case of this one. ◁

Example 6.10. Let G be a topological group. For each subgroup $H \subset G$ we may consider the group $\ell(H)$ of homeomorphisms of G, whose elements are the left translations $\ell_h : G \to G$, $\ell_h(x) = h \cdot x$, defined by elements $h \in H$. (See Example 6.7 where $G = \mathbb{R}^n$ e $H = \mathbb{Z}^n$.) The group of homeomorphisms $\ell(H)$ is properly discontinuous if, and only if, H is a discrete subgroup of G. One part of the statement is obvious: if $\ell(H)$ is properly discontinuous, the orbit of each element of G is a discrete set. In particular, H is discrete because it is the orbit of the neutral element of G. Conversely, suppose that $H \subset G$ is discrete. Then there exists a neighborhood U of the neutral element $e \in G$ such that $U \cap H = \{e\}$. Since the map $(x, y) \mapsto xy^{-1}$, of $G \times G$ into G, is continuous, there exists a neighborhood $V \ni e$ such that $x, y \in V \Rightarrow xy^{-1} \in U$. We assert that, for every $h \in H$, with $h \neq e$, we have $(\ell_h \cdot V) \cap V = \varnothing$. In fact, if there existed $x \in (\ell_h \cdot V) \cap V$ then there would exist $y \in V$ with $x = hy$ and from this $h = xy^{-1} \in U \cap H$, hence $h = e$. Note that the orbits of the group of homeomorphisms $\ell(H)$ are the cosets $H \cdot x$, determined by the subgroup $H \subset G$. Example 6.7 is a particular case of this situation. ◁

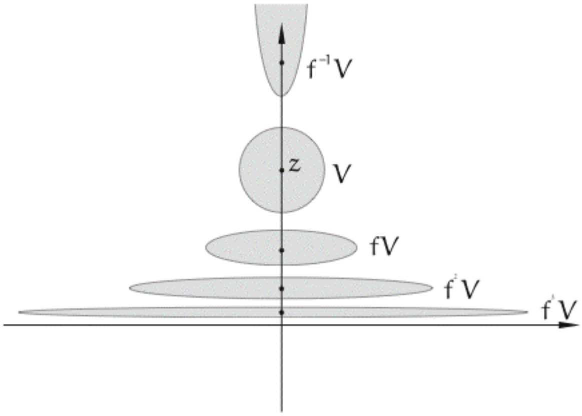


Figure 6.5.

Example 6.11. Let $X = \mathbb{R}^2 - \{0\}$. The homeomorphism $f\colon X \to X$, defined by $f(x, y) = (ax, a^{-1}y)$, where $a > 1$ is a constant, generates a group G of homeomorphisms of X, whose elements are the powers f^n, $n \in \mathbb{Z}$, where $f^n = f \circ f \circ \cdots \circ f$ (n times) if $n > 0$, $f^n = f^{-1} \circ \ldots \circ f^{-1}$ ($|n|$) times) if $n < 0$ and $f^0 = id_X$. The group G is properly discontinuous. In fact, for every $n \in \mathbb{Z}$, we have $f^n(x, y) = (a^n \cdot x, a^{-n} \cdot y)$. If G were not properly discontinuous, there would exist a point $z = (x, y) \in X$, a sequence of points $z_n = (x_n, y_n) \in X$ and a sequence of integers $k_n \neq 0$ such that $\lim_n z_n = z$ and $\lim_n f^{k_n}(z_n) = z$. In order to fix ideas, suppose that $x \neq 0$. Then, from the hypothesis $\lim(x_n, y_n) = (x, y) = \lim(a^{k_n} \cdot x_n, a^{-k_n} \cdot y_n)$, it would result that $\lim_n a^{k_n} = 1$, which contradicts $k_n \in \mathbb{Z} - \{0\}$.

Figure 6.5 shows the orbit of a point $z = (0, y)$, whose elements are the points $z_n = (0, a^{-n} \cdot y)$, and the sets $f^n V$, $n \in \mathbb{Z}$, where V is a disk with center z. ◁

Given a group G of homeomorphisms of X, we denote by X/G the quotient space of X by the equivalence relation whose equivalence classes are the orbits Gx, $x \in X$. The canonical projection $p\colon X \to X/G$ associates to each point $x \in X$ its orbit $p(x) = G \cdot x$. The open sets of the topology of X/G are the sets $A \subset X/G$ such that $p^{-1}(A)$ is open in X. Thus, the open sets of X/G are the images $p(U)$ where $U \subset X$ is an open set which is a union of orbits.

The continuous map $p\colon X \to X/G$ is open because if $V \subset X$ is open, then $p^{-1}(p(V)) = \bigcup_{g \in G} g{\cdot}V$ is open in X.

Proposition 6.6. *Let G be a group of homeomorphisms freely operating in the space X. The following statements are equivalent:*

1. *G is properly discontinuous.*
2. *The canonical projection $p\colon X \to X/G$ is a covering map.*
3. *$p\colon X \to X/G$ is locally injective.*

Proof. $1 \Rightarrow 2$: Let $y = p(x)$ be an arbitrary point in X/G. Take a convenient neighborhood $U \ni x$. Since p is an open map, $V = p(U)$ is an open neighborhood of y. We have that

$$p^{-1}(V) = \bigcup_{g \in G} g{\cdot}U$$

is the union of pairwise disjoint open sets (because U is a convenient neighborhood), and the restriction of the continuous map p to each of these open sets is injective; therefore, it is a homeomorphism onto $p(g{\cdot}U) = p(U) = V$. Hence 1 implies 2.
$2 \Rightarrow 3$: Obvious.
$3 \Rightarrow 1$: From 3, we conclude that each point $x \in X$ belongs to an open set U in which there are no two points in the same orbit. Then U is a convenient neighborhood of x and this proves that $3 \Rightarrow 1$. □

Corollary 6.7. *Let $f\colon G \to H$ be a surjective continuous homomorphism between two topological groups. In order that f be a covering map, it is necessary and sufficient that it be a local homeomorphism or, equivalently, that f be continuous, open, and its kernel be a discrete subgroup.*

In fact, under these conditions, denoting by $K = f^{-1}(e)$ the kernel of f, the group $\ell(K)$ of left translations by elements of K operates in a properly discontinuous mode in G (see Example 6.10). Hence, the quotient map $\pi\colon G \to G/K$ is a covering. By passing to the quotient, there exists a homeomorphism $\bar{f}\colon G/K \to H$ such that $\bar{f} \circ \pi = f$. Hence, f is a covering. The converse is obvious.

The quotient space X/G of a Hausdorff space X by a properly discontinuous group of homeomorphisms G is locally Hausdorff because it is locally homeomorphic to X. But, globally, X/G may or may not be a Hausdorff space. Since $p\colon X \to X/G$ is open, the necessary and sufficient condition in order that X/G be Hausdorff is that the set $\Gamma = \{(x, gx); x \in X, g \in G\}$, graph of the equivalence relation determined by G, be closed in $X \times X$. When G is finite, then Γ is the union of a finite number of closed subsets of $X \times X$ (the graphs of the homeomorphisms $g \in G$). Hence, X/G is Hausdorff. In particular, when the Hausdorff space X is compact, the quotient space X/G of X by a properly discontinuous group of homeomorphisms is Hausdorff because G is necessarily finite.

In Example 6.11 above, the quotient space X/G is not Hausdorff. In fact, the points $w = (0, 1)$ and $z = (1, 0)$ do not belong to the same orbit. Nevertheless, for any disks $U \ni w$ and $V \ni z$, we have that $f^n U$ is, for large values of $n > 0$, a long flattened oval, close to the x-axis. (On the other hand, for large $n < 0$, $f^n U$ is a long vertical oval, close to the y-axis.) This forces $f^n U \cap V \neq \varnothing$ for $n > 0$ sufficiently large. From this, it follows that the points $G \cdot w$ and $G \cdot z$ in X/G do not have disjoint neighborhoods.

Another way to verify that X/G is not a Hausdorff space is to consider the sequences $w_n = (a^{-n}, 1)$ and $z_n = (1, a^{-n})$. For each $n \in \mathbb{N}$, w_n and z_n belong to the same orbit because $z_n = f^n w_n$. But $\lim w_n = (0, 1) = w$ and $\lim z_n = (1, 0) = z$ belong to distinct orbits of G. This says that, in the quotient space X/G, the sequence $p(w_n) = p(z_n)$ has two distinct limits $p(w)$ and $p(z)$. Hence, X/G is not Hausdorff. The reader may imagine the quotient space X/G as the union of four cylinders and four circles in $\mathbb{R}^3$, with a topology different from the usual.

In Example 6.7, the quotient space $\mathbb{R}^n/G = \mathbb{R}^n/\mathbb{Z}^n$ is the n-dimensional torus. For $n = 1$, we obtain the circle S^1 and, in general, $\mathbb{R}^n/\mathbb{Z}^n$is homeomorphic to the Cartesian product $S^1 \times \ldots \times S^1$ of n copies of the circle. Thus, even with G being infinite, the quotient space is Hausdorff.

In Example 6.8, the quotient space S^n/G is the n-dimensional projective space.

6.4 Path Lifting and Homotopies

A continuous and surjective map $f\colon X \to Y$ is said to have the *path lifting property* when, for any arbitrary path $a\colon J \to Y$, with $J = [s_0, s_1]$, and each point $x \in X$ such that $f(x) = a(s_0)$, there exists a path $\widetilde{a}\colon J \to X$ such that $\widetilde{a}(s_0) = x$ and $f \circ \widetilde{a} = a$.

We know that not every local homeomorphism $f\colon X \to Y$ has the path lifting property but, when X is a Hausdorff space, the lifting $\widetilde{a}\colon J \to X$ of a path $a\colon J \to Y$ is completely determined by a and the initial point $x = \widetilde{a}(s_0)$.

When f is surjective and, for any arbitrary path $a\colon J \to Y$ and any point $x \in X$ with $f(x) = a(s_0)$, there exists a unique path $\widetilde{a}\colon J \to X$ such that $f \circ \widetilde{a} = a$ and $\widetilde{a}(s_0) = x$, we say that $f : X \to Y$ has the *unique path lifting property.*

Even when $\widetilde{X}$ is not Hausdorff, a covering $p : \widetilde{X} \to X$ has the unique path lifting property. This is the content of the proposition below, according to which the analytic continuation of the local inverse of p along a path is always possible when the local homeomorphism p is a covering map. The reader should not forget that the unique path lifting property requires, first of all, that f be surjective, by definition.

Proposition 6.7. *Let $p : \widetilde{X} \to X$ be a covering map. Given a path $a : J \to X$, $J = [s_0, s_1]$ and a point $\widetilde{x} \in \widetilde{X}$ such that $p(\widetilde{x}) = a(s_0)$, there exists a unique path $\widetilde{a} : J \to \widetilde{X}$ with $\widetilde{a}(s_0) = \widetilde{x}$ and $p \circ \widetilde{a} = a$. (In other words: p has the unique path lifting property.)*

Proof. Assume initially that $a(J) \subset V$, where V is a distinguished neighborhood. Then, since $\widetilde{x} \in p^{-1}(V)$, there exists an open set $U \ni \widetilde{x}$ which is mapped homeomorphically by p onto V. Let $f = (p|U)^{-1} : V \to U$, and set $\widetilde{a} = f \circ a$. Next, consider the case where $J = J_1 \cup J_2$ is the union of two compact intervals with an endpoint s_* in common, in such a way that the proposition holds for the restrictions $a_1 = a|J_1$ and $a_2 = a|J_2$. We choose $\widetilde{a}_1 : J_1 \to \widetilde{X}$ in such a way that $\widetilde{a}_1(s_0) = \widetilde{x}$ and $p \circ \widetilde{a}_1 = a_1$. After this, we obtain $\widetilde{a}_2 : J_2 \to \widetilde{X}$ such that $p \circ \widetilde{a}_2 = a_2$ and $\widetilde{a}_2(s_*) = \widetilde{a}_1(s_*)$, which is possible because $p(\widetilde{a}_1(s_*)) = a_1(s_*) = a_2(s_*)$. Then we define $\widetilde{a} : J \to \widetilde{X}$ by $\widetilde{a}|J_1 = \widetilde{a}_1$ and $\widetilde{a}|J_2 = \widetilde{a}_2$. The existence of $\widetilde{a}$ in the general case reduces to the two particular cases considered because, by the continuity of $a : J \to X$ and the compactness of J, there exists a decomposition $J = J_1 \cup \ldots \cup J_n$ of J as the union of consecutive intervals, in such a way that $a(J_i) \subset V_i$, a distinguished neighborhood, for $i = 1, 2, \ldots, n$. The uniqueness results from Proposition 6.3. □

Now we prove that if a local homeomorphism has the unique path lifting property then the lifting $\widetilde{a}$ depends continuously of a and the initial point $\widetilde{x} = \widetilde{a}(0)$. For this purpose, we present an appropriate description of the compact-open topology for paths.

Let X be a topological space and $C(I; X)$ be the set of paths $a : I \to X$. Given the open sets $U_1, \ldots, U_n$ in X and a partition $0 = t_0 < t_1 < \ldots < t_n = 1$, we use the notation

$$A(t_0, t_1, \ldots, t_n; U_1, \ldots, U_n)$$

to represent the set of all paths $a : I \to X$ such that $a([t_{i-1}, t_i]) \subset U_i$ for $i = 1, \ldots, n$. These sets constitute the basis for a topology. From now on, the symbol $C(I;X)$ means the topological space obtained by taking this topology in the set of paths $a\colon I \to X$. We remark (but we will not use this fact) that if the topology of X comes from a metric d then the topology that we have just defined in $C(I;X)$ is induced by the metric $d(a,b) = \sup_{0\le s\le 1} d(a(s), b(s))$.

The following remark can be easily verified: If B is a basis of open sets in X then the open sets $A(t_0, t_1, \ldots, t_n; U_1, \ldots, U_n)$, comprised only of open sets U_i belonging to the basis B, also constitute a basis for $C(I;X)$. This fact is used in the proof of the proposition below.

Proposition 6.8. *Let $f\colon X \to Y$ be a local homeomorphism with the unique path lifting property. Given a path $a\colon I \to Y$ and a point $x \in X$ with $f(x) = a(0)$, there exists a unique path $\widetilde{a}\colon I \to X$ such that $\widetilde{a}(0) = x$ and $f \circ \widetilde{a} = a$. The lifted path $\widetilde{a}$ depends continuously on a and the initial point x. More precisely: let $\Omega \subset C(I;Y) \times X$ be the subspace whose elements are the pairs (a,x) such that $a(0) = f(x)$. Then the map $L\colon \Omega \to C(I;X)$, given by $L(a,x) = \widetilde{a}$, is continuous.*

Proof. Consider in X the basis B whose elements are the open sets U which are homeomorphically mapped by f onto open sets $V \subset Y$. Let $A = A(t_0, t_1, \ldots, t_n; U_1, \ldots, U_n)$ be an open set of the corresponding basis in $C(I;X)$, containing the path $\widetilde{a}$, and set $V_i = f(U_i)$ and $\varphi_i = (f|U_i)^{-1}$. Then the set $A(t_0, t_1, \ldots, t_n; V_1, \ldots, V_n)$ is a neighborhood of the path $a = f \circ \widetilde{a}$. We state that if the path $b\colon I \to Y$ belongs to this neighborhood and if $x' \in U_1$ then $\widetilde{b} = L(b, x')$ belongs to A. In fact, for $i = 1, 2, \ldots, n$, we have $\widetilde{b}([t_{i-1}, t_i]) = \varphi_i b([t_{i-1}, t_i]) \subset U_i$, by the uniqueness of the lifting of the restriction $b|[t_{i-1}, t_i]$ from the initial point $\widetilde{b}(t_{i-1})$. This concludes the proof. □

We will obtain the homotopy lifting property as a consequence of Proposition 6.8. In Chapter 1, we interpreted a homotopy $H\colon Z \times I \to Y$ as a path in the space $C(Z;Y)$. Now we use a dual interpretation. To each homotopy $H\colon Z \times I \to Y$ we associate a map $h\colon Z \to C(I;Y)$ which associates to each point $z \in Z$ the path $h_z\colon I \to Y$, defined by $h_z(t) = H(z,t)$. Imagining $Z \times I$ as a cylinder, union of vertical line segments, the path h_z is the restriction of H to the vertical segment $z \times I$.

Proposition 6.9. *$H\colon Z \times I \to Y$ is continuous if, and only if, $h\colon Z \to C(I,Y)$ is continuous.*

Proof. Suppose that h is continuous. Given $(z_0, t_0) \in Z \times I$, let V be a neighborhood of $H(z_0, t_0)$ in Y. We must obtain a neighborhood U of

z_0 in Z and in interval $J \subset I$, containing t_0 as an interior point (in I), such that $H(U \times J) \subset V$. By the continuity of the path h_{z_0}, there exists a closed interval $J \subset I$, containing t_0 as an interior point (in I), such that $H(z_0, t) = h_{z_0}(t) \in V$ for all $t \in J$. Let A the set of all paths $a \in C(I; Y)$ such that $a(J) \subset V$. It is obvious that A is a neighborhood of h_{z_0} in $C(I; Y)$. By the continuity of h, there exists a neighborhood U of z_0 in Z such that $h_z \in A$ for every $z \in U$; that is, $H(z, t) \in V$ for all $z \in U$ and $t \in J$.

Conversely, let H be continuous. To prove the continuity of h, let $z_0 \in Z$ and consider the basic neighborhood

$$A(t_0, \ldots, t_n; V_1, \ldots, V_n)$$

of h_{z_0}. We must find a neighborhood U of z_0 in such a way that $z \in U$ and $t_{i-1} \leq t \leq t_i$ imply $H(z, t) \in V_i (1 \leq i \leq n)$. Now, $H^{-1}(V_i)$ is a neighborhood of $z_0 \times [t_{i-1}, t_i]$ in $Z \times I$. Since $[t_{i-1}, t_i]$ is compact, there exists an open set $U_i \subset Z$, containing z_0, such that $U_i \times [t_{i-1}, t_i] \subset H^{-1}(V_i)$. We set $U = U_1 \cap \ldots \cap U_n$. This concludes the proof. □

Proposition 6.10. *Let $\varphi \colon X \to Y$ be a local homeomorphism with the unique path lifting property. Given a homotopy $H \colon Z \times I \to Y$ between two continuous maps $f, g \colon Z \to Y$, if f has a lifting $\widetilde{f} \colon Z \to X$, then g also admits a lifting $\widetilde{g} \colon Z \to X$, which is homotopic to $\widetilde{f}$. More precisely, the homotopy H admits a unique lifting $\widetilde{H} \colon Z \times I \to X$ such that $\widetilde{H}(z, 0) = \widetilde{f}(z)$ for every $z \in Z$; $\widetilde{g}$ is then defined by $\widetilde{g}(z) = \widetilde{H}(z, 1)$.*

Proof. Let $h \colon Z \to C(I; Y)$ be obtained from H as in the previous proposition. For each $z \in Z$, let $\widetilde{h}_z = L(h_z, \widetilde{f}(z))$ be the unique path lifting h_z with origin at the point $\widetilde{f}(z)$. Since L is continuous, we see that $z \mapsto \widetilde{h}_z$ defines a continuous map $\widetilde{h} \colon Z \to C(I; X)$ and therefore a homotopy $\widetilde{H} \colon Z \times I \to X$, satisfying $\widetilde{H}(z, 0) = \widetilde{h}_z(0) = \widetilde{f}(z)$ and $\varphi \widetilde{H}(z, t) = \varphi \widetilde{h}_z(t) = h_z(t) = H(z, t)$. This concludes the proof. □

Proposition 6.11. *Let $f \colon X \to Y$ be a local homeomorphism with the unique path lifting property. If the paths $a, b \colon I \to Y$, with the same endpoints y_0, y_1, are homotopic then their liftings $\widetilde{a}, \widetilde{b} \colon I \to X$, starting at the same point x_0, end at the same point x_1 and, moreover, they are homotopic.*

Proof. Let $H \colon I \times I \to Y$ be a homotopy between a and b and $\widetilde{H} \colon I \times I \to X$ be the lifting of H such that $\widetilde{H}(s, 0) = \widetilde{a}(s)$ for every $s \in I$. Since $f(\widetilde{H}(0, t)) = H(0, t) = y_0$ and $f(\widetilde{H}(1, t)) = H(1, t) = y_1$ do not depend on t, it follows that $\widetilde{H}(0, t) = x_0$ and $\widetilde{H}(1, t) = x_1$ also do not depend on t

because the fibers $f^{-1}(y_0)$ and $f^{-1}(y_1)$ are discrete. The path $s \mapsto \widetilde{H}(s,1)$ in X is a lifting of b starting at x_0. It follows from the unique path lifting property that $\widetilde{H}(s,1) = \widetilde{b}(s)$ for all $s \in I$. Thus, we have $\widetilde{H}\colon \widetilde{a} \cong \widetilde{b}$. □

Corollary 6.8. *Let $f\colon X \to Y$ be a local homeomorphism with unique path lifting property. If the closed path $a\colon I \to Y$ is homotopic to a constant, any lifting $\widetilde{a}\colon I \to X$ is closed and homotopic to a constant.*

Note that in Proposition 6.11, nothing prevents the paths a and b from being closed. Also note that the fact that $\widetilde{a}$ is closed does not imply that a is homotopic to a constant. (See Proposition 7.2.)

Example 6.12. Now we use Proposition 6.11 to exhibit an example of a space whose fundamental group is not abelian. Our space X is the union of two circles with a point x_0 in common. It is convenient to think of X as the union of the great parallel of the torus and a meridian of the same torus, which cuts the parallel at the point x_0. We denote by a a closed path that covers the parallel homeomorphically with the exception, of course, of the endpoints that are mapped onto x_0; b denotes an analogous path defined over the meridian. We introduce a covering space $\widetilde{X}$, which is the subset of the plane sketched in Figure 6.6. To obtain $\widetilde{X}$ we take on the rectangular axis, starting from the origin, four segments of length 1. From the free endpoint of each of the four segments, we take three segments of length 1/2, parallel to the axis. From the free endpoint of each of these twelve segments, we take three segments of length 1/4, and so on. The covering space $\widetilde{X}$ is the union of the segments (an infinite number) thus constructed. The covering map $p\colon \widetilde{X} \to X$ sends each horizontal segment onto a and each vertical segment onto b in such a way that the increasing order of

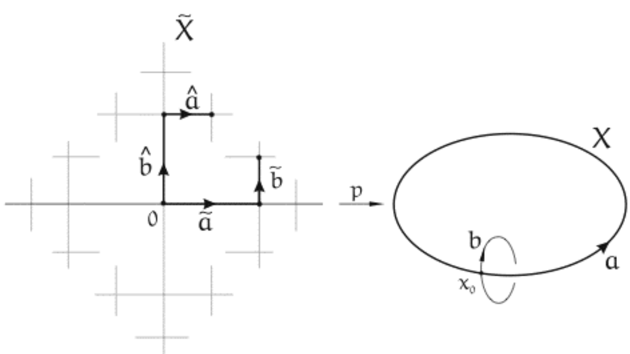


Figure 6.6.

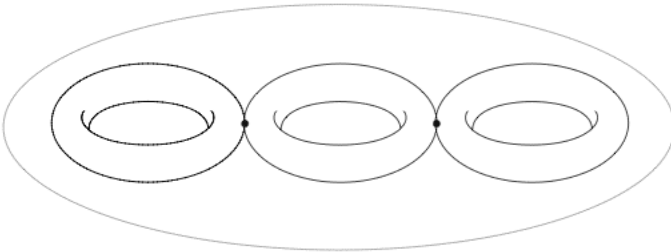


Figure 6.7.

the coordinate that varies in each one of these segments agrees with the orientations of the paths a and b respectively and that the endpoints are mapped onto x_0. ◁

Now we show that the closed paths ab and ba, with bases at the point x_0, are not homotopic in X. For this, we just have to consider their respective liftings $\widetilde{ab}$ and $\widehat{ba}$ in $\widetilde{X}$, with origin at the point O. The final point of $\widetilde{ab}$ is $(1, 1/2)$, while $\hat{b}\hat{a}$ ends at the point $(1/2, 1)$. If ab and ba were homotopic in X, their liftings from the point O would end at the same point of $\widetilde{X}$, because of Proposition 6.11.

The space X above is known as the *figure 8* space because it is homeomorphic to the graphical sign of the digit eight. Now we can exhibit other spaces with non-abelian fundamental group. For example, the union of a list (finite or infinite) of circles, each one of them with a point in common only with the previous and the following circles in the list. If the number of circles is ≥ 2, such an space has the figure eight as a retract, hence its fundamental group is not abelian. (See Proposition 2.10.) Also, a compact non-orientable surface of genus $g \geq 2$ has a non abelian fundamental group because it admits as a retract a union of g circles with $g-1$ points of tangency. (In Figure 6.7, $g = 3$.)

We should also mention the complement of a set of two points in $\mathbb{R}^2$. This space has the same homotopy type of the figure eight space, so its fundamental group is nonabelian.

Example 6.13. The fundamental group of the figure eight space is generated by the homotopy classes $\alpha = [a]$ and $\beta = [b]$. This follows from Proposition 2.11. In fact, by denoting the figure eight space by X, we have $X = X_1 \cup X_2$, where X_1 and X_2 are circles with a point x_0 in common. We cannot directly apply the mentioned proposition because neither X_1 nor X_2 are open sets in X. But, if we take the points $x_1 \in X_1$ and $x_2 \in X_2$, both different from x_0, and set $U = X - \{x_1\}$, $V = X - \{x_2\}$, we see that the inclusions $X_2 \to U$ and $X_1 \to V$ are homotopy equivalences. From this, it follows that the homotopy class of a is a generator of the infinite cyclic group $\pi_1(V)$ and the class of b generates the infinite cyclic group $\pi_1(U)$. The same proposition, applied to $X = U \cup V$, states that $\pi_1(X, x_0)$ is generated by α and β. In fact, we may state a sharper result: The generators α and β are *free*; that is, no monomial of the type $\alpha^m \beta^n \alpha^p \ldots$, product of a finite number of alternating powers of α and β, can be reduced to the neutral element of $\pi_1(X, x_0)$ except when the exponents $m, n, p, \ldots \in \mathbb{Z}$ are all null. This fact will be proved in the next chapter. ◁

Example 6.14. It follows from Example 6.12 that a compact orientable surface of genus > 1 does not admit a topological group structure. In fact, from Proposition 2.12, the fundamental group of a topological group is abelian. It remains to consider the compact orientable surfaces of genus 0 and 1. The torus $T = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$ has genus 1, and it is obviously a topological group; the sphere S^2 has genus 0, and it does not admit a structure of topological group, but for a completely different reason, which can be explained as follows: Suppose that S^2 is a topological group, with neutral element e. By fixing a point $a \in S^2$, close to e but satisfying $a \neq e$, we would have $a \cdot x \neq -x$ and $a \cdot x \neq x$ for all $x \in S^2$. Then, by defining $v\colon S^2 \to \mathbb{R}^3$ by setting $v(x) = \langle x, a \cdot x\rangle x - a \cdot x$, the map would be continuous, with $v(x) \neq 0$ and $\langle x, v(x)\rangle = 0$ for all $x \in S^2$, in contradiction with Proposition 4.4. Thus, we conclude that the torus is the only compact orientable surface that admits a topological group structure. In Example 7.18, we show that no compact surface (orientable or not), except the torus, can be a topological group. ◁

Proposition 6.12. *Let $f\colon X \to Y$ a local homeomorphism with the unique path lifting property. If X is pathwise connected and Y is simply connected, then f is a homeomorphism.*

Proof. We just have to prove that f (which is already continuous, open, and surjective) is also injective. Consider $x_0, x_1 \in X$ such that $f(x_0) = f(x_1)$. Now take a path $\widetilde{a}\colon I \to X$ whose initial point is x_0 and final point is x_1. The path $a = f \circ \widetilde{a}$ is closed in Y, therefore it is homotopic to a constant. By Corollary 6.8, its lifting $\widetilde{a}$ is closed; hence, $x_0 = x_1$. □

Corollary 6.9. *Let $f\colon X \to Y$ be a local homeomorphism with the unique path lifting property. If Y is simply connected and locally pathwise connected, then f maps each connected component of X homeomorphically onto Y.*

Since the space X is locally homeomorphic to Y, it is locally pathwise connected. Thus, every connected component $C \subset X$ is pathwise connected and open. Hence, $p|C$ is a local homeomorphism and, as can be easily proved, $p|C\colon C \to Y$ has the unique path lifting property. It follows from Proposition 6.12 that $p|C$ is a homeomorphism from C onto Y.

It follows from Proposition 6.12 that every pathwise connected covering of a simply connected space is a homeomorphism.

For example, let $U \subset \mathbb{R}^n$ be an open connected and bounded set. Given a class C^1 map $f\colon U \to \mathbb{R}^n$, suppose that $f'(x)\colon \mathbb{R}^n \to \mathbb{R}^n$ is, for all $x \in U$, an isomorphism. By the Inverse Function theorem, f is a local homeomorphism. It may happen that f is not a covering of the open set $V = f(U)$. But if f is such that $x_k \to x \in \partial U \Rightarrow f(x_k) \to y \in \partial V$, then f extends to a continuous map $\bar{f}\colon \overline{U} \to \overline{V}$ such that $\bar{f}(\partial U) \subset \partial V$. The map f is, in this case, proper, and therefore, it is a covering $f\colon U \to V$. If we know that V is simply connected (for example, V convex), then we may conclude that f is injective and therefore, it is a C^1 diffeomorphism from U onto V.

Analogously, let $f\colon M^m \to N^m$ be a class C^1 map where M^m and N^m are differentiable surfaces (without boundary) of dimension m. Suppose that the derivative $f'(x)\colon T_xM \to T_{f(x)}N$ is an isomorphism at each point $x \in M$. If M is compact and connected and N is simply connected, then f is bijective and therefore, it is a diffeomorphism from M onto N. The case where M is not compact will be covered in Section 5.

Example 6.15. A local homeomorphism from a connected space onto a simply connected space may not be injective (if it is not a covering map). For example: Let $X = \mathbb{C} - \{1, -1\}$, $Y = \mathbb{C}$ and define $f\colon X \to Y$ by $f(z) = z^3 - 3z$. Since $f'(z) \neq 0$ for all $z \in X$, we see that f is a local homeomorphism (Inverse Function theorem), that it is surjective because the values 2 and -2, of the polynomial $z^3 - 3z$ at the points 1 and -1, are also attained at the points 2 and -2, which belong to X. But f is not injective, even though its image $Y = \mathbb{C}(= \mathbb{R}^2)$ is simply connected. In fact, $f(0) = f(\sqrt{3}) = f(-\sqrt{3}) = 0$. Therefore, $f\colon X \to \mathbb{C}$ is a non-injective local homeomorphism onto a simply connected space. ◁

Corollary 6.10. *Let $f\colon X \to Y$ be a local homeomorphism with the unique path lifting property and $V \subset Y$ an open connected and pathwise locally connected set, such that every closed path contained in V is homotopic to a constant in Y. Then each connected component of $U = f^{-1}(V)$ is mapped homeomorphically by f onto V.*

If V were simply connected, we would have a particular case of Corollary 6.9. In the general case, we just have to observe that, with the assumed hypothesis on V, the lifting of every closed path contained in V is a closed path. This is enough to assure that $f|U$ is injective in each connected component of $U = f^{-1}(V)$ and the proof of Corollary 6.9 applies, word by word.

Corollary 6.10 says that in every set V to the above type we can define several "branches" of the inverse of f, one branch for each connected component of $f^{-1}(V)$. For example, if we take $f\colon \mathbb{R}^2 \to \mathbb{R}^2 - \{0\}$ given by $f(z) = e^z$, we reobtain the well known fact that in each simply connected region $V \subset \mathbb{R}^2 - \{0\}$ it is possible to define an infinite number of branches of the logarithm.

Corollary 6.11. *Let $p\colon \widetilde{X} \to X$ be a covering map. If an open set $V \subset X$ is connected and locally pathwise connected and, moreover, every closed path in V is homotopic to a constant in X, then V is a distinguished neighborhood.*

A topological space X is called *semi-locally simply connected* when every point $x \in X$ has a neighborhood V such that every closed path in V is homotopic to a constant in X.

Important cases of semi-locally simply connected spaces are the topological manifolds and the polyhedra. In fact, in these spaces, every point has a simply connected neighborhood, so these spaces are actually locally simply connected.

Example 6.16. We give now an example of a space Y that is semi-locally simply connected, but contains a point that does not have any simply connected neighborhood.

We start with a space X, pathwise connected, which is not semi-locally simply connected: For each $n \in \mathbb{N}$, let X_n be the circle of center $(0, 1/n)$ and radius $1/n$, in the plane. (X_n is tangent to the x-axis at the origin.)

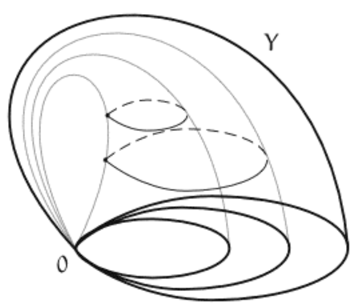


Figure 6.8.

We take $X = \bigcup_n X_n$. The space Y is obtained by taking the cone with base X and identifying the vertex of this cone with the origin O, the tangency point of the circles X_n (see Figure 6.8). ◁

Proposition 6.13. *Let X be a locally pathwise connected and semi-locally simply connected space. A map $p\colon \widetilde{X} \to X$ is a covering if, and only if, it is a local homeomorphism with the unique path lifting property.*

Proof. The "If" part is the Corollary 6.10 above. The "Only if" part is Proposition 6.7. □

Corollary 6.12. *Let X be a locally pathwise connected and semi-locally simply connected space. If $p\colon \widetilde{X} \to X$ and $q\colon \widehat{X} \to \widetilde{X}$ are covering maps, the composite map $p \circ q\colon \widehat{X} \to X$ is also a covering map.*

In fact, it is obvious that if the maps p and q have the unique path lifting property, then the composite map $p \circ q$ also has the property.

Remark. If p has a finite number of leaves then, as we can see from the definition, its composite map $p \circ q$, with another covering map q, is still a covering map, even without imposing to one of the spaces (and therefore to all of them) the condition of being semi-locally simply connected.

6.4.1 An Application

Let G, H be topological groups. A *local homomorphism* from G to H is a *continuous* map $f\colon U \to H$, defined in a neighborhood U of the neutral element $e \in G$, such that if $x, y, x \cdot y \in U$, then $f(x \cdot y) = f(x) \cdot f(y)$. As an application of Proposition 6.12, we prove the following.

If the group G is simply connected and locally pathwise connected, then every local homomorphism $f\colon U \to H$, from G into a topological group H, extends to a continuous homomorphism $\bar{f}\colon G \to H$.

In fact, restricting f, if necessary, we may suppose that its domain U is pathwise connected. Let $A \subset G \times H$ be the subgroup of the product $G \times H$ generated by the graph of f. We define on A the topological group topology according to which a fundamental system of neighborhoods of the neutral element is given by the sets $\widetilde{V} = \{(x, f(x)); x \in V\}$, where $V \subset U$ is a neighborhood of the neutral element. Let $p\colon A \to G$ be the restriction of the projection $\pi_G\colon G \times H \to G$. The continuous homomorphism p maps the graph $\widetilde{U}$ of f homeomorphically onto U. Since G is connected, and therefore generated by U, p is surjective (and a local homeomorphism). By Corollary 6.10, $p\colon A \to G$ is a covering. Now, G is simply connected and A is connected, because it is generated by the connected neighborhood $\widetilde{U}$ = graph of f. Hence, p is a homeomorphism from A onto G. The inverse homeomorphism $p^{-1}: G \to A$ is given by $p^{-1}(x) = (x, \bar{f}(x))$. The continuous homomorphism $\bar{f}: G \to H$, thus defined, is the extension of f that we have been searching for.

6.5 Differentiable Coverings

First, we examine what happens when a surjective local homeomorphism $f: X \to Y$ does not have the path lifting property.

This means that there exist $x \in X$ and a path $a: I \to Y$ such that $a(0) = f(x)$ but a cannot be lifted to a path in X starting at the point x. We suppose that X is Hausdorff, which gives us the uniqueness of the liftings that might there exist.

Since f is a local homeomorphism, for $\varepsilon > 0$ sufficiently small the restriction $a|[0, \varepsilon]$ has a lifting starting at the point x. Therefore, there exists a number r, $0 < r \leq 1$, such that, for all r' with $0 < r' < r$, the path $a|[0, r']$ has a lifting starting at the point x but $a|[0, r]$ does not have. This means (because of the uniqueness of the lifting) that $a|[0, r)$ has a lifting $\widetilde{a}: [0, r) \to X$ but, when $s \to r$, $\widetilde{a}(s)$ does not have an adherence value (hence no limit) in X. (In fact, if $x' \in X$ were an adherence value of $\widetilde{a}(s)$ when $s \to r$, the continuity of f would imply that $f(x')$ would be the adherence value for $a(s)$ when $s \to r$ and therefore $f(x') = a(r)$. Then, by taking a neighborhood of x' mapped homeomorphically by f onto a neighborhood of $a(r)$, we would conclude that $a|[0, r]$ would have a lifting.) The non-existence of an adherence value of $\widetilde{a}(s)$ in X when $s \to r$, results, in particular, that the set $\{\widetilde{a}(s); 0 \leq s < r\}$ is closed in X, while its image by f, that is, $\{a(s); 0 \leq s < r\}$, is not closed in Y. Hence we can state the

Proposition 6.14. *Let X be a Hausdorff space. If a surjective local homeomorphism $f: X \to Y$ is a closed map then f has the unique path lifting property. In particular if, moreover, Y is locally pathwise connected and semi-locally simply connected, then f is a covering map.*

Remark. Under very general conditions, if a local homeomorphism $f: X \to Y$ is a closed map, then f is proper; that is, $f^{-1}(y)$ is a finite set, for all $y \in Y$. (See Proposition A.6 in the Appendix.)

We provide now a sufficient condition in order that a map be a covering within the scope of the differential calculus.

Proposition 6.15. *Let $f: \mathbb{R}^m \to \mathbb{R}^m$ be a map of class C^1, whose values are contained in a open connected set $Y \subset \mathbb{R}^m$. Suppose that there exists* *a covering of* Y *by open sets* V*, and to each of these sets is associated a number* $\varepsilon_V > 0$*, in such a way that* $f(x) \in V$ *implies* $|f'(x) \cdot u| \geq \varepsilon_V \cdot |u|$ *for all* $u \in \mathbb{R}^m$*. Then* $f(\mathbb{R}^m) = Y$ *and* $f\colon \mathbb{R}^m \to Y$ *is a covering map.*

Proof. First we show that if $a\colon [0,1] \to Y$ is a path of class C^1 in Y and $b\colon [0,1) \to \mathbb{R}^m$ is such that $f(b(s)) = a(s)$, $0 \leq s < 1$, then b is of class C^1 and there exists $\lim\limits_{s\to 1} b(s)$ in $\mathbb{R}^m$. The fact that $b \in C^1$ follows easily from the fact that f is a local diffeomorphism of class C^1. Next, let $y_1 = a(1)$ and consider $V \ni y_1$, $\varepsilon_V > 0$ as in the statement of the proposition. There exists $\delta > 0$ such that $1 - \delta < s < 1 \Rightarrow f(b(s)) = a(s) \in V$ and therefore $|f'(b(s))\cdot b'(s)| \geq \varepsilon_V \cdot |b'(s)|$. On the other hand, $f'(b(s)) \cdot b'(s) = a'(s)$, hence $|b'(s)| \leq |a'(s)|/\varepsilon_V$ when $1 - \delta < s < 1$. Since the interval $[0,1]$ is compact and a if of class C^1, there exists $A > 0$ such that $|a'(s)| \leq A \cdot \varepsilon_V$ for all $s \in [0,1]$. Therefore, if $1 - \delta < s_1$, $s_2 < 1$, we have:

$$|b(s_2) - b(s_1)| = |\int_{s_1}^{s_2} b'(s)ds| \leq |s_2 - s_1| \cdot A.$$

By the Cauchy criterion in the complete metric space $\mathbb{R}^m$, it follows that the limit $\lim\limits_{s\to 1} b(s)$ exists.

Now we prove that every rectilinear path contained in Y, starting at an arbitrary point $y_0 \in f(\mathbb{R}^m)$, can be lifted from any point $x_0 \in f^{-1}(y_0)$. In fact, it this were not true, there would exist a rectilinear path $a(s) = (1-s)y_0 + sy_1$ in Y such that the restriction $a|[0,1)$ would have a lifting $b\colon [0,1) \to \mathbb{R}^m$, with $b(0) = x_0$, and such that the limit $\lim\limits_{s\to 1} a(s)$ would not exist. But this contradicts what we have proved above.

Now we verify that $f(\mathbb{R}^m)$ is a closed subset of the open set Y. In fact, every y_1 that belongs to the closure of $f(\mathbb{R}^m)$, relatively to Y, can be connected to a point $y_0 \in f(\mathbb{R}^m)$ by a rectilinear path contained in Y, which can be lifted to $\mathbb{R}^m$, in such a way that $y_1 \in f(\mathbb{R}^m)$. Since Y is connected and $f(\mathbb{R}^m)$ is obviously open, it follows that $f(\mathbb{R}^m) = Y$.

Therefore, every rectilinear path in Y can be lifted, and the proposition follows from Lemma 6.1 below. □

Lemma 6.1. *Let* $Y \subset \mathbb{R}^m$ *be an open set. In order to verify the path lifting property relative to a local homeomorphism* $f\colon X \to Y$*, it suffices to consider the rectilinear paths in* Y*; that is, the paths* $a\colon I \to Y$ *defined by* $a(s) = (1-s)y_0 + sy_1$*.*

Proof. Suppose initially that Y is convex. If every rectilinear path a in Y has a lifting $\widetilde{a}$ in X, starting at an arbitrary point x in $f^{-1}(a(0))$ then, naturally, $\widetilde{a}$ is unique and depends continuously on a. (See Propositions 6.2

and 6.8.) Therefore, if $a\colon I \to Y$ is any path in Y, let $a_t\colon I \to Y$, $0 \le t \le 1$, the rectilinear path that connects $a(0)$ to $a(t)$; that is, $a_t(s) = (1-s)a(0) + sa(t)$, $0 \le s \le 1$. Given $x \in f^{-1}(a(0))$, let $\widetilde{a}_t$ be the lifting of a_t that starts at the point x. We define a path $\widetilde{a}\colon I \to X$ by setting $\widetilde{a}(t) = \widetilde{a}_t(1)$. Then $\widetilde{a}$ is a lifting of a starting at the point x.

In the general case, Y can be covered by open balls and the above argument shows that every path contained in one of these balls can be lifted to X. Now we observe that any path in Y can be decomposed in a finite sequence of smaller paths, such that each one of them is contained in an open ball and therefore, it can be lifted. It follows that the whole path can be lifted, which proves the lemma. □

Corollary 6.13. *Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ a map of class C^1. If there exists $\alpha > 0$ such that $|f'(x) \cdot v| \ge \alpha |v|$ for all x and every v in $\mathbb{R}^m$, then f is a bijection and therefore, it is a diffeomorphism from $\mathbb{R}^m$ onto itself.*

In fact, take $Y = V = \mathbb{R}^m$ and $\varepsilon_V = \alpha$ in the proposition. Then f is a covering of $\mathbb{R}^m$. Since $\mathbb{R}^m$ is simply connected, it follows from Proposition 6.12 that f is a bijection and therefore it is a diffeomorphism.

Corollary 6.14. *Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a map of class C^1 such that $|f'(x) \cdot v| = |v|$ for all x and every v in $\mathbb{R}^m$. Then f is an isometry; that is, $|f(x) - f(y)| = |x - y|$ for any $x, y \in \mathbb{R}^m$. (As we know from linear algebra, this implies that there exist a linear orthogonal transformation $T\colon \mathbb{R}^m \to \mathbb{R}^m$ and a vector $c \in \mathbb{R}^m$ such that $f(x) = T \cdot x + c$ for every $x \in \mathbb{R}^m$.)*

In fact, by Corollary 6.13, f is a diffeomorphism. The Mean Value theorem applied to f gives us $|f(x) - f(y)| \le |x - y|$ for any $x, y \in \mathbb{R}^m$. The same theorem applied to f^{-1} gives us $|x - y| \le |f(x) - f(y)|$. Hence, f is an isometry.

Proposition 6.15 can be stated in a global scope, by considering Riemannian manifolds instead of open sets in Euclidean space. The proof follows precisely the same argument, substituting the rectilinear paths by geodesics and the convex subsets of $\mathbb{R}^m$ by geodesically convex sets. Corollary 6.13 is valid only for complete, simply connected manifolds (same proof) and Corollary 6.14 is false (see $\xi\colon \mathbb{R} \to S^1$, $\xi(t) = e^{it}$). The statement of the global version of Proposition 6.15 follows:

Proposition 6.16. *Let M^m, N^m be Riemannian manifolds of the same dimension m, with M^m complete and N^m connected. Suppose that there exists a map $f\colon M \to N$, of class C^1, and a covering of N by open sets*

V, and to each of these open sets it is associated a number $\varepsilon_V > 0$ such that $x \in M$, $f(x) \in V \Rightarrow |f'(x)\cdot u| \geq \varepsilon_V \cdot |u|$ for every $u \in T_xM$. Then $f\colon M \to N$ is a covering map.

This proposition, with the same proof, is still valid for Banach manifolds, by omitting the sentence "of the same dimension m" and by requiring that the derivative $f'(x)\colon T_xM \to T_yN$, $y = f(x)$, be an isomorphism, for every $x \in M$.

6.6 Exercises

1. Give the following examples:

   a) A continuous bijection which is not a local homeomorphism;

   b) A continuous surjective map $f\colon \mathbb{R} \to \mathbb{R}$ such that $f^{-1}(y)$ is discrete for every $y \in \mathbb{R}$ but f is not locally injective;

   c) A counter-example to Proposition 6.2 with Z disconnected.

2. If $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to Y$ are coverings, then the map $p \times q\colon \widetilde{X} \times \widetilde{Y} \to X \times Y$, defined by $(p \times q)(x, y) = (p(x), q(y))$, is also a covering.

3. Consider the covering map $p\colon \widetilde{X} \to X$, and let $Y \subset X$ be an arbitrary subset. Set $\widetilde{Y} = p^{-1}(Y)$ and $q = p|\widetilde{Y}$. Show that $q\colon \widetilde{Y} \to Y$ is a covering.

4. Let $p\colon \widetilde{X} \to X$ be a covering where the base X is connected and locally connected. For every connected component $C \subset \widetilde{X}$, we have $p(C) = X$. Conclude that $p|C\colon C \to X$ is a covering.

5. Contrary to the function $f\colon S^1 \to S^1$, $f(z) = z^2$, there does not exist a continuous map $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that $\varphi^{-1}(y)$ has exactly two points, for every $y \in \mathbb{R}$.

6. Given the polynomial $p(z) = 2z^3 - 9z^2 + 12z + 1$, obtain two finite subsets $F_1 \subset \mathbb{C}$ e $F_2 \subset \mathbb{C}$ such that $p\colon \mathbb{C} - F_1 \to \mathbb{C} - F_2$ is a covering with three leaves.

7. Let X be a connected space and G a properly discontinuous group of homeomorphisms of X. Suppose that a continuous map $f\colon X \to X$ has the following property: For every $x \in X$, there exists $g \in G$ such that $f(x) = gx$. Prove that f is a homeomorphism.

8. Let E be an equivalence relation in the Hausdorff space X, such that the quotient map $\varphi\colon X \to X/E$ is open. Prove that X/E is a Hausdorff

space if, and only if, the graph $T = \{(x, y) \in X \times X; xEy\}$ is a closed set in $X \times X$.

9. Let G be any homeomorphism group of the space X. Consider in G the discrete topology and show that the map $\varphi: G \times X \to X$, $\varphi(g, x) = gx$, is a covering.

10. Let $f: X \to Y$ be a local homeomorphism with the unique path lifting property. Suppose that Y is simply connected and locally pathwise connected. Show that, for every $x_0 \in X$ with $f(x_0) = y_0$, there exists a section $\sigma: Y \to X$, such that $\sigma(y_0) = x_0$. Derive from this again Proposition 6.12.

11. Consider the neighborhood $U = \{e^{it}; -\pi < t < \pi\}$ of the neutral element of S^1, and define the local homomorphism $f: U \to S^1$, $f(e^{it}) = e^{it/2}$. Show that f does not extend to a continuous homomorphism $\bar{f}: S^1 \to S^1$.

12. Let $p: X \to X$ be a covering and a, $b: I \to X$ be freely homotopic closed paths. If b, has a closed lifting $\widetilde{b}$, then a also has a closed lifting $\widetilde{a}$, which is freely homotopic to $\widetilde{b}$.

13. Let G be a simply connected and locally pathwise connected topological group. If a connected topological group K is locally isomorphic to G, then K is isomorphic to a quotient of G by a discrete subgroup H (necessarily contained in the center of G).

14. A compact and connected hypersurface $M^n \subset \mathbb{R}^{n+1}$ of class C^∞ whose Gaussian curvature is different from zero at every point is diffeomorphic to the sphere S^n. (The Gaussian curvature is the Jacobian determinant of the normal map $M^n \to S^n$.)

15. Given the covering $p: \widetilde{X} \to X$ and the continuous map $f: Z \to X$, let $\widetilde{Z} = \{(z, \widetilde{x}) \in Z \times \widetilde{X}; f(z) = p(\widetilde{x})\}$. Prove that the map $q: \widetilde{Z} \to Z$, defined by $q(z, \widetilde{x}) = z$ is a covering. Prove also that f admits a continuous lifting $\sigma: Z \to \widetilde{X}$ if, and only if, there exists a continuous section $\sigma: Z \to \widetilde{Z}$ for q.

16. Show that Exercise 3 follows from Exercise 15.

17. Let U be the set of quaternions $w = t + xi + yj + zk$ where $t > 0$ and X the set of real quaternions ≤ 0. By setting $V = \mathbb{R}^4 - X$, prove that the map $f: U \to V$, defined by $f(w) = w^2$, is a surjective proper local diffeomorphism, and conclude that f is a diffeomorphism (global) from U onto V.

18. Let H be a locally pathwise connected, closed subgroup of the connected group G. If G/H is simply connected, prove that H is connected.

(Suggestion: Consider H_0, a connected component of the neutral element. Observe that H_0 is the normal subgroup of H, H/H_0 is a discrete subgroup of G/H_0, and the natural projection from G/H_0 onto its quotient by H/H_0 induces a covering $G/H_0 \to G/H$; hence, $H = H_0$.)

19. Let $p\colon \widetilde{X} \to X$ be a covering with $\widetilde{X}$ connected and $p^{-1}(x)$ finite, for every $x \in X$. If there exists a continuous map $f\colon \widetilde{X} \to \mathbb{R}$, injective in each fiber $p^{-1}(x)$, then p is a homeomorphism.

Chapter 7

Covering Maps and Fundamental Groups

7.1 The Conjugate Class of a Covering Map

Given a covering map $p\colon \widetilde{X} \to X$, take $\widetilde{x} \in \widetilde{X}$ and set $x = p(\widetilde{x})$. We use the notation $H(\widetilde{x})$ to represent the image of the homomorphism $p_{\#}\colon \pi_1(\widetilde{X}, \widetilde{x}) \to \pi_1(X, x)$, induced by the covering projection p.

The subgroup $H(\widetilde{x}) \subset \pi_1(X, x)$ is, as we show in this chapter, the most important algebraic tool to characterize the covering $p\colon \widetilde{X} \to X$.

If $\widetilde{X}$ is simply connected, then $H(\widetilde{x}) = \{0\}$ for all $\widetilde{x} \in \widetilde{X}$. The converse is also true and follows from Proposition 1 below.

First we should recall that, by fixing an element g in a group G, the map $x \mapsto g \cdot x \cdot g^{-1}$ is an automorphism of G, called *conjugation* by g. If H is a subgroup of G, its image by this automorphism is the subgroup $g \cdot H \cdot g^{-1} = \{g \cdot x \cdot g^{-1}; x \in H\}$, isomorphic to H, called a *conjugate subgroup of* H. The conjugate class of H in G is the set of all subgroups $g \cdot H \cdot g^{-1}$, conjugate of H, obtained when we vary g in G. The subgroup H is said to be *normal* when $g \cdot H \cdot g^{-1} = H$ for every $g \in G$; that is, when its conjugate class has only one element, namely, the group H itself. This happens, for example, when G is abelian.

Proposition 7.1. *Let $p\colon \widetilde{X} \to X$ be a covering. For any $x_0 \in X$, $\widetilde{x}_0 \in p^{-1}(x_0)$, the induced homomorphism $p_{\#}\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective. If $\widetilde{X}$ is pathwise connected, then, when $\widetilde{x}$ varies in the fiber $p^{-1}(x_0)$, the image $H(\widetilde{x}) = p_{\#}\pi_1(\widetilde{X}, \widetilde{x})$ describes all conjugate classes of the subgroup $H(\widetilde{x}_0)$.*

Proof. By Corollary 6.8, the induced homomorphism $p_{\#}\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective. Given any other point $\widetilde{x} \in p^{-1}(x_0)$, there exists in $\widetilde{X}$ a path $\widetilde{c}$, with origin $\widetilde{x}$ and final point $\widetilde{x}_0$. Then $c = p \circ \widetilde{c}$ is a closed path in X, with base at the point x_0. By Proposition 2.5, every element $\alpha \in \pi_1(\widetilde{X}, \widetilde{x})$ has the form $\alpha = [\widetilde{c}\widetilde{b}\widetilde{c}^{-1}]$, where $[\widetilde{b}] \in \pi_1(\widetilde{X}, \widetilde{x}_0)$. Hence, $p_{\#}(\alpha) = \gamma p_{\#}([\widetilde{b}])\gamma^{-1}$, where $\gamma = [c]$. From this, $H(\widetilde{x}) = \gamma \cdot H(\widetilde{x}_0) \cdot \gamma^{-1}$.

Conversely, let $H = \gamma \cdot H(\widetilde{x}_0) \cdot \gamma^{-1}$ be any conjugate subgroup of $H(\widetilde{x}_0)$ in $\pi_1(X, x_0)$, and set $\gamma = [c]$. By lifting the closed path c^{-1} from the point $\widetilde{x}_0$, we obtain a path $\widetilde{c}^{-1}$ in $\widetilde{X}$, whose final point we denote by $\widetilde{x}$. Then $\widetilde{x} \in p^{-1}(x_0)$ and the path $\widetilde{c}$, in $\widetilde{X}$, begins at $\widetilde{x}$ and ends at $\widetilde{x}_0$, with $p \circ \widetilde{c} = c$. From what we just saw, this gives us $H(\widetilde{x}) = \gamma \cdot H(\widetilde{x}_0) \cdot \gamma^{-1}$ and therefore $H = H(\widetilde{x})$. □

Proposition 7.2. *Consider a covering $p\colon \widetilde{X} \to X$. Let $a, b\colon I \to X$ be paths that start at the same point x and end at the same point y, and $\widetilde{a}$, $\widetilde{b}\colon I \to \widetilde{X}$ their liftings from a point $\widetilde{x} \in \widetilde{X}$. In order that $\widetilde{a}(1) = \widetilde{b}(1)$, it is necessary and sufficient that $[ab^{-1}] \in H(\widetilde{x})$.*

Proof. Assume that $[ab^{-1}] \in H(\widetilde{x})$. Then the lifting $\widetilde{c}$ of the path ab^{-1} from the point $\widetilde{x}$ is closed. The paths $\widetilde{a}, \widetilde{b}\colon I \to \widetilde{X}$, defined by $\widetilde{a}(s) = \widetilde{c}(s/2)$ and $\widetilde{b}(s) = \widetilde{c}(1 - s/2)$, start at the point $\widetilde{x}$, end at the same point $\widetilde{c}(1/2)$ and are, respectively, liftings of a and b. The converse is obvious. □

The following corollary will be important later on.

Corollary 7.1. *Let $p\colon \widetilde{X} \to X$ be a covering. Given a closed path $a\colon I \to X$, with base at the point x, its lifting $\widetilde{a}\colon I \to \widetilde{X}$, from a point $\widetilde{x} \in p^{-1}(x)$, is closed if, and only if, $[a] \in H(\widetilde{x})$.*

This follows from Proposition 7.2 by taking $b = e_x$.

Corollary 7.2. *Let $p\colon \widetilde{X} \to X$ be a covering, with $\widetilde{X}$ simply connected. A closed path $a\colon I \to X$ is homotopic to a constant if, and only if, some of its lifting $\widetilde{a}\colon I \to \widetilde{X}$ is closed. (And then all of its liftings are closed.)*

More generally: *Still supposing that $\widetilde{X}$ is simply connected, consider the paths a, $b\colon I \to X$ with the same endpoints x_0, x_1. Let $\widetilde{a}$, $\widetilde{b}\colon I \to \widetilde{X}$ be their liftings from the point $\widetilde{x}_0 \in p^{-1}(x_0)$. We have $a \cong b$ if, and only if, $\widetilde{a}$ and $\widetilde{b}$ end at the same point $\widetilde{X}$.*

Corollary 7.3. *Let $p\colon \widetilde{X} \to X$ be a covering, with $\widetilde{X}$ pathwise connected. By fixing a point $x_0 \in X$, the following statements are equivalent:*

1. *For some $\widetilde{x}_0 \in p^{-1}(x_0)$, the subgroup $H(\widetilde{x}_0) \subset \pi_1(X, x_0)$ is normal;*

2. *The subgroups $H(\widetilde{x}) \subset \pi_1(X, x_0)$, when $\widetilde{x}$ varies in $p^{-1}(x_0)$, are normal and they are all the same;*

3. *Given a closed path $a\colon I \to X$, with base at x_0, either all of the liftings of a from the points $\widetilde{x} \in p^{-1}(x_0)$ are closed or none of them is closed.*

In fact, by Proposition 7.2, Condition 3 above is equivalent to stating that $[a] \in H(\widetilde{x}_a) \Leftrightarrow [a] \in H(\widetilde{x})$ for all $\widetilde{x} \in p^{-1}(x_0)$. This means that all of the groups $H(\widetilde{x})$ are equal, when $\widetilde{x}$ varies in the fiber $p^{-1}(x_0)$. But these groups constitute a conjugate class; hence, they are equal if, and only if, one of them is normal, and therefore all of them are normal.

When $\widetilde{X}$ is pathwise connected and one of the conditions in the above corollary is satisfied (and therefore, all of the conditions are satisfied), we say that $p\colon \widetilde{X} \to X$ is a *regular covering.*

Remark. When $\widetilde{X}$ and X are pathwise connected, the reader can easily prove that the regularity of the covering $p\colon \widetilde{X} \to X$ does not depend on the point $x_0 \in X$ fixed above.

When the covering $p\colon \widetilde{X} \to X$ is regular, we may use $H(x_0)$, instead of $H(\widetilde{x}_0)$, in order to identify the image $p_{\#}\pi_1(\widetilde{X}, \widetilde{x}_0)$, $\widetilde{x}_0 \in p^{-1}(x_0)$.

Example 7.1. If the fundamental group of X is abelian, then every covering $p\colon \widetilde{X} \to X$, with $\widetilde{X}$ pathwise connected, is regular. ◁

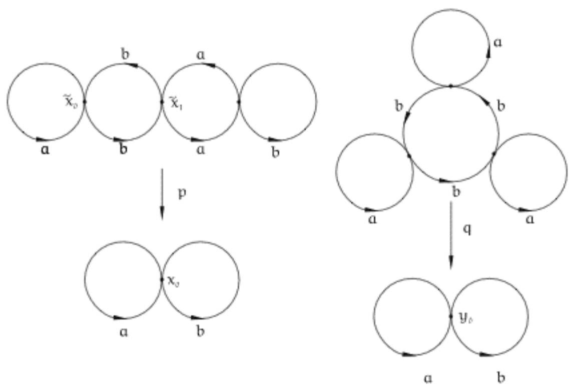


Figure 7.1.

Example 7.2. Let's consider now the two coverings of the figure eight space shown in Figure 7.1. Each one of them is a covering with three leaves. (We will see in what follows that every covering with two leaves is regular; therefore, this example is the simplest possible.) ◁

In each of these examples, the space $\widetilde{X}$ is a union of circles. In the example on the left, the circles on the extremities are applied homeomorphically onto the two circles that form the figure eight space, in the manner indicated by the letters and arrows, while each one of the circles in the middle covers twice the corresponding circle of the base. In the example on the right, the central circle of the space $\widetilde{Y}$ covers three times the circle b of the base, while each one of the three outside circles is mapped homeomorphically onto the other circle of the base. The covering $p\colon \widetilde{X} \to X$ on the left is not regular. In fact, the lifting of the closed path a is open if it starts at the point $\widetilde{x}_1$ and it is closed if its initial point is $\widetilde{x}_0$. On the other hand, the covering $q\colon \widetilde{Y} \to Y$ on the right is regular. In fact, the liftings of a are always closed and the liftings of b are always open. Since $\pi_1(Y, y_0)$ is generated by $[a]$ and $[b]$, Condition 3 of Corollary 7.3 holds.

Example 7.3. Let G be a properly discontinuous group of homeomorphisms of the pathwise connected topological space X. (See Section 6.3.) The quotient map $\pi\colon X \to X/G$ is a regular covering. In fact, choose $x_0, x_1 \in X$ such that $\pi(x_0) = \pi(x_1)$. Then $x_1 = gx_0$, $g \in G$. By the uniqueness of the lifting, the paths $\widetilde{a}$, $\hat{a}\colon I \to X$, with $\widetilde{a}(0) = x_0$, $\hat{a}(0) = x_1$, are liftings of the same path in X/G if, and only if, $\hat{a}(s) = g(\widetilde{a}(s))$ for every $s \in I$. Hence, $\hat{a}$ is closed if, and only if, $\widetilde{a}$ is closed. ◁

Before stating the next proposition, we recall some concepts from algebra.

Let S be an arbitrary set and G a group. A *right action* of the group G on the set S is a map $S \times G \to S$, which maps each pair $(x, g) \in S \times G$ to an element $xg \in S$, in such a way that the following conditions hold:

1. $x(gh) = (xg)h$;

2. $x \cdot e = x$, for any $x \in S$; $e =$ neutral element of G.

In this case, we say that the group G *operates* on *acts* on the right in the set S. A left action is defined in a similar way.

If G acts on the right in the set S, the *orbit* of an element $x \in S$ is the set $xG = \{xg; g \in G\}$. The group G is said to *operate transitively* in S when the orbit of an element of S (and therefore of all elements of S) is the set S itself. This means that, given any two elements x, $y \in S$, there exists $g \in G$ such that $y = xg$.

When G acts on the right in S, given an element $x \in S$, the set $H(x) = \{g \in G; xg = x\}$ is a subgroup of G, called *isotropy group* (or *stabilizer*) of the point x. If $y = xh$ then $yg = y \Leftrightarrow x(hgh^{-1}) = x$; that is, $H(x) = h \cdot H(y) \cdot h^{-1}$. In sum: If two elements x, $y \in S$ belong to the same orbit of G then their isotropy groups are conjugate.

Suppose that G acts transitively on the right in the set S. By fixing a point $x_0 \in S$, the map $\varphi\colon G \to S$, given by $\varphi(g) = x_0 g$ is surjective and satisfies $\varphi(g) = \varphi(h) \Leftrightarrow hg^{-1} \in H(x_0)$, where $H(x_0)$ is the isotropy group of x_0. Therefore, by passing to the quotient, φ induces a bijection

$$\overline{\varphi}\colon G/H(x_0) \to S.$$

In particular, the cardinal number of S is equal to the index $[G\colon H(x_0)]$ of the subgroup $H(x_0)$ in G; that is, the cardinal number of the set $G/H(x_0)$ of the cosets $H(x_0) \cdot g$, $g \in G$.

Proposition 7.3. *Let $p\colon \widetilde{X} \to X$ be a covering, with $\widetilde{X}$ pathwise connected. For each $x \in X$, the fundamental group $\pi_1(X, x)$ acts transitively on the right in the fiber $p^{-1}(x)$. The isotropy group of each point $\widetilde{x} \in p^{-1}(x)$ is $H(\widetilde{x}) = p_\# \pi_1(\widetilde{X}, \widetilde{x})$.*

Proof. Given $\alpha \in \pi_1(X, x)$ and $\widetilde{x} \in p^{-1}(x)$, we define $\widetilde{x}\alpha \in p^{-1}(x)$ as follows: we choose $a \in \alpha$, lift the path a from the initial point $\widetilde{x}$, take the final point $\widetilde{y}$ of this lifting and set $\widetilde{x}\alpha = \widetilde{y}$. It is easy to verify that this procedure defines (without ambiguities) an operation of $\pi_1(X, x)$ on the right in the fiber $p^{-1}(x)$. We have $\widetilde{x}\alpha = \widetilde{x}$ if, and only if, the lifting of the path a, from $\widetilde{x}$, is closed. By Corollary 7.1, this occurs if, and only if, $\alpha \in H(\widetilde{x})$. The transitivity results from the fact that $\widetilde{X}$ is pathwise connected: Given $\widetilde{x}$, $\widetilde{y} \in p^{-1}(x)$, let $\widetilde{a}$ be a path in $\widetilde{X}$ starting at $\widetilde{x}$ and ending at $\widetilde{y}$. Then $a = p \circ \widetilde{a}$ is a closed path in X with base at the point x. Let $\alpha = [a]$. It is obvious that $\widetilde{y} = \widetilde{x}\alpha$. □

Corollary 7.4. *If $\widetilde{X}$ is pathwise connected then, for any $\widetilde{x} \in \widetilde{X}$, and $x = p(\widetilde{x})$, the number of leaves of p is equal to the index of the subgroup $H(\widetilde{x}) \subset \pi_1(X, x)$.*

Corollary 7.5. *If $\widetilde{X}$ is pathwise connected, every covering $p\colon \widetilde{X} \to X$ with two leaves is regular.*

In fact, every subgroup of index two is normal.

Corollary 7.6. *Let $\widetilde{X}$ be pathwise connected. The covering projection $p\colon \widetilde{X} \to X$ is a homeomorphism if, and only if, the induced homeomorphism $p_{\#}$ is an isomorphism.*

In fact, this is the condition we should have in order that the number of leaves be one.

Corollary 7.7. *If $\widetilde{X}$ is simply connected, then the number of leaves of the covering is equal to the number of elements of $\pi_1(X, x)$. When these two numbers are finite, the equality between them implies that $\widetilde{X}$ is simply connected.*

Remark. The permutations of the fiber $p^{-1}(x)$ of the form $\widetilde{x} \mapsto \widetilde{x}{\cdot}\alpha$, where $\alpha \in \pi_1(X, x)$, form a group $M(x)$, called the *monodromy group* of the covering $p\colon \widetilde{X} \to X$ at the point x. For all $x \in X$, $M(x)$ is a homomorphic image of $\pi_1(X, x)$. More precisely, we have

$$M(x) \approx \pi_1(X, x)/H_0,$$

where

$$H_0 = \bigcap_{p(\widetilde{x})=x} H(\widetilde{x}).$$

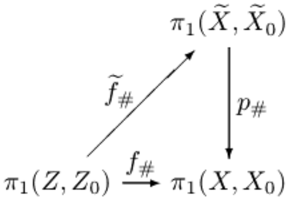
If the covering is regular, we have $H_0 = H(\widetilde{x})$ for all $\widetilde{x} \in p^{-1}(x)$.

7.2 The Fundamental Lifting Theorem

In this section, we show how the fundamental group allows us to give an algebraic answer to the topological problem of knowing whether a continuous map $f\colon Z \to X$, taking values at the base of a covering, admits a lifting $\widetilde{f}\colon Z \to \widetilde{X}$. It is convenient here to use pairs (X, x_0); that is, spaces with a base point.

Proposition 7.4. *Let $p\colon \widetilde{X} \to X$ be a covering, of the pathwise connected space X. Let Z be a connected and locally pathwise connected space (hence pathwise connected) and $f\colon (Z, z_0) \to (X, x_0)$ a continuous map. Given $\widetilde{x}_0 \in p^{-1}(x_0)$, in order that f have a lifting $\widetilde{f}\colon (Z, z_0) \to (\widetilde{X}, \widetilde{x}_0)$, it is necessary and sufficient that $f_{\#}\pi_1(Z, z_0) \subset H(\widetilde{x}_0)$.*

Proof. If there exists $\widetilde{f}\colon (Z, z_0) \to (\widetilde{X}, x_0)$ continuous such that $p \circ \widetilde{f} = f$, then, considering the homomorphisms induced by f, p and $\widetilde{f}$, we see that the diagram below is commutative:



From $f_\# = p_\# \circ \widetilde{f}_\#$, it follows that the image of $f_\#$ is contained in the image of $p_\#$, which is $H(\widetilde{x}_0)$. Hence the inclusion is necessary for the existence of $\widetilde{f}$.

Conversely, suppose that $f_\# \pi_1(Z, z_0) \subset H(\widetilde{x}_0)$. We define $\widetilde{f}\colon Z \to \widetilde{X}$, by setting $\widetilde{f}(z_0) = \widetilde{x}_0$ and, for an arbitrary $z \in Z$, we take a path $a\colon I \to Z$, from z_0 to z, we denote by $\widetilde{a}\colon I \to \widetilde{X}$ the lifting of $f \circ a\colon I \to X$ from the point $\widetilde{x}_0$ and we set $\widetilde{f}(z) = \widetilde{a}(1)$. Now let us show that $\widetilde{f}$ is well defined. In fact, if $b\colon I \to Z$ is another path from z_0 to z, then ba^{-1} is a closed path with base z_0. From this, $(f \circ b)(f \circ a)^{-1} = f \circ (ab^{-1})$ is a closed path, with base x_0, whose homotopy class belongs to the image of $f_\#$ and therefore (because of the hypothesis) to $H(\widetilde{x}_0)$. From this it results that the paths $\widetilde{a}$ e $\widetilde{b}$, liftings of $f \circ a$ and $f \circ b$, respectively, from $\widetilde{x}_0$, end at the same point. (See Proposition 7.2.) Evidently, we have $p \circ \widetilde{f} = f$. It remains to prove only that $\widetilde{f}$ is continuous at an arbitrary point $z \in Z$. Here we use the fact that Z is locally pathwise connected. Let V be a neighborhood of $\widetilde{f}(z)$ in $\widetilde{X}$. We may suppose that $p|V$ is a homeomorphism onto a neighborhood U of $f(z)$ in X. Let W be a pathwise connected neighborhood of z in Z such that $f(W) \subset U$. We claim that $\widetilde{f}(W) \subset V$. This will prove the continuity of $\widetilde{f}$ at the point z. We know that $\widetilde{f}(z)$ is the final point of a path $\widetilde{a}$ in $\widetilde{X}$ that starts at $\widetilde{x}_0$, with $p \circ \widetilde{a} = f \circ a$, where a is a path in Z, starting at z_0 and ending at z. Given $w \in W$, we take a path b in W, starting at z and ending at w. Since $p|V$ is a homeomorphism onto U, there exists a path $\widetilde{b}$ in V, starting at $\widetilde{f}(z)$ and ending at a certain point $v \in V$, with $p \circ \widetilde{b} = f \circ b$. Then $\widetilde{a}\widetilde{b}$ is a path in $\widetilde{X}$, that starts at $\widetilde{x}_0$, such that $p \circ (\widetilde{a}\widetilde{b}) = (p \circ \widetilde{a})(p \circ \widetilde{b}) = (f \circ a)(f \circ b) = f \circ (ab)$. Since ab connects z_0 to w in Z, it follows from the definition of $\widetilde{f}$ that $\widetilde{f}(w) = (\widetilde{a}\widetilde{b})(1) = v$; therefore, $\widetilde{f}(w) \in V$. □

Second Proof of Proposition 7.4. The experienced topologist, faced with an argument where, in order to define a map, he has to make arbitrary choices that turn out to be irrelevant, always suspects that such map might somehow be obtained by passing to the quotient. In the present case, the suspicion is true, as we show now.

Let $C(Z;z_0) \subset C(I;Z)$ be the subspace (in the compact-open topology) whose elements are the paths with origin at the point z_0. A similar interpretation is given to the notations $C(X;x_0)$ and $C(\widetilde{X};\widetilde{x}_0)$. The following three maps are continuous: $f_*: C(Z;z_0) \to C(X;x_0)$, $L: C(X;x_0) \to C(\widetilde{X};\widetilde{x}_0)$, and $v: C(\widetilde{X};\widetilde{x}_0) \to \widetilde{X}$, given by $f_*(a) = f \circ a$, $L(c) = \widetilde{c} =$ lifting of c from $\widetilde{x}_0$, and $v(\widetilde{a}) = \widetilde{a}(1)$. Therefore the composite map $\hat{f} = v \circ L \circ f_*: C(Z;z_0) \to \widetilde{X}$ is also continuous. On the other hand, the surjection $u: C(Z;z_0) \to Z$, defined by $u(a) = a(1)$, is continuous and, moreover, it is open (which follows from the fact that Z is pathwise locally connected). Hence u is a quotient map. Now we observe that the hypothesis on the image of $f_\#$ gives us: $u(a) = u(a') \Rightarrow \hat{f}(a) = \hat{f}(a')$. Thus, $\hat{f}$ is compatible with the equivalence relation defined by u. By passing to the quotient, there exists therefore a unique continuous map $\widetilde{f}: Z \to \widetilde{X}$ such that $\widetilde{f} \circ u = \hat{f}$. The map $\widetilde{f}$ is the lifting of f we have been looking for. □

Corollary 7.8. *Let X be pathwise connected and Z be simply connected and locally pathwise connected. Every continuous map $f: (Z, z_0) \to (X, x_0)$ admits a lifting $\widetilde{f}: (Z, z_0) \to (\widetilde{X}, \widetilde{x}_0)$, where $\widetilde{x}_0 \in p^{-1}(x_0)$ is chosen arbitrarily.*

The above corollary explains why it is always possible to lift a path: I is simply connected.

As an application, we use Proposition 7.4 to establish the conditions under which a continuous complex function has a continuous logarithm.

Example 7.4. **(Logarithm of a function)** Let $U \subset \mathbb{C}$ be an open connected set and $f: U \to \mathbb{C} - \{0\}$ a continuous map. In order that there exists $g: U \to \mathbb{C}$ continuous such that $f(z) = e^{g(z)}$ for all $z \in U$, it is necessary and sufficient that, for every closed path $a: I \to U$, the number or turns of the closed path $f \circ a: I \to \mathbb{C} - \{0\}$ around the point 0 be equal to zero. When f is holomorphic, the function g is necessarily holomorphic. In fact, the condition on the number of turns of the path $f \circ a$ means that the induced homomorphism $f_\#: \pi_1(U, u_0) \to \pi_1(\mathbb{C} - \{0\})$ is null. Considering the covering map $p: \mathbb{C} \to \mathbb{C} - \{0\}$, given by $p(z) = e^z$, we see that the induced homomorphism $p_\#: \pi_1(\mathbb{C}) \to \pi_1(\mathbb{C} - \{0\})$ is null because $\pi_1(\mathbb{C}) = \{0\}$. Thus, f has a lifting relatively to p if, and only if, $f_\# = 0$. Now, the fact that g is a lifting of f relatively to p means that $f(z) = e^{g(z)}$ for all $z \in U$. Since g is continuous, and p, f are holomorphic, with $p'(z) \neq 0$, it follows from the Inverse Function Theorem that $f(z) = p(g(z))$

for every z implies that g is holomorphic. Note, in particular, that when U is simply connected, $f_\#$ is always null; hence, every continuous function (respectively holomorphic) that is non-null in a simply connected domain always admits a continuous logarithm. (It is usual to call every continuous function g such that $f(z) = e^{g(z)}$ a "branch of $\log f(z)$".) ◁

In an analogous way, we study the existence of the k-th root of a map in the example below.

Example 7.5. Considering the covering map $p\colon \mathbb{C}-\{0\} \to \mathbb{C}-\{0\}$, $p(z) = z^k$, $k \in \mathbb{N}$, (Example 6.6), we show that, *given a continuous function* $f\colon U \to \mathbb{C} - \{0\}$, *defined in an open, connected set* $U \subset \mathbb{C}$, *there exists* $g\colon U \to \mathbb{C} - \{0\}$ *continuous (called a "branch of* $\sqrt[k]{f(z)}$*") such that* $f(z) = g(z)^k$ *for every* $z \in U$ *if, and only if, every closed path* $a\colon I \to U$ *is mapped by* f *in a path* $f \circ a\colon I \to \mathbb{C} - \{0\}$, *whose number of turns around the origin* 0 *is a multiple of* k. (Again, if f is holomorphic, g is also holomorphic.) We just have to observe that, for the covering $p(z) = z^k$, the image of the homomorphism $p_\#$ is the subgroup of $\pi_1(\mathbb{C} - \{0\}) = \mathbb{Z}$ formed by the multiples of k. The condition on the number of turns of the path $f \circ a$ means that the image of the homomorphism $f_\#\colon \pi_1(U, u_0) \to \pi_1(\mathbb{C} - \{0\})$ is contained in the image of $p_\#$. Hence, such a condition is necessary and sufficient in order that f have a lifting relative to p. Now, g is such a lifting if, and only if, $f(z) = g(z)^k$ for all $k \in U$. Again, we remark that, in particular, if U is simply connected, for every continuous function $f\colon U \to \mathbb{C} - \{0\}$, there always exists a branch of $\sqrt[k]{f(z)}$ defined on U. ◁

The following proposition, which is also a direct application of Proposition 7.4, expresses, grosso modo, that every covering of a topological group is still a topological group.

Proposition 7.5. *Let* G *be a locally pathwise connected topological group, with neutral element* e. *Given a covering* $p\colon \widetilde{G} \to G$, *with* $\widetilde{G}$ *connected, and a point* $\widetilde{e} \in p^{-1}(e)$, *there exists a unique topological group structure in* $\widetilde{G}$, *such that* $\widetilde{e}$ *is the neutral element and* p *is a homomorphism.*

Proof. Let $p{\cdot}p\colon (\widetilde{G} \times \widetilde{G}, (\widetilde{e}, \widetilde{e})) \to (G, e)$ be the continuous map defined by $(p{\cdot}p)(\widetilde{x}, \widetilde{y}) = p(\widetilde{x}) \cdot p(\widetilde{y})$. The essential point consists in proving that $p{\cdot}p$ has a lifting $m\colon (\widetilde{G} \times \widetilde{G}, (\widetilde{e}, \widetilde{e})) \to (\widetilde{G}, \widetilde{e})$. The image of the induced homomorphism $(p \cdot p)_\#$ is the set of homotopy classes of all paths in G of the form $a{\cdot}b$, where $[a]$ and $[b]$ belong to the image of $p_\#$. Thus, the lifting $m\colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}$, with $m(\widetilde{e}, \widetilde{e}) = \widetilde{e}$ and $p(m(\widetilde{x}, \widetilde{y})) = p(\widetilde{x}){\cdot}p(\widetilde{y})$, exists. For the sake of simplicity, we write $m(\widetilde{x}, \widetilde{y}) = \widetilde{x}{\cdot}\widetilde{y}$. Therefore, we have a continuous multiplication in $\widetilde{G}$, which turns p into an homomorphism. It remains to verify that it turns $\widetilde{G}$ into a group, in which $\widetilde{e}$ is the neutral element. First, the continuous maps $\widetilde{x} \mapsto \widetilde{x} \cdot \widetilde{e}$ and $\widetilde{x} \mapsto \widetilde{x}$, of $\widetilde{G}$ into $\widetilde{G}$, are liftings of the same map p, which have the same value at the point $\widetilde{e}$. Hence, they coincide; that is, we have $\widetilde{x} \cdot \widetilde{e} = \widetilde{x}$ for all $\widetilde{x} \in \widetilde{G}$. In an analogous way, we see that $\widetilde{e} \cdot \widetilde{x} = \widetilde{x}$; therefore, $\widetilde{e}$ is neutral element for the multiplication of $\widetilde{G}$. The associativity is proved by observing that the maps $(\widetilde{x}, \widetilde{y}, \widetilde{z}) \mapsto (\widetilde{x} \cdot \widetilde{y}) \cdot \widetilde{z}$ and $(\widetilde{x}, \widetilde{y}, \widetilde{z}) \mapsto \widetilde{x} \cdot (\widetilde{y} \cdot \widetilde{z})$ are liftings of the map $(\widetilde{x}, \widetilde{y}, \widetilde{z}) \mapsto p(\widetilde{x}) \cdot p(\widetilde{y}) \cdot p(\widetilde{z})$, which coincide at the point $(\widetilde{e}, \widetilde{e}, \widetilde{e})$. By virtue of Proposition 2.12, the map $\widetilde{x} \mapsto p(\widetilde{x})^{-1}$ has a lifting $i \colon \widetilde{G} \to \widetilde{G}$, such that $i(\widetilde{e}) = \widetilde{e}$ and $p(i(\widetilde{x})) = p(\widetilde{x})^{-1}$, so $p(i(\widetilde{x}) \cdot \widetilde{x}) = e$, and from this $i(\widetilde{x}) \cdot \widetilde{x} \in p^{-1}(e)$ for all $\widetilde{x} \in \widetilde{G}$. Since $\widetilde{G}$ is connected and the fiber $p^{-1}(e)$ is discrete, the product $i(\widetilde{x}) \cdot \widetilde{x}$ is constant when $\widetilde{x}$ varies in $\widetilde{G}$. Now we observe that $i(\widetilde{e}) \cdot \widetilde{e} = \widetilde{e}$. Hence, $i(\widetilde{x}) \cdot \widetilde{x} = \widetilde{e}$, which gives us $i(\widetilde{x}) = \widetilde{x}^{-1}$. □

Let $p \colon \widetilde{G} \to G$ be a *homomorphic covering*; that is, a covering map that is also a homomorphism between the topological groups $\widetilde{G}$ and G.

When $\widetilde{G}$ (and therefore, G) is pathwise connected, the covering $p \colon \widetilde{G} \to G$ is regular, because $\pi_1(G)$ is abelian. Moreover, the following proposition holds.

Proposition 7.6. *Let $K = p^{-1}(e)$ be the kernel of the homomorphic covering $p \colon \widetilde{G} \to G$, where $\widetilde{G}$ is pathwise connected. There exists a natural isomorphism $\pi_1(G)/\pi_1(\widetilde{G}) \approx K$.*

Proof. Above, we are identifying $\pi_1(\widetilde{G})$ with its image $H(\widetilde{e})$ using the induced homomorphism $p_{\#}$. In order to obtain the isomorphism, we define $\varphi \colon \pi_1(G) \to K$ by setting $\varphi(\alpha) = \widetilde{a}(1)$, where $\widetilde{a}$ is the lifting, from $\widetilde{e}$, of a path $a \in \alpha$. (In the notation of Proposition 7.3, $\varphi(\alpha) = \widetilde{e} \cdot \alpha$.) We claim that φ is a homomorphism. In order to verify this, we consider operation of the group $\pi_1(G)$ as being $\alpha \cdot \beta$. (See Proposition 2.12.) If $\widetilde{a}$ and $\widetilde{b}$ are liftings, from $\widetilde{e}$, of the paths $a \in \alpha$ and $b \in \beta$ respectively, the fact that p is a homomorphism gives us $\widetilde{a \cdot b} = \widetilde{a} \cdot \widetilde{b}$. Hence $\varphi(\alpha \cdot \beta) = (\widetilde{a} \cdot \widetilde{b})(1) = \widetilde{a}(1) \cdot \widetilde{b}(1) = \varphi(\alpha) \cdot \varphi(\beta)$. The homomorphism φ is surjective, because $\pi_1(G)$ acts transitively in the fiber $p^{-1}(e) = K$. The kernel of φ is the set of elements $\alpha = [a] \in \pi_1(G)$ such that $\widetilde{a}$ is closed. By Proposition 7.2, this kernel is $\pi_1(\widetilde{G})$. By passing to the quotient, we obtain an isomorphism $\overline{\varphi} \colon \pi_1(G)/\pi_1(\widetilde{G}) \to K$. □

Corollary 7.9. *Let $p \colon \widetilde{G} \to G$ be a homomorphic covering. If $\widetilde{G}$ is simply connected, then $\pi_1(G)$ is isomorphic to the kernel $K = p^{-1}(e)$ of the homomorphism p.*

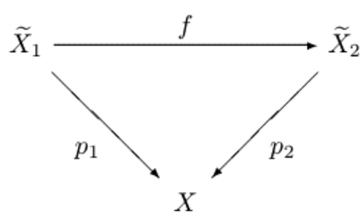
Example 7.6. The covering map $p \colon \mathbb{R} \to S^1$, given by $p(x) = e^{2\pi i x}$, is a homomorphism of the (additive) group $\mathbb{R}$ over the (multiplicative) group

S^1, with kernel $\mathbb{Z}$. Since $\mathbb{R}$ is simply connected, we reobtain the fact that $\pi_1(S^1) \approx \mathbb{Z}$. Analogously, we may consider the homomorphic covering $p\colon \mathbb{R}^n \to T^n = S^1 \times \cdots \times S^1$, $p(x_1, \ldots, x_n) = (e^{2\pi i x_1}, \ldots, e^{2\pi i x_n})$, whose kernel is $\mathbb{Z}^n$, and find that the fundamental group of the n-dimensional torus T^n is isomorphic to $\mathbb{Z}^n$. Finally, the homomorphism $\varphi\colon S^3 \to \mathrm{SO}(3)$, defined in Section 1 of Chapter 4, has kernel $\{-1, 1\} \approx \mathbb{Z}_2$ and it is surjective. Therefore, it is a homomorphic covering. Since S^3 is simply connected, it follows that $\pi_1(\mathrm{SO}(3)) \approx \mathbb{Z}_2$. ◁

Example 7.7. We can compute again $\pi_1(\mathrm{SO}(4))$ by taking a homomorphic covering whose base is $\mathrm{SO}(4)$ and whose domain is simply connected. For this, we define $p\colon S^3 \times S^3 \to \mathrm{SO}(4)$ by associating to each pair $(x, y) \in S^3 \times S^3$ the linear transformation $p_{x,y}\colon \mathbb{R}^4 \to \mathbb{R}^4$, given by $p_{x,y}(w) = xwy^{-1}$ (quaternion multiplication). Since $|x| = |y| = 1$, the linear transformation $p_{x,y}$ preserves norms; since x and y may be connected to 1 by paths in S^3, $p_{x,y}$ can be connected to the identity transformation by a path formed by linear transformations that preserve norms. Hence $p_{x,y} \in \mathrm{SO}(4)$. It is obvious that $p\colon S^3 \times S^3 \to \mathrm{SO}(4)$, thus defined, is an infinitely differentiable homomorphism. If (x, y) belongs to the kernel of p, then $x \cdot w \cdot y^{-1} = w$ for all $w \in \mathbb{R}^4$. In particular, $x \cdot 1 \cdot y^{-1} = 1$, so $x = y$. It follows that $x \cdot w \cdot x^{-1} = w$ for all $w \in \mathbb{R}^4$. As in Section 4.1, we conclude that $x = \pm 1$. Thus, the kernel of the homomorphism $p\colon S^3 \times S^3 \to \mathrm{SO}(4)$ consists of two elements $(1, 1)$ and $(-1, -1)$. Finally, since $\dim(S^3 \times S^3) = \dim \mathrm{SO}(4) = 6$ and the kernel of p is discrete, the rank theorem, from analysis in Euclidean space, tells us that p has rank 6 and therefore, it is an open map. Since $S^3 \times S^3$ is compact and $\mathrm{SO}(4)$ is Hausdorff connected, this implies that p is surjective. The surjective homomorphism p is a covering because it has a discrete kernel. Its domain, $S^3 \times S^3$, is simply connected. Hence, $\pi_1(\mathrm{SO}(4))$ has two elements and therefore, it is isomorphic to $\mathbb{Z}_2$. ◁

7.3 Homomorphisms of Covering Spaces

Let $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$ be two coverings with the same base space X. A *homomorphism* between them is a continuous map $f\colon \widetilde{X}_1 \to \widetilde{X}_2$ such that $p_2 \circ f = p_1$. This means that the following diagram is commutative.



If $p_3\colon \widetilde{X}_3 \to X$ is another covering with base space X and $g\colon \widetilde{X}_2 \to \widetilde{X}_3$ is a homomorphism, the composite map $g \circ f\colon \widetilde{X}_1 \to \widetilde{X}_3$ is still a homomorphism. We say that $f\colon \widetilde{X}_1 \to \widetilde{X}_2$ is an *isomorphism* when f is a homeomorphism such that $p_2 \circ f = p_1$. Then $f^{-1}\colon \widetilde{X}_2 \to \widetilde{X}_1$ is also an isomorphism. In this case, the coverings p_1 and p_2 are said to be *isomorphic*.

An *endomorphism* is an homomorphism of a covering into itself. Given a covering $p\colon \widetilde{X} \to X$, an endomorphism is, therefore, a continuous map $f\colon \widetilde{X} \to \widetilde{X}$ such that $p \circ f = p$.

When the endomorphism f is a homeomorphism of $\widetilde{X}$ onto itself, we say that f is an *automorphism*. The set $G(\widetilde{X}|X)$ of the covering automorphisms $p\colon \widetilde{X} \to X$ is a group under the operation of map composition. Sometimes automorphisms are called *covering transformations* or *covering translations*.

The condition $p_2 \circ f = p_1$ means that f maps each fiber $p_1^{-1}(x)$ into the fiber $p_2^{-1}(x)$. In particular, an endomorphism $f\colon \widetilde{X} \to \widetilde{X}$ maps each fiber $p^{-1}(x)$ into itself. An isomorphism f induces, for each $x \in X$, a bijection of the fiber $p_1^{-1}(x)$ onto the fiber $p_2^{-1}(x)$. An automorphism, therefore, determines a permutation of each fiber $p^{-1}(x)$.

Note that a homomorphism $f\colon \widetilde{X}_1 \to \widetilde{X}_2$ is a lifting of the continuous map $p_1\colon \widetilde{X}_1 \to X$ with respect to the covering $p_2\colon \widetilde{X}_2 \to X$. Thus, when $\widetilde{X}_1$ is connected, two homomorphisms that coincide at a point $\widetilde{x}_1 \in \widetilde{X}_1$ are equal.

Example 7.8. Consider the covering maps $p_1 : \mathbb{R}^2 \to T^2$, from the plane onto the torus and $p_2 : S^1 \times \mathbb{R} \to T^2$, from the cylinder onto the torus, given by $p_1(s,t) = (e^{2\pi is}, e^{2\pi it})$ and $p_2(z,t) = (z, e^{2\pi it})$. The map $f\colon \mathbb{R}^2 \to S^1 \times \mathbb{R}$, from the plane onto the cylinder, given by $f(s,t) = (e^{2\pi is}, t)$, satisfies the condition $p_2 \circ f = p_1$; hence, it is a covering homomorphism. Note that the subgroup $H_1 = \{0\} \subset \pi_1(T^2)$ is associated to the covering p_1, and the subgroup $H_2 = \mathbb{Z} \oplus \{0\}$ of $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ is associated to the covering p_2. We have $H_1 \subset H_2$. It is this inclusion that makes it possible the existence of f. ◁

Example 7.9. Let G be a properly discontinuous group of homeomorphisms of the connected topological space X. We claim that the group of covering automorphisms of the covering $p\colon X \to X/G$ is precisely the group G. In fact, if $g \in G$ then, for all $x \in X$, we have $p(gx) = G\cdot gx = G\cdot x = p(x)$, hence $p \circ g = p$ and from this $g \in G(X|X/G)$. Conversely, given an automorphism $f\colon X \to X$, we fix $x_0 \in X$ and take $x_1 = f(x_0)$. Then x_1 belongs to the same fiber that x_0, hence there exists $g \in G$ with $gx_0 = x_1$. Therefore f and g are liftings of p that coincide at the point x_0. Since X is connected, we have $f = g$, so $f \in G$. Thus, for example, the automorphisms of the covering $p\colon \mathbb{R}^2 \to T^2$, of the torus by the plane, $p(s,t) = (e^{2\pi is}, e^{2\pi it})$, are translations $(s,t) \mapsto (s+m, t+n)$ where $m, n \in \mathbb{Z}$. Another example, that generalizes this one of the torus, is the following: let $p\colon \widetilde{G} \to G$ a homomorphic covering of the connected topological group $\widetilde{G}$ onto G. The kernel $K = p^{-1}(e)$, being a normal and discrete subgroup of the connected group $\widetilde{G}$, is central, that is, their elements commute with every other element in G. The automorphisms of the covering $p\colon \widetilde{G} \to G$ are translations $f_k\colon \widetilde{G} \to \widetilde{G}$, $f_k(x) = k \cdot x = x \cdot k$, $k \in K$. In fact, the set of the translations f_k, $k \in K$, is a properly discontinuous group of homeomorphisms of $\widetilde{G}$, isomorphic to K, and the quotient space $\widetilde{G}/K$ is homeomorphic to G. $\triangleleft$

Proposition 7.7. *Let $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$ be coverings with the same base space X. If $\widetilde{X}_2$ is connected and locally pathwise connected, every homomorphism $f\colon \widetilde{X}_1 \to \widetilde{X}_2$ is a covering. In particular, f is surjective.*

Proof. Take $\widetilde{x}_1 \in \widetilde{X}_1$. Let $\widetilde{x}_2 = f(\widetilde{x}_1)$. If $a\colon I \to \widetilde{X}_2$ is any path starting at $\widetilde{x}_2$, we set $a_0 = p_2 \circ a$ and consider $\widetilde{a}\colon I \to \widetilde{X}_1$, the lifting of a_0 with respect to the covering p_1, starting at the point $\widetilde{x}_1$. Then $f \circ \widetilde{a}\colon I \to \widetilde{X}_2$ is a lifting of a_0 with respect to p_2, starting at the point $\widetilde{x}_2$. If follows that $f \circ \widetilde{a} = a$. In particular, $f(\widetilde{a}(1)) = a(1)$. Since $\widetilde{X}_2$ is pathwise connected, any one of its points is of the form $a(1)$, for some path a starting at $\widetilde{x}_2$. Hence f is surjective. The same argument also shows that f has the unique path lifting property. Since the relation $p_2 \circ f = p_1$ implies that the continuous map f is a local homeomorphism, the proposition is already proved in the case where one of the spaces $\widetilde{X}_1$, $\widetilde{X}_2$, X is semi-locally simply connected (and the same happens with the other two).

In the general case, let $\widetilde{x}_2 \in \widetilde{X}_2$ be an arbitrary point. Take a connected neighborhood U of the point $x_0 = p_2(\widetilde{x}_2)$, which is distinguished with respect to the coverings p_2 and p_1. (Observe that the spaces $\widetilde{X}_1$, $\widetilde{X}_2$, and X are locally homeomorphic; hence, they are locally connected.) Let V be the connected component of $p_2^{-1}(U)$ that contains the point $\widetilde{x}_2$. We claim that V is a distinguished neighborhood of $\widetilde{x}_2$, relative to f. We have

$$p_1^{-1}(U) = \bigcup_{\lambda} \widetilde{U}_\lambda,$$

a union of disjoint open sets where, for each λ, $p_1|\widetilde{U}_\lambda$ is a homeomorphism onto U. If, for some λ, we have $f(\widetilde{U}_\lambda) \cap V \neq \varnothing$, then, since the connected set $f(\widetilde{U}_\lambda)$ is contained in the set $p_2^{-1}(U)$, of which V is a connected component, it follows that $f(\widetilde{U}_\lambda) \subset V$ and from this, $f|\widetilde{U}_\lambda = (p_2|V)^{-1} \circ (p_1|\widetilde{U}_\lambda)$; hence,

f is a homeomorphism from $\widetilde{U}_\lambda$ onto V. Therefore, by setting

$$L_0 = \{\lambda; f(\widetilde{U}_\lambda) \cap V \neq \varnothing\},$$

we see that $L_0 \neq \varnothing$ (because f is surjective), that

$$f^{-1}(V) = \bigcup_{\lambda \in L_0} \widetilde{U}_\lambda$$

and that $f|\widetilde{U}_\lambda$ is a homeomorphism onto V, for all $\lambda \in L_0$, thus concluding the proof. □

Corollary 7.10. *Let $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ be coverings over the same base space X, with $\widetilde{X}_1$ and $\widetilde{X}_2$ connected and locally pathwise connected. A homomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$ is an isomorphism if, and only if, $f_\# : \pi_1(\widetilde{X}_1, \widetilde{x}_1) \to \pi_1(\widetilde{X}_2, f(\widetilde{x}_1))$ is surjective (and therefore an isomorphism between the fundamental groups).*

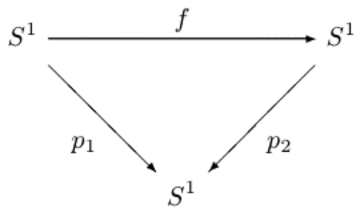
In fact, this results from Corollary 7.6, by taking into account Proposition 7.7.

In the following proposition, we have the coverings $p_1 : \widetilde{X}_1 \to X, p_2 : \widetilde{X}_2 \to X$. Given the points $\widetilde{x}_1 \in \widetilde{X}_1$ and $\widetilde{x}_2 \in \widetilde{X}_2$ with $p_1(\widetilde{x}_1) = p_2(\widetilde{x}_2) = x_0$, we denote by $H_1(\widetilde{x}_1)$ and $H_2(\widetilde{x}_2)$, respectively, the subgroups of $\pi_1(X, x_0)$ which are images of the induced homomorphisms $(p_1)_\# : \pi_1(\widetilde{X}_1, \widetilde{x}_1) \to \pi_1(X, x_0)$ and $(p_2)_\# : \pi_1(\widetilde{X}_2, \widetilde{x}_2) \to \pi_1(X, x_0)$.

Proposition 7.8. *Let $\widetilde{X}_1$, $\widetilde{X}_2$ be connected and locally pathwise connected. In order that there exists a homomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$ with $f(\widetilde{x}_1) = \widetilde{x}_2$ it is necessary and sufficient that $H_1(\widetilde{x}_1) \subset H_2(\widetilde{x}_2)$.*

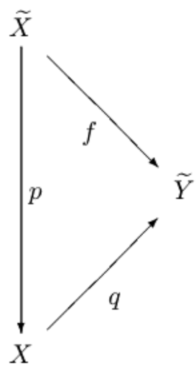
Proof. This follows from Proposition 7.4, because a homomorphism f is a lifting of p_1 relatively to the covering p_2. □

Example 7.10. Consider the coverings $p_1 : S^1 \to S^1$ and $p_2 : S^1 \to S^1$, given by $p_1(z) = z^{12}$ and $p_2(z) = z^3$. The corresponding subgroups are $H_1 = 12\mathbb{Z}$ and $H_2 = 3\mathbb{Z}$. (Since the fundamental groups are abelian, we write H instead of $H(\widetilde{x})$.) Evidently, we have $H_1 \subset H_2$. Hence, by taking any z_1, $z_2 \in S^1$ with $p_1(z_1) = p_2(z_2)$, there exists a homomorphism $f : S^1 \to S^1$ such that $f(z_1) = z_2$. We choose $z_1 = z_2 = 1$. Then $f(z) = z^4$ is the homomorphism between the given coverings. This is illustrated by the commutative diagram below.



◁

Corollary 7.11. *Let $p\colon \widetilde{X} \to X$ be a covering whose domain $\widetilde{X}$ is simply connected and locally pathwise connected. For every covering $q\colon \widetilde{Y} \to X$ with $\widetilde{Y}$ connected, there exists a covering $f\colon \widetilde{X} \to \widetilde{Y}$ such that $q \circ f = p$. This is illustrated by the commutative diagram below.*



In fact, for any $\widetilde{x} \in \widetilde{X}$ and $\widetilde{y} \in \widetilde{Y}$ with $p(\widetilde{x}) = q(\widetilde{y})$ we have $\{0\} = p_{\#}\pi_1(\widetilde{X}, \widetilde{x}) \subset q_{\#}\pi_1(\widetilde{Y}, \widetilde{y})$.

Because of the above corollary, a covering $p\colon \widetilde{X} \to X$ with $\widetilde{X}$ simply connected and locally pathwise connected is called a *universal covering*, since $\widetilde{X}$ covers any other covering $\widetilde{Y}$ of the space X.

Corollary 7.12. *Under the hyphotesis of Proposition 7.8, the homomorphism $f\colon \widetilde{X}_1 \to \widetilde{X}_2$, with $f(\widetilde{x}_1) = \widetilde{x}_2$, is an isomorphism if, and only if, $H_1(\widetilde{x}_1) = H_2(\widetilde{x}_2)$.*

The "Only if" part is obvious. We just have to prove the "if" part.

In fact, in this case, Proposition 7.8 guarantees the existence of a homomorphism $\widetilde{g}\colon \widetilde{X}_2 \to \widetilde{X}_1$ with $g(\widetilde{x}_2) = \widetilde{x}_1$. Then $g \circ f\colon \widetilde{X}_1 \to \widetilde{X}_1$ is an endomorphism such that $\widetilde{x}_1$ is a fixed point, so it coincides with the identity map of $\widetilde{X}_1$. In a similar way we prove that $f \circ g\colon \widetilde{X}_2 \to \widetilde{X}_2$ is the identity, hence f is a homeomorphism.

Corollary 7.13. *Two simply connected coverings of a locally pathwise connected path are isomorphic.*

The above propositions can be summarized as follows:

A. Let X be a connected and locally pathwise connected space and $x_0 \in X$. To each covering $p\colon \widetilde{X} \to X$, with connected domain $\widetilde{X}$, there corresponds a conjugate class $\mathcal{H}(x_0)$, of subgroups of $\pi_1(X, x_0)$, that consists of the subgroups $H(\widetilde{x}) = p_\#\pi_1(\widetilde{X}, \widetilde{x})$, $\widetilde{x} \in p^{-1}(x_0)$. Two coverings, $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$, with connected and locally pathwise connected domains, are isomorphic if, and only if, the corresponding conjugate classes $\mathcal{H}_1(x_0)$ and $\mathcal{H}_2(x_0)$ are equal.

B. Under the same conditions, there exists a homomorphism $f\colon \widetilde{X}_1 \to \widetilde{X}_2$ if, and only if, every subgroup $H_1 \in \mathcal{H}_1(x_0)$ is contained in some subgroup $H_2 \in \mathcal{H}_2(x_0)$.

These two results show how the coverings can be classified by means of the subgroups of the fundamental group of the base space. It remains to establish an important complement, according to which, if X is semilocally simply connected, every conjugate class of subgroups in $\pi_1(X, x_0)$ is the class of some covering $p\colon \widetilde{X} \to X$. This will be proved later.

Example 7.11. What are all the coverings $p\colon \widetilde{X} \to S^1$, of the circle S^1, with $\widetilde{X}$ connected? If we identify the fundamental group of S^1 with $\mathbb{Z}$, their subgroups take the form $n\mathbb{Z}$, $n = 0, 1, 2, \ldots$. The coverings $p_n\colon S^1 \to S^1$, $p_n(z) = z^n$ determine the subgroups $n\mathbb{Z}$ with $n > 0$, while $p_0\colon R \to S^1$, $p_0(t) = e^{2\pi i t}$ determine the subgroup $\{0\}$. Any other covering $p\colon \widetilde{X} \to S^1$, with $\widetilde{X}$ connected, is isomorphic to one of these. Analogously, we prove that a covering of the real projective space P^n, with connected domain, must be a homeomorphism or is isomorphic to the covering of two leaves $p\colon S^n \to P^n$, since the fundamental group $\pi_1(P^n) = \mathbb{Z}_2$ has only two subgroups. ◁

7.4 Covering Automorphisms

The discussion of the previous section will now be restricted to the case of only one covering $p\colon \widetilde{X} \to X$.

We assume, in all of this section, that $\widetilde{X}$ is connected and locally pathwise connected.

Let $\widetilde{x}_0$, $\widetilde{x}_1 \in p^{-1}(x_0)$. As we proved above, there exists an endomorphism $f\colon \widetilde{X} \to \widetilde{X}$ such that $f(\widetilde{x}_0) = \widetilde{x}_1$ if, and only if, $H(\widetilde{x}_0) \subset H(\widetilde{x}_1)$. We also know that $H(\widetilde{x}_1) = \alpha^{-1}H(\widetilde{x}_0)\alpha$, where $\alpha \in \pi_1(X, x_0)$ is the homotopy class of $a = p \circ \widetilde{a}$ and $\widetilde{a}$ is a path in $\widetilde{X}$ that starts at $\widetilde{x}_0$ and ends at $\widetilde{x}_1$.

Example 7.12. Consider the nonregular covering $p\colon \widetilde{X} \to X$ presented in Example 7.2, and the points $\widetilde{x}_0$, $\widetilde{x}_1$ used in the same example. There does not exist an endomorphism $f\colon \widetilde{X} \to \widetilde{X}$ such that $f(\widetilde{x}_0) = \widetilde{x}_1$. In fact, if f existed, by letting $\widetilde{a}$ be the lifting of the path a from the point $\widetilde{x}_0$, $f \circ \widetilde{a}$ would be the lifting of a from the point $\widetilde{x}_1$. Now, we observe that, since $\widetilde{a}$ is closed, $f \circ \widetilde{a}$ also would be closed, but we saw in that example that the lifting of a starting at $\widetilde{x}_1$ is open. ◁

It also follows from the previous discussion that the endomorphism $f\colon \widetilde{X} \to \widetilde{X}$, with $f(\widetilde{x}_0) = \widetilde{x}_1$, is an automorphism if, and only if, $H(\widetilde{x}_0) = H(\widetilde{x}_1)$. Again, $H(\widetilde{x}_1) = \alpha^{-1}H(\widetilde{x}_0)\alpha$, $\alpha \in \pi_1(X, x_0)$ being the homotopy class of a path $a = p \circ \widetilde{a}$, where the path $\widetilde{a}$ starts at $\widetilde{x}_0$ and ends at $\widetilde{x}_1$, in the space $\widetilde{X}$.

If the covering $p\colon \widetilde{X} \to X$ is regular, it follows that, given any two points $\widetilde{x}_0$, $x_1 \in \widetilde{X}$ belonging to the same fiber $p^{-1}(x_0)$, there exists an endomorphism $f\colon \widetilde{X} \to \widetilde{X}$ such that $f(\widetilde{x}_0) = \widetilde{x}_1$. Besides this, every endomorphism of a regular covering is an automorphism. (See Corollary 7.12.)

In order to express the fact that, given any two points of the same fiber of a regular covering $p\colon \widetilde{X} \to X$, there exists an automorphism that maps one onto the other, we say that the automorphism group $G(\widetilde{X}|X)$ of a regular covering acts *transitively* in the fibers. This transitivity, besides being necessary, is also sufficient in order that $p\colon \widetilde{X} \to X$ be regular because it implies that if $\widetilde{a}$ is a lifting of the closed path a, the other liftings of a have the form $f \circ \widetilde{a}$, with $f \in G(\widetilde{X}|X)$; therefore, they are all open or all closed, according whether $\widetilde{a}$ is open or closed.

Another case where every endomorphism is an automorphism is the one of a covering $p\colon \widetilde{X} \to X$, with a finite number m of leaves. In fact, by Proposition 7.7, every endomorphism $f\colon \widetilde{X} \to \widetilde{X}$ is a covering, with a number n of leaves, necessarily finite. The equality $p \circ f = p$ implies that $p^{-1}(x) = f^{-1}(p^{-1}(x))$ for all $x \in X$, which gives us $m = nm$; hence, $n = 1$ and f is an automorphism.

Example 7.20 shows the existence of an endomorphism that is not an automorphism.

The *normalizer* of the subgroup H in a group G is the set $N(H)$ of all elements $g \in G$ such that $g^{-1}Hg = H$. The normalizer $N(H)$ is the largest subgroup of G that contains H as a normal subgroup. H is a normal subgroup of the group G if, and only if, $N(H) = G$.

In Proposition 7.13, it was established that, given the covering $p\colon \widetilde{X} \to X$, the fundamental group $\pi_1(X, x_0)$ acts transitively on the right in the fiber $p^{-1}(x_0)$, the isotropy group of $\widetilde{x} \in p^{-1}(x_0)$ being equal to $H(\widetilde{x})$. The action of an element $\alpha \in \pi_1(X, x_0)$ on $\widetilde{x} \in p^{-1}(x_0)$ is denoted by $\widetilde{x}\cdot\alpha$. We recall that $\widetilde{x}\cdot\alpha = \widetilde{a}(1)$ where $\widetilde{a}\colon I \to \widetilde{X}$ is the lifting, starting from $\widetilde{x}$, of a path $a\colon I \to X$ such that $\alpha = [a]$.

In terms of these notions, the existence of an endomorphism $f\colon \widetilde{X} \to \widetilde{X}$ with $f(\widetilde{x}_0) = \widetilde{x}_1$, where $\widetilde{x}_1 = \widetilde{x}_0\cdot\alpha$, is equivalent to the statement that $H(\widetilde{x}_0) \subset \alpha^{-1}H(\widetilde{x}_0)\alpha$. Moreover, f is an automorphism if, and only if, $H(\widetilde{x}_0) = \alpha^{-1}H(\widetilde{x}_0)\alpha$; that is, if, and only if, $\alpha \in N(H(\widetilde{x}_0))$.

In particular, for each $\alpha \in N(H(\widetilde{x}_0))$, there exists a unique automorphism $f\colon \widetilde{X} \to \widetilde{X}$ such that $f(\widetilde{x}_0) = \widetilde{x}_0 \cdot \alpha$.

This is the crucial remark in order to prove the following proposition, which establishes an isomorphism between the group $G(\widetilde{X}|X)$ of the covering automorphisms $p\colon \widetilde{X} \to X$ and the quotient group $N(H(\widetilde{x}_0))/H(\widetilde{x}_0)$. In order to prove it, it is convenient to start with the following lemma.

Lemma 7.1. *Let $f\colon \widetilde{X} \to \widetilde{X}$ be an endomorphism of the covering $p\colon \widetilde{X} \to X$. For any $\widetilde{x} \in X$ and $\alpha \in \pi_1(X, p(\widetilde{x}))$, we have $f(\widetilde{x}\cdot\alpha) = f(\widetilde{x})\cdot\alpha$.*

Proof. Let $\alpha = [a]$ and $\widetilde{a}$ be a lifting of a starting at the point $\widetilde{x}$. Then $\widetilde{x}\cdot\alpha = \widetilde{a}(1)$, so $f(\widetilde{x}\cdot\alpha) = f(\widetilde{a}(1))$. On the other hand, $f \circ \widetilde{a}$ is a lifting of a that starts at the point $f(\widetilde{x})$. Hence

$$f(\widetilde{x})\cdot\alpha = (f \circ \widetilde{a})(1) = f(\widetilde{a}(1)) = f(\widetilde{x}\cdot\alpha),$$

which proves the lemma. □

Proposition 7.9. *Let $p\colon \widetilde{X} \to X$ be a covering, with $\widetilde{X}$ connected and locally pathwise connected. For each $\widetilde{x}_0 \in \widetilde{X}$, there exists a group isomorphism $G(\widetilde{X}|X) \approx N(H(\widetilde{x}_0))/H(\widetilde{x}_0)$.*

Proof. We define a map $\varphi\colon N(H(\widetilde{x}_0)) \to G(\widetilde{X}|X)$ by setting, for each $\alpha \in N(H(\widetilde{x}_0))$, $\varphi(\alpha) = f$, where $f\colon \widetilde{X} \to \widetilde{X}$ is the automorphism such that $f(\widetilde{x}_0) = \widetilde{x}_0\cdot\alpha$. If $\varphi(\alpha) = f$ e $\varphi(\beta) = g$, we have $\widetilde{x}_0\cdot\alpha = f(\widetilde{x}_0)$ and $\widetilde{x}_0\cdot\beta = g(x_0)$. By the above lemma,

$$(f \circ g)(\widetilde{x}_0) = f(g(\widetilde{x}_0)) = f(\widetilde{x}_0\cdot\beta) = f(x_0)\cdot\beta = x_0\cdot\alpha\beta.$$

Therefore, $f \circ g = \varphi(\alpha\beta)$ and φ is a group homomorphism. We have $\varphi(\alpha) = id_{\widetilde{X}}$; that is, $\widetilde{x}_0\cdot\alpha = \widetilde{x}_0$, if, and only if, $\alpha \in H(\widetilde{x}_0)$. Thus, $H(\widetilde{x}_0)$ is the kernel of φ. We claim that φ is surjective. In fact, given $f \in G(\widetilde{X}|X)$, let $f(\widetilde{x}_0) = \widetilde{x}_1$. Since $\pi_1(X, x_0)$ acts transitively on $p^{-1}(x_0)$, there exists $\alpha \in \pi_1(X, x_0)$ such that $\widetilde{x}_1 = \widetilde{x}_0 \cdot \alpha$. Since f is an automorphism, we have $\alpha \in N(H(\widetilde{x}_0))$. Since $f(\widetilde{x}_0) = \widetilde{x}_0 \cdot \alpha$, it follows that $f = \varphi(\alpha)$. The isomorphism theorem for groups yields, by passing to the quotient an isomorphism $\overline{\varphi}\colon N(H(\widetilde{x}_0))/H(\widetilde{x}_0) \to G(\widetilde{X}|X)$. □

Corollary 7.14. *If $\widetilde{X}$ is connected, locally pathwise connected, and the covering is regular, we have an isomorphism $G(\widetilde{X}|X) \approx \pi_1(X, x_0)/H(\widetilde{x}_0)$ for each $\widetilde{x}_0 \in p^{-1}(x_0)$.*

Corollary 7.15. *If $\widetilde{X}$ is simply connected, and locally pathwise connected then the group $G(\widetilde{X}|X)$ of automorphisms of the covering $p\colon \widetilde{X} \to X$ is isomorphic to the fundamental group $\pi_1(X, x_0)$.*

The isomorphism $\pi_1(X, x_0) \to G(\widetilde{X}|X)$ mentioned in Corollary 7.15 is defined by choosing a point $\widetilde{x}_0 \in \widetilde{X}$. It maps the element $\alpha \in \pi_1(X, x_0)$ into the automorphism $f\colon \widetilde{X} \to \widetilde{X}$, thus described: Given $\widetilde{x} \in \widetilde{X}$, we connect $\widetilde{x}$ to $\widetilde{x}_0$ by a path $\widetilde{b}$ in $\widetilde{X}$. Let $b = p \circ \widetilde{b}$, $a \in \alpha$ and $x = p(\widetilde{x})$. Then bab^{-1} is a closed path with base at the point x. The lifting of bab^{-1} from the point $\widetilde{x}$ ends at a point $\widetilde{y} \in p^{-1}(x)$. We set $f(\widetilde{x}) = \widetilde{y}$.

Example 7.13. **(Klein bottle)** Let G be the group generated by the homeomorphisms $f, g\colon \mathbb{R}^2 \to \mathbb{R}^2$, where $f(x, y) = (x, y+1)$ and $g(x, y) = (x+1, 1-y)$. Since $gf = f^{-1}g$, we can write the elements of G in the form $f^m \cdot g^n$, where m, $n \in \mathbb{Z}$. Now, $f^m g^n(x, y) = (x + n, y + m)$ if n is even and $f^m g^n(x, y) = (x + n, 1 - y + m)$ if n is odd. G is a properly discontinuous group of homeomorphisms of $\mathbb{R}^2$. In fact, given $z = (x, y)$, let V be the open square with center z, with sides of length 1, parallel to the axes. Then, for every $h \neq id$ in G, we have $V \cap hV \neq \varnothing$. The quotient space $K = \mathbb{R}^2/G$ is called the *Klein bottle*. The quotient map $p\colon \mathbb{R}^2 \to K$ exhibits the plane as a universal covering of the Klein bottle; therefore, the fundamental group of K is isomorphic to the group G of the automorphisms of this covering. Thus, the fundamental group of K has two generators, f, g, that satisfy the relation $gf = f^{-1}g$. (For other descriptions of the Klein bottle, see Examples 8.10 and 8.11, and Section 7 in this chapter.) ◁

Example 7.14. Now we give an example of a space whose fundamental group is $\mathbb{Z}_n$ (integers mod n). For this, we just have to consider a Hausdorff simply connected space Y which is also locally pathwise connected, and a group G de homeomorphisms of Y, isomorphic to $\mathbb{Z}_n$, such that none of them, with the exception of id_Y, has fixed points. (Being finite, G is properly discontinuous.) The quotient space Y/G will have fundamental group isomorphic to $\mathbb{Z}_n$. Let $D = \{z \in \mathbb{R}^2; |z| \leq 1\}$ be the unit disk of the plane and $X = \{(z,i); z \in D, i = 1,2,\dots,n\}$ be the union of the n horizontal disjoint disks $D \times \{1\},\dots,D \times \{n\}$. Y is the quotient space of X by the equivalence relation that identifies the points (z,i) and (z,j) when $|z| = 1$. If $n = 3$, Y is homeomorphic to the union of the sphere S^2 with its equatorial disk. For $n > 3$, we need $n-2$ curved "equatorial disks," all of them having in common the circle S^1. If $n = 2$, Y is homeomorphic to the sphere S^2. The space Y is simply connected, as we can see by induction, using Corollary 2.9. In order to define the homeomorphism $\varphi\colon Y \to Y$, we denote by $[z,i] \in Y$ the equivalence class of $(z,i) \in X$. Let $u = e^{2\pi i/n}$. Set $\varphi[z,i] = [u \cdot z, i+1]$ if $i < n$ and $\varphi[z,n] = [u \cdot z, 1]$. Note that φ does not have fixed points and the group generated by φ is $G = \{id_Y, \varphi, \dots, \varphi^{n-1}\}$, isomorphic to $\mathbb{Z}_n$. Therefore, the quotient space Y/G has fundamental group isomorphic to $\mathbb{Z}_n$. When $n = 2$, φ is the antipodal map and Y/G is the real projective plane. ◁

Now we give an example, simpler than the previous one, of a space with fundamental group $\mathbb{Z}_n$, in dimension 3.

Example 7.15. Again, let $u = e^{2\pi i/n}$. The sphere S^3 is the set of pairs (z,w) of complex numbers such that $|z|^2 + |w|^2 = 1$. The homeomorphism $\varphi\colon S^3 \to S^3$, defined by $\varphi(z,w) = (u \cdot z, u \cdot w)$, generates the group $G = \{id, \varphi, \varphi^2, \dots, \varphi^{n-1}\}$ of homeomorphisms without fixed points in S^3. G is a properly discontinuous group, isomorphic to $\mathbb{Z}_n$, so the quotient space $X = S^3/G$ has fundamental group isomorphic to $\mathbb{Z}_n$. When $n = 2$, X is the real projective space P^3. For any $n \in \mathbb{N}$, X is known as the *lens space* $L_{n,1}$. ◁

Example 7.16. Let X be the figure eight space, that is the union of two circles with a point in common. In Example 6.12 we showed a covering $p\colon \widetilde{X} \to X$. In that example, $\widetilde{X}$ is a tree; that is, a graph without cycles. We know that every tree is a contractible space. In particular, $\widetilde{X}$ is simply connected. Therefore, the fundamental group of X is isomorphic to the group of automorphisms $G(\widetilde{X}|X)$ of the covering $p\colon \widetilde{X} \to X$. Now, by taking the origin $\widetilde{x}_0$ as base point, an automorphism $f\colon \widetilde{X} \to \widetilde{X}$ is completely determined by the image $\widetilde{x}_1 = f(\widetilde{x}_0)$, which must be one of the "crossing points" of $\widetilde{X}$, and it can be any one of them. (Such points form the fiber over the point $x_0 \in X$, intersection of the two circles.) Now, the crossing points in $\widetilde{X}$ are in $1-1$ correspondence with the words $a^m b^n a^p \dots$, because there exists a unique way to go from $\widetilde{x}_0$ to any other crossing point, along the segments of $\widetilde{X}$, without moving along the same segment twice. We conclude from this that the fundamental group of X is isomorphic to the free nonabelian group with two generators; each one of these generators is the homotopy class of one of the circles that form the space X. Considering a construction analogous to that of Example 12, Chapter 6, in three dimensions, we can show that the fundamental group of the union of three circles with one point in common is a free group with three generators. More generally, using the same method, it is possible to prove that if $X = X_1 \cup \ldots \cup X_n$ is the union of n circles with a point x_0 in common, then $\pi_1(X, x_0)$ is a free group with n generators. ◁

7.5 Properly Discontinuous Groups and Regular Coverings

We have seen that if G is a properly discontinuous group of homeomorphisms of a topological space X, then the quotient map $\pi\colon X \to X/G$, onto the space of orbits of G, is a covering (Section 6.3) and when X is pathwise connected this covering is regular. (Example 6.3 of that chapter.) We now show that, conversely, every regular covering is essentially the quotient map onto the space of orbits of a properly discontinuous group of homeomorphisms.

Proposition 7.10. *Let $\widetilde{X}$ be a connected space. The automorphism group of a covering $p\colon \widetilde{X} \to X$ is a properly discontinuous group of homeomorphisms of the space $\widetilde{X}$.*

Proof. Given $\widetilde{x} \in \widetilde{X}$, let U be a distinguished neighborhood of $x = p(\widetilde{x})$ and V a neighborhood of $\widetilde{x}$ such that $p|V$ is a homeomorphism onto U. If the automorphism $f\colon \widetilde{X} \to \widetilde{X}$ is different from the identity, then $f(v) \neq v$ for every $v \in V$. Since v and $f(v)$ belong to the same fiber of the covering p and $p|V$ is injective, it follows that $f(v) \notin V$. Hence, $V \cap f(V) = \varnothing$, which proves that $G(\widetilde{X}|X)$ is properly discontinuous. □

Given the covering $p\colon \widetilde{X} \to X$, with $\widetilde{X}$ connected, consider the covering $\pi\colon \widetilde{X} \to \widetilde{X}/G$, where $G = G(\widetilde{X}|X)$ is the properly discontinuous group whose elements are the automorphisms of the covering p.

Proposition 7.11. *Let X be a connected and locally pathwise connected space. If the covering $p\colon \widetilde{X} \to X$ is regular, there exists a homeomorphism $\xi\colon \widetilde{X}/G \to X$ which makes the diagram below commutative.*

$$\begin{array}{ccc} & \widetilde{X} & \\ {}^{\pi}\swarrow & & \searrow^{p} \\ \widetilde{X}/G & \xrightarrow{\ \xi\ } & X \end{array}$$

Proof. Given $\widetilde{x}$, $\widetilde{y} \in \widetilde{X}$, we have that $p(\widetilde{x}) = p(\widetilde{y}) \Leftrightarrow \widetilde{x}, \widetilde{y}$ belong to the same fiber of $p \Leftrightarrow$ there exists $f \in G$, such that $f(\widetilde{x}) = \widetilde{y}$ (because the covering is regular if, and only if, G is transitive in the fibers) $\Leftrightarrow G \cdot \widetilde{x} = G \cdot \widetilde{y} \Leftrightarrow \pi(\widetilde{x}) = \pi(\widetilde{y})$. Hence, the equivalence relations determined by p and π in $\widetilde{X}$ coincide. This gives us a continuous bijection $\xi \colon \widetilde{X}/G \to X$ such that $\xi \circ \pi = p$. Since p is open, the same happens with ξ. Hence, ξ is a homeomorphism. □

7.6 Existence of Coverings

We start with the following question: Which topological spaces have a simply connected covering? It is easy to obtain a necessary condition. If $p \colon \widetilde{X} \to X$ is a covering with $\widetilde{X}$ simply connected, then X must be semi-locally simply connected. In fact, if $V \subset X$ is a distinguished neighborhood, then every closed path a, contained in V, has a closed lifting $\widetilde{a}$. Since $\widetilde{X}$ is simply connected, $\widetilde{a}$ is homotopic to a constant in $\widetilde{X}$ and from this we conclude that $a = p \circ \widetilde{a}$ is also homotopic to a constant in X.

We show below (for locally pathwise connected spaces), that this condition is also sufficient for the existence of a "universal" covering $p \colon \widetilde{X} \to X$; that is, a covering with $\widetilde{X}$ simply connected. Moreover, we prove that if X is locally pathwise connected and semi-locally simply connected then, for every subgroup $H \subset \pi_1(X, x_0)$ there exists a covering $p \colon \widetilde{X} \to X$ such that $p_{\#}\pi_1(\widetilde{X}, \widetilde{x}_0) = H$. This will complete the discussion we started at the end of Section 3. Given a connected, locally pathwise connected and semi-locally simply connected space X, we fix $x_0 \in X$. To each covering $p \colon \widetilde{X} \to X$, with $\widetilde{X}$ connected, corresponds a conjugate class $\mathcal{H}(x_0)$ of subgroups of $\pi_1(X, x_0)$. Two connected coverings with base X are isomorphic if, and only if, to them corresponds the same conjugate class. Now we see that such correspondence between classes of isomorphic coverings with base X and conjugate classes of subgroups of $\pi_1(X, x_0)$ is surjective and therefore bijective, under these topological hypothesis on X.

Proposition 7.12. *Let X be a connected, locally pathwise connected, and semi-locally simply connected space. Given $x_0 \in X$ and a subgroup $H \subset \pi_1(X, x_0)$, there exists a covering $p \colon \widetilde{X} \to X$, with $\widetilde{X}$ connected, and a point $\widetilde{x}_0 \in \widetilde{X}$ such that $p_{\#}\pi_1(\widetilde{X}, \widetilde{x}_0) = H$.*

Proof. Let a, b be two paths in X, starting at the point x_0. We say that a and b are equivalent, and we write $a \equiv b$, when $a(1) = b(1)$ and $[ab^{-1}] \in H$. We use the notation $\langle a \rangle$ to represent the equivalence class of the path a. Let $\widetilde{X}$ be the set of all equivalence classes $\langle a \rangle$ of the paths a in X that start at the point x_0. We define a map $p\colon \widetilde{X} \to X$ by setting $p(\langle a\rangle) = a(1)$. In order to introduce a topology in $\widetilde{X}$, we consider the basis of X formed by all pathwise connected open sets $U \subset X$ such that every path in U is homotopic to a constant in X. For each point $\langle a\rangle \in \widetilde{X}$ and each open set $U \in \mathcal{U}$ such that $a(1) \in U$, we set

$$\widetilde{U}\langle a\rangle = \{\langle ab\rangle; b(I) \subset U\}.$$

The sets $\widetilde{U}\langle a\rangle$ form the basis for a topology in $\widetilde{X}$, according to which $p\colon \widetilde{X} \to X$ is continuous and open. Besides this, $p|\widetilde{U}\langle a\rangle$ is a bijection (and therefore, a homeomorphism) onto U. For each $U \in \mathcal{U}$, the inverse image $p^{-1}(U)$ is the union of the sets $\widetilde{U}\langle a\rangle$, where a varies among the paths in X with origin x_0 and final point in U. Two of these sets either coincide or are disjoint. It follows that $p\colon \widetilde{X} \to X$ is a covering. Given a path $a\colon I \to X$, with origin x_0, for each $t \in I$ let $a_t\colon I \to X$ be the path given by $a_t(s) = a(st)$. Then $\widetilde{a}\colon I \to \widetilde{X}$, defined by $\widetilde{a}(t) = \langle a_t\rangle$, is the lifting of a starting at the point $\widetilde{x}_0 = \langle e_{x_0}\rangle$, where e_{x_0} is the constant path in X, at the point x_0. Note that every point $\langle a\rangle \in \widetilde{X}$ can be connected to the point $\widetilde{x}_0$ by the path $t \mapsto \langle a_t\rangle$. Hence, $\widetilde{X}$ is pathwise connected. We have $[a] \in p_{\#}\pi_1(\widetilde{X}, \widetilde{x}_0) \Leftrightarrow \widetilde{a}$ is a closed path $\Leftrightarrow \langle a\rangle = \langle a_1\rangle = \langle e_{x_0}\rangle \Leftrightarrow [a] \in H$. This completes the proof. □

Corollary 7.16. *Every connected, locally pathwise connected, and semi-locally simply connected topological space X admits a covering $p\colon \widetilde{X} \to X$, with $\widetilde{X}$ simply connected. Any two of these coverings are isomorphic.*

Example 7.17. By Corollary 7.16, the group $\mathrm{SO}(n)$ has a covering $p\colon \widetilde{X} \to \mathrm{SO}(n)$, with $\widetilde{X}$ simply connected. Since $\pi_1(\mathrm{SO})(n) = \mathbb{Z}_2$, this covering has two leaves. (See Corollary 7.7.) By Proposition 7.5, there exists a group structure in $\widetilde{X}$ which turns the projection $p\colon \widetilde{X} \to \mathrm{SO}(n)$ into a homomorphism. This group $\widetilde{X}$ is called the *group of spinors of order n* and it is denoted by $\mathrm{Spin}(n)$. We have $\mathrm{Spin}(3) = S^3$, and $\mathrm{Spin}(4) = S^3 \times S^3$. The group structure in $\mathrm{Spin}(n)$, obtained by means of the requirement that $p\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ be an homomorphism, is unique, provided that we choose one of the two elements that are mapped by p in the identity matrix to be the neutral element. ◁

7.7 Fundamental Group of a Compact Surface

In order to finish this chapter, we shall determine, in terms of generators and relations, the fundamental group of a compact surface. By "surface,"

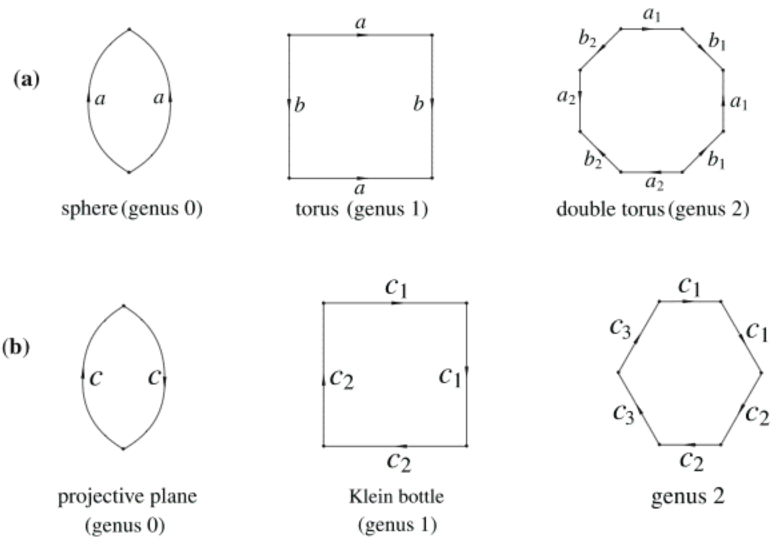


Figure 7.2. Some identification schemes for surface construction.

we mean a topological manifold of dimension 2; that is, a Hausdorff topological space that is locally homeomorphic to the Euclidean plane $\mathbb{R}^2$.

It is proven in combinatorial topology (see Seifert & Threlfall (1980)) that every compact surface is the quotient space of a plane polygon by an equivalence relation according to which the sides that make up the boundary of the polygon are identified two by two, according to schemes such as those illustrated in Figure 7.2.

There are three identifying schemes. The first is the one of the *orientable surface of genus zero*, which is homeomorphic to the sphere S^2, in which the "polygon" has two sides, which must be glued one to the other, as indicated in the leftmost picture in Figure 7.2(a).

The second type of scheme is that of an *orientable surface of genus* $g \geq 1$. The polygon has $4g$ sides, labeled $a_1, b_1, a_1, b_1, \ldots, a_g, b_g$. Each of these sides is oriented by means of an arrow. Moving along the boundary of the polygon in the clockwise sense, the directions of the arrows provides a "word" $\omega = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_g^{-1} b_g^{-1}$, which represents a closed path in the surface. This is again illustrated for $g = 1$ and $g = 2$ in Figure 7.2(a). (middle and right pictures). The picture on the left in Figure 7.3 illustrates the general scheme.

The third type of identifying scheme is that of a *nonorientable surface of genus* $g = h - 1$. The polygon has $2h$ sides, labeled $c_1, c_1, c_2, c_2, \ldots,$ c_h, c_h. These sides are oriented by arrows in such a way that, by moving

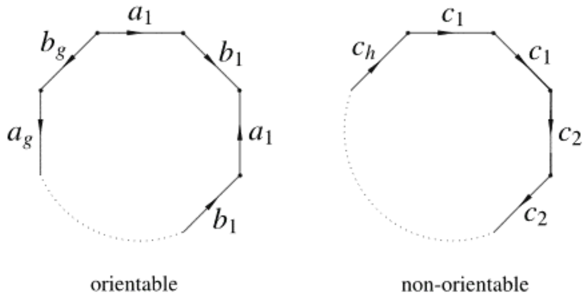


Figure 7.3.

along the boundary of the polygon in the clockwise sense, the arrows point in our direction. This gives us the word $\lambda = c_1^2 c_2^2 \dots c_h^2$, which represents a closed path in the surface. This is illustrated for $g = 0$, $g = 1$ and $g = 2$ in Figure 7.2(b). The picture on the right in Figure 7.3 illustrates the general scheme.

Note that, both the image (by the quotient map) of the path ω of the second scheme as well as the one of the path λ of the third scheme are homotopic to constant paths in the corresponding surfaces, because ω and λ are homotopic to a constant on their polygons.

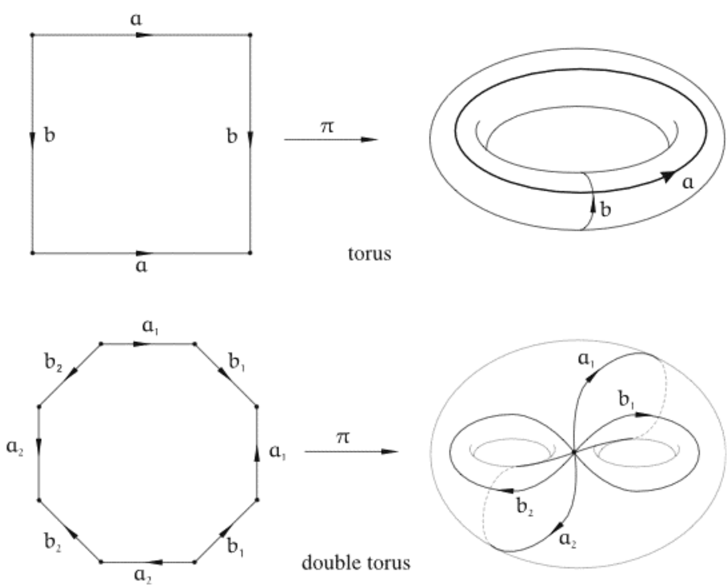


Figure 7.4.

If $\pi \colon P \to S$ is the quotient map of the polygon P onto the surface S, the boundary of P is transformed by π into a union of circles with a point in common. The number of circles is $2g$ for an orientable surface of genus $g \geq 1$ and $h = g + 1$ for a nonorientable surface of genus $g \geq 1$. The interior of the polygon P is mapped by π homeomorphically onto the complementary set of this union of circles on the surface. This is illustrated in Figure 7.4 for the orientable surfaces of genus 1 (torus) and genus 2 (double torus).

Proposition 7.13. *The fundamental group of a compact orientable surface of genus $g \geq 1$ has $2g$ generators $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g$, β_g and only one relation,*

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\ldots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1} = 1.$$

The fundamental group of a compact nonorientable surface of genus g has $h = g + 1$ generators $\gamma_1, \gamma_2, \ldots, \gamma_h$ and only one relation,

$$\gamma_1^2\gamma_2^2\ldots\gamma_h^2 = 1.$$

The above proposition is a particular case of the following situation: we have a space X, a closed subset $A \subset X$, and a continuous map $f \colon D \to X$, of the unit disk $D = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 \leq 1\}$ on X, satisfying the following conditions:

1. $f(S^1) \subset A$;
2. $f|\operatorname{int}.D$ is a homeomorphism onto $X - A$.

In this case, we say that X is obtained from A by the *adjunction of a two-dimensional cell.* Let $e = f(D)$, $a = f|S^1$. We then write $X = A\bigcup_a e$. By choosing base points $u_o \in S^1$ and $x_0 = f(u_0) \in A$, the homotopy class $[a]$ is an element of the fundamental group $\pi_1(A, x_0)$. When X is a compact surface, the subset A is the union of a finite number of circles with a point in common and f is the quotient map.

More generally, suppose that we have a family of continuous maps $a_\lambda \colon S^1 \to A$, $\lambda \in L$, all of them taking values in a topological space A, and consider the disjoint union $Z = A \cup (\bigcup_{\lambda \in L} D_\lambda)$, where each D_λ is a copy of the disk D. We consider in Z the "sum" topology, in which A and each D_λ are open and closed. We introduce in Z the equivalence relation that identifies each point $z \in S^1_\lambda$ (boundary of D_λ) with its image $a_\lambda(z) \in A$. Let X be the quotient space of Z by this equivalence relation. We set

$$X = A\bigcup_{a_\lambda}\{e_\lambda\},$$

where e_λ is the image of D_λ by the quotient map $f\colon Z \to X$. Let $f_\lambda = f|D_\lambda$ and $\dot{e}_\lambda = f_\lambda(S^1_\lambda) = a_\lambda(S^1_\lambda) = e_\lambda \cap A$. We say that X *is obtained from* A *by the adjunction of the* 2*-dimensional cells* e_λ. The following properties hold:

1. For each $\lambda \in L$, f_λ is a homeomorphism of the interior of the disk D_λ onto the open set $e_\lambda - \dot{e}_\lambda$ of the space X;

2. If $\lambda \neq \mu$ then $e_\lambda - \dot{e}_\lambda$ and $e_\mu - \dot{e}_\mu$ are disjoint;

3. The set $S \subset X$ is closed (respectively open) if, and only if, for each $\lambda \in L$, the intersection $S \cap e_\lambda$ is closed (respectively open) in e_λ.

In the proposition below, which contains Proposition 7.13, we suppose that A is pathwise connected, so that the fundamental group $\pi(A, x_0)$ does not depend essentially of the base point x_0.

Proposition 7.14. *Given a pathwise connected and semi-locally simply connected space* A*, let* $X = A \bigcup_a e$ *be the space obtained from* A *by the adjunction of the* 2*-dimensional cell* e*, by means of the continuous map* $a\colon S^1 \to A$*. By setting* $x_0 = a(u_0)$*,* $u_0 \in S^1$*, the fundamental group* $\pi_1(X, x_0)$ *is isomorphic to the quotient group of* $\pi_1(A, x_0)$ *by the normal subgroup generated by* $[a]$*.*

Proof. Let $j\colon A \to X$ be the inclusion map. We must show:

A. The induced homomorphism $j_\#\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ is surjective;

B. The kernel of $j_\#$ is the normal subgroup generated by $[a]$.

Proof of A. Let $y \in X - A$ (for example, the center of the cell e). Set $U = X - A$ and $V = X - \{y\}$. Then U and V are open sets in X, $U \cap V$ is homeomorphic to the open disk minus a point (and therefore it is pathwise connected) and $X = U \cup V$. It follows from Proposition 2.11 that $\pi_1(X, x_0)$ is generated by the images of $\pi_1(U)$ and $\pi_1(V)$, induced by the inclusions $U \to X$ and $V \to X$. Now, U is contractible because it is homeomorphic to an open disk, and the inclusion $A \to V$ is a homotopy equivalence: Its homotopic inverse $V \to A$ is the retraction that projects each point of $V - A$ radially from the point y and leave fixed the points of A. Hence, $\pi_1(U)$ is trivial and the homomorphisms $\pi_1(A, x_0) \to \pi_1(V, x_0)$ and $\pi_1(V, x_0) \to \pi_1(X, x_0)$, induced by inclusions, are surjective; hence, the composite homomorphism $j_\#$ is also surjective.

Proof of B. Let N be the normal subgroup of $\pi_1(A, x_0)$ generated by the class $[a]$. By Proposition 7.12, there exists a covering $p\colon \widetilde{A} \to A$ such that $p_\# \pi_1(\widetilde{A}, \widetilde{x}_0) = N$, for all $\widetilde{x}_0 \in p^{-1}(x_0)$. Since $[a] \in N$, the map $a\colon S^1 \to A$

has a lifting $\widetilde{a}_\lambda\colon S^1 \to \widetilde{A}$, with $\widetilde{a}_\lambda(u_0) = \lambda$, for each $\lambda \in p^{-1}(x_0)$. Let $\widetilde{X}$ be the space obtained from $\widetilde{A}$ by adjunction of two-dimensional cells through the maps $\widetilde{a}_\lambda$. There exists a covering map $q\colon \widetilde{X} \to X$ such that $q|\widetilde{A} = p$. (In order to see this, note that the covering $p\colon \widetilde{A} \to A$ is regular; hence, $A = \widetilde{A}/G$, where $G = G(\widetilde{A}|A)$. Now, every $g \in G$ extends in an evident way to a homeomorphism of $\widetilde{X}$ and G becomes a properly discontinuous group in $\widetilde{X}$, with $\widetilde{X}/G = X$.) Now we take, arbitrarily, a closed path b in A, with base at the point x_0, such that b is homotopic to a constant path in X (this means that $[b] \in \pi_1(A, x_0)$ is in the kernel of $j_\#$). Then the lifting $\widetilde{b}$ of the path b, from any point $\lambda \in p^{-1}(x_0)$, is a closed path. But $\widetilde{b}$ is a path in $\widetilde{A}$. Since it is closed, this means that $[b] \in N$. (See Corollary 7.1.) Thus, the kernel of the homomorphism $j_\#\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ is contained in the normal subgroup N, generated by $[a]$. But it is obvious that $[a]$ belongs to the kernel of $j_\#$, which is a normal subgroup; hence, such a kernel contains N. This concludes the proof. □

Proposition 7.13 follows from Proposition 7.14 by virtue of Example 7.16, according to which the fundamental group of the space that consists of n circles with a point in common is a free subgroup with n generators.

Remark. With the exception of the sphere and the projective plane, the universal covering space of a compact surface is the plane $\mathbb{R}^2$. This follows from the fact that the fundamental groups of these surfaces are infinite; hence, their universal covering spaces are not compact. Since the covering map is a local homeomorphism, each covering space of a surface is also a surface. But, the only noncompact simply connected surface is the plane $\mathbb{R}^2$. The proof of this fact is surprisingly nontrivial. (See Seifert & Threlfall (1980), page 332. Another proof can be given as a consequence of the Koebe uniformization theorem for Riemann surfaces.)

Example 7.18. Completing the discussion that we started in Example 14 of Chapter 6, we see that, with the exception of the torus, no compact surface admits the structure of a topological group. In fact, the fundamental group of a compact surface is nonabelian, with three exceptions: the sphere, the projective plane and the torus. We have already seen (in the example mentioned above) that the sphere is not a topological group. By Proposition 7.5, it follows that the projective plane also is not a topological group, because it is covered by the sphere. The torus is a group, so all the cases are now covered. ◁

Example 7.19. As a consequence of Proposition 7.13, the fundamental group of the Klein bottle has two generators, say c, d, that satisfy the unique relation $c^2d^2 = 1$, which is equivalent to $cd = c^{-1}d^{-1}$. By setting $a = c$ and $b = dc$, we see that the same group admits the generators a, b, which satisfy the unique relation $ab = b^{-1}a$ (because $ab = cdc = c^{-1}d^{-1}c = b^{-1}a$). This takes us back to the description of the same group given in Example 7.12 and illustrates the fact that the presentation of a group by using generators and relations can, in general, be done in different ways, apparently distinct. ◁

Example 7.20. By virtue of Proposition 7.8 and its Corollary 7.12, one obtains an endomorphism $f\colon \widetilde{X} \to \widetilde{X}$ which is not an automorphism when the covering $p\colon \widetilde{X} \to X$ has the following property: there exist $x_0 \in X$, $\widetilde{x}_0 \in p^{-1}(x_0)$, and $\alpha \in \pi_1(X, x_0)$ such that $\alpha^{-1} \cdot H(\widetilde{x}_0) \cdot \alpha$ is a proper subgroup of $H(\widetilde{x_0})$. With this purpose, we take X as the space obtained by the adjunction of a two-dimensional cell to the figure eight space through the map $aba^{-1}b^{-2}\colon S^1 \to$ figure eight, where a and b are the two canonical closed paths in the figure eight space. By Proposition 7.14, the fundamental group of X (with base in x_0, crossing point in the figure eight) has the generators $[a] = \alpha$ and $[b] = \beta$, with the unique relation $\alpha\beta\alpha^{-1} = \beta^2$. By Proposition 7.12, there exists a covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that $H(\widetilde{x}_0) = p_{\#}\pi_1(\widetilde{X}, x_0)$ is the cyclic subgroup of $\pi_1(X, x_0)$ generated by β. Then $\alpha^{-1} \cdot H(\widetilde{x}_0) \cdot \alpha = H(\widetilde{x}_1)$, where $\widetilde{x}_1 = \widetilde{x}_0 \cdot \alpha$. It follows that $H(\widetilde{x}_1)$ is the proper subgroup of $H(\widetilde{x}_0)$ whose elements are the powers of β with an even exponent. Therefore, there exists an endomorphism $f\colon (\widetilde{X}, \widetilde{x}_1) \to (\widetilde{X}, \widetilde{x}_0)$, which is not an automorphism. ◁

7.8 Exercises

1. Let X be the figure eight space and $\widetilde{X}$ the subset of the upper half-plane, formed by the horizontal axis, along with the circles of radius $1/3$, tangent to this axis at the points $(n, 0)$, $n \in Z$. Define a covering map of $\widetilde{X}$ onto X. Determine whether this covering is regular or not. Take $x_0 = (0, 0)$, describe the conjugate class of $\pi_1(X, x_0)$ defined by this covering, and determine $G(\widetilde{X}|X)$.

2. In Exercise 3 of Chapter 6, show that p regular implies q regular.

3. Let X be an arbitrary topological space. Given the covering $\xi\colon \mathbb{R} \to S^1$, $\xi(t) = e^{it}$, prove that a continuous map $f\colon X \to S^1$ has a lifting with respect to ξ if, and only if, it is homotopic to a constant.

4. If $n \geq 2$, then every continuous map $f\colon P^n \to S^1$ is homotopic to a constant.

5. Every continuous map from the sphere S^2 to the torus T^2 is homotopic to a constant. The same occurs with maps from S^2 to S^1.

6. Let M be a compact orientable surface of genus $g > 1$. Show that there exists $f\colon M \to S^1$ continuous, nonhomotopic to a constant.

7. Find a covering of the figure eight space such that $H(\widetilde{x}_0)$ is a cyclic group. Is this covering regular? What is the group $G(\widetilde{X}|X)$ of the automorphisms of this covering?

8. In Exercise 15 of Chapter 6, if $p\colon \widetilde{X} \to X$ is a regular covering of n leaves, and $f = p$, show that $\widetilde{Z}$ is the union of n disjoint copies of $\widetilde{X}$.

9. Let X be the union of two circles tangent at the point x_0 and $\widetilde{X}$ be the grid consisting of the points of the plane that have at least one integral coordinate. Define a covering map $p\colon \widetilde{X} \to X$ such that, by fixing a point $\widetilde{x}_0 \in p^{-1}(x_0)$, $H(\widetilde{x}_0)$ is the commutator subgroup (generated by the elements of the form $\alpha\beta\alpha^{-1}\beta^{-1}$) in $\pi_1(X, x_0)$ and $G(\widetilde{X}|X) = \mathbb{Z} \oplus \mathbb{Z}$.

10. Determine all of the connected coverings of the torus $T^n = S^1 \times \ldots \times S^1$.

11. Let $p\colon \widetilde{X} \to X$ be a covering with $\widetilde{X}$ connected, $G(\widetilde{X}|X)$ be the automorphism group and $\pi\colon \widetilde{X} \to \widetilde{X}/G(\widetilde{X}|X)$ the quotient map. There exists a continuous map $q\colon \widetilde{X}/G(\widetilde{X}|X) \to X$ such that $q \circ \pi = p$. Both π and q are covering maps.

12. Let $X = S^1 \cup S^2$ be the union of a circle and a sphere, with $S^1 \cap S^2 = \{x_0\}$. Obtain the universal covering of X.

13. What is the universal covering of the space formed by the union of a torus with a circle that has a point in common with it?

14. Let $p\colon \widetilde{G} \to G$ be a homomorphic covering. If $\widetilde{G}$ is simply connected, then $\pi_1(G)$ is isomorphic to the kernel of p.

15. Restate, in terms of the complex integral

$$\int_a \frac{f'(z)}{f(z)} dz,$$

the conditions of the Examples 7.4 and 7.5, in order that the holomorphic function $f\colon U \to \mathbb{C} - \{0\}$ have, respectively, a branch of $\log f(z)$ and a branch of $\sqrt[k]{f(z)}$ defined globally in U.

16. We say that a topological space is triangulable when it is homeomorphic to a polyhedron. Let $p\colon \widetilde{X} \to X$ be a covering. If X if triangulable, prove that $\widetilde{X}$ is also triangulable.

17. Let α_i be the number of simplices of dimension i of the polyhedron P. If X is a space homeomorphic to P, the number

$$\chi(X) = \sum_{i \geq 0} (-1)^i \alpha_i$$

is called the *Euler characteristic* of X. If $p\colon \widetilde{X} \to X$ is a covering with n leaves of the triangulable space X, prove that $\chi(\widetilde{X}) = n \cdot \chi(X)$. By taking into account that $\chi(S^{2n}) = 2$, show that a covering $p\colon S^{2n} \to X$ has at most two leaves. Therefore, if the universal covering of X is S^{2n}, then X is homeomorphic to S^{2n} or $\pi_1(X) = \mathbb{Z}_2$. Conclude also that the only finite group that acts freely on S^{2n} is $\mathbb{Z}_2$.

18. Let $p\colon \widetilde{M} \to M$ be a differentiable covering (by this we mean, in particular, that p is a local diffeomorphism between the manifolds $\widetilde{M}$ and M). If M admits a continuous non-null tangent vector field, the same happens to $\widetilde{M}$.

19. Use the first part of Exercise 15 to prove the converse of Exercise 16 in the case where $\widetilde{M}$ is compact. (Admit the theorem from differential topology that relates $\chi(M)$ with the existence of continuous and non-null tangent vector fields to M.)

20. Let ω be a closed differential form of degree 1 and class C^∞ in the manifold M and x_0 a point of M. Consider the subgroup $H \subset \pi_1(M, x_0)$ whose elements are the homotopy classes $\alpha = [a]$ of the paths $a\colon (I, \partial I) \to (M, x_0)$ such that $\int_a \omega = 0$ and the covering $p\colon \widetilde{M} \to M$ such that $p_{\#}\pi_1(M, \widetilde{x}_0) = H$ for some $\widetilde{x}_0 \in p^{-1}(x_0)$. Prove that there exists a function $f\colon \widetilde{M} \to \mathbb{R}$, of class C^∞, such that $p^*\omega = df$.

21. In Exercise 20, show that the covering $p\colon \widetilde{M} \to M$ is regular and that its automorphism group $G(\widetilde{M}|M)$ is isomorphic to the group of "periods" of ω; that is, the additive group of real numbers $\int_a \omega$, where $[a] \in \pi_1(M, x_0)$.

22. Let Z be the simply connected (but not locally connected) space defined in Exercise 16 of Chapter 2, and $Y \subset Z$ the arc there mentioned. Denote by X' the interval $[0, 1/\pi]$ of the horizontal axis and let $W = X' \cup Y$. Define a continuous bijection $\varphi\colon Z \to W$ by setting $\varphi(x, sen(1/x)) = x$ if $x \in [0, 1/\pi]$, and $\varphi(x, y) = (x, y)$ if $(x, y) \in Y$. Show that $\varphi^{-1}\colon W \to Z$ is not continuous. Consider a homeomorphism $h\colon W \to S^1$ and show that the continuous bijection $g = h \circ \varphi\colon Z \to S^1$ does not have a continuous lifting $\widetilde{g}\colon Z \to \mathbb{R}$ with respect to the universal covering $\xi\colon \mathbb{R} \to S^1$. Conclude that $g\colon Z \to S^1$ is not homotopic to a constant. (See Exercise 2, above.)

Chapter 8

Oriented Double Covering

In this chapter, we study an example of covering that has applications to topology and geometry. Since we will treat this topic in detail, we decided to include it as a separate chapter, which justifies itself because there are few comprehensive expositions of this subject in the literature.

8.1 Orientation of a Vector Space

Let E be a vector space of dimension m over the field of real numbers.

A *basis* in E is an ordered list $\mathcal{E} = (e_1, \ldots, e_m)$ of m linearly independent vectors. If $\mathcal{F} = (f_1, \ldots, f_m)$ is another basis in E, there exists a unique $m \times m$ invertible real matrix, $A = (a_{ij})$ such that

$$f_j = \sum_{i=1}^{m} a_{ij} e_i$$

for every $j = 1, 2, \ldots, m$. A is called the *transition matrix* from the basis $\mathcal{E}$ to the basis $\mathcal{F}$.

Given two bases $\mathcal{E}$ and $\mathcal{F}$ in E, we say that $\mathcal{E}$ and $\mathcal{F}$ are *equally oriented*, and we denote this by $\mathcal{E} \equiv \mathcal{F}$, when the transition matrix from $\mathcal{E}$ to $\mathcal{F}$ has a positive determinant. The relation $\mathcal{E} \equiv \mathcal{F}$ is an equivalence relation in the set of bases of the space E. Since a transition matrix has either a positive or a negative determinant, this equivalence relation has precisely two equivalence classes.

Each one of these equivalence classes is called an *orientation* of the vector space E.

Thus, an orientation in the vector space E is a set $\mathcal{O}$ of bases of E with the following property: if a basis $\mathcal{E}$ belongs to $\mathcal{O}$ and $\mathcal{F}$ is any basis in E, then $\mathcal{F} \in \mathcal{O}$ if, and only if, the transition matrix from $\mathcal{E}$ to $\mathcal{F}$ has a positive determinant.

Given an orientation $\mathcal{O}$ in the vector space E, the other orientation of E is called the *opposite orientation* of $\mathcal{O}$ and it will be denoted by $-\mathcal{O}$.

Every basis $\mathcal{E}$ in E defines an orientation $\mathcal{O}$ in the space E. $\mathcal{O}$ consists of the bases of E that are equally oriented with respect to the basis $\mathcal{E}$.

In order that two bases $\mathcal{E} = (e_1, \dots, e_m)$ and $\mathcal{F} = (f_1, \dots, f_m)$ be equally oriented it is necessary and sufficient that there exist m paths $h_j : I \to E$ such that, for every $t \in I$ the list $\mathcal{H}(t) = (h_1(t), \dots, h_m(t))$ is a basis of E, with $\mathcal{H}(0) = \mathcal{E}$ and $\mathcal{H}(1) = \mathcal{F}$. To prove this, we should recall that the set of $m \times m$ matrices with positive determinant is pathwise connected. If we have

$$f_j = \sum_{i=1}^{m} a_{ij} e_i, \quad j = 1, \dots, m, \quad \text{with} \quad \det(a_{ij}) > 0,$$

then we take a matrix path $A(t) = (a_{ij}(t))$, satisfying $\det A(t) > 0$ for all $t \in I$, $A(0) = m \times m$ identity matrix, $A(1) = (a_{ij})$, and we set

$$h_j(t) = \sum_{i=1}^{m} a_{ij}(t) e_i.$$

Thus, two bases of E are equally oriented if, and only if, one of them can be continuously deformed onto the other, in such a way that at each instant of the deformation we have a basis of E. (For topological facts in E, we take an arbitrary norm of the space. It is well known that all norms in a finite dimensional vector space define the same topology.)

An *oriented vector space* is a pair $(E, \mathcal{O})$, where E is a vector space of finite dimension over the field of real numbers and $\mathcal{O}$ is an orientation of E. It is very common to refer to such a space by using only the vector space notation E; that is, the orientation does not appear explicitly.

The vector space $\mathbb{R}^m$ will always be considered with the orientation defined by the canonical basis $\mathcal{E} = (e_1, \dots, e_m)$, where $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$.

In an oriented vector space $(E, \mathcal{O})$, the bases that belong to $\mathcal{O}$ are called *positive* and the other bases are called *negative*.

Let E, F be oriented vector spaces with the same dimension m. An isomorphism $f : E \to F$ is said to be *positive* when it transforms any positive basis of E onto a positive basis of F. In order that this occur, it suffices for f to transform *one* positive basis of E onto a positive basis of F. When an isomorphism from E onto F is not positive, we say that it is *negative*.

A linear transformation $T : \mathbb{R}^m \to \mathbb{R}^m$ is a positive isomorphism if, and only if, its matrix with respect to the canonical basis of $\mathbb{R}^m$ has determinant > 0.

When an isomorphism $f\colon E \to F$, between two oriented vector spaces, is positive, we say that f is *orientation preserving*. If only one the these vector spaces is oriented, the requirement that the isomorphism $f\colon E \to F$ be positive determines in a unique way an orientation in the other space.

8.2 Orientable Manifolds

By "manifold," we mean differentiable manifold (Hausdorff with countable basis), of a certain class C^k which we specify when it is necessary. The differentiable maps will belong to this class C^k.

Let M, N be manifolds of the same dimension. For each $x \in M$ and each $y \in N$ we choose, arbitrarily, an orientation $\mathcal{O}_x$ in T_xM (the vector space tangent to M at the point x) and an orientation $\mathcal{O}'_y$ in T_yN. Let $f\colon M \to N$ be a local diffeomorphism. We say that f is *positive* (with respect to the chosen orientations) when, for each $x \in M$, the linear isomorphism $f'(x)\colon T_xM \to T_{f(x)}N$ is positive. Analogously, we define a *negative* local diffeomorphism; in this case, we must require that, for all $x \in M$, the linear isomorphism $f'(x)\colon T_xM \to T_{f(x)}N$ reverses the orientation. We must observe that there may exist local diffeomorphisms which are neither positive nor negative.

Evidently, it is not interesting to choose, in a random way, an orientation in each tangent vector space of a manifold without any correlation with one another. We impose now that this choice be, in a certain sense, continuous.

An *orientation* $\mathcal{O}$ in a differentiable manifold M is a correspondence that associates to each point $x \in M$ an orientation $\mathcal{O}_x$ in the tangent vector space T_xM, in such a way that every point $x \in M$ belongs to the domain U of a positive coordinate system $\varphi\colon U \to \mathbb{R}^m$. (That is, for each $x \in U$, the derivative $\varphi'(x)\colon (T_xM, \mathcal{O}_x) \to \mathbb{R}^m$ preserves orientation.)

An *oriented manifold* is a pair $(M, \mathcal{O})$, where M is a differentiable manifold and $\mathcal{O}$ is an orientation in M.

A manifold is said to be *orientable* when it is possible to define some orientation in it.

Let $\mathcal{O}$ be an orientation in a manifold M. We denote by $-\mathcal{O}$ the correspondence that associates to each $x \in M$ the orientation $-\mathcal{O}_x$ in T_xM, opposite to $\mathcal{O}_x$. It is easy to see that $-\mathcal{O}$ is an orientation of M, called *opposite orientation* of $\mathcal{O}$.

Example 8.1. The Euclidean space $\mathbb{R}^m$ is orientable. In fact, $\mathbb{R}^m$ is oriented: We always consider it with its natural orientation. More generally, every Lie group G is orientable: We choose an arbitrary orientation in the tangent space T_eG (e is the neutral element of G) and we extend it to each tangent space T_gG by requiring that the linear isomorphism $L_g\colon T_eG \to T_gG$, the derivative of the left translation $h \mapsto gh$, be positive at the point e. ◁

Example 8.2. Every open subset U of an orientable manifold M is an orientable manifold. In fact, for each $x \in U$, we have $T_xU = T_xM$. An orientation of M determines, in a natural way, an orientation in U, called the *induced orientation*. This example is a particular case of the next one. ◁

Example 8.3. Let $f\colon M \to N$ be a local diffeomorphism. An orientation $\mathcal{O}'$ in N determines, by means of f, an orientation $\mathcal{O}$ in M, characterized by the property of turning $f\colon (M, \mathcal{O}) \to (N, \mathcal{O}')$ into a positive map; that is, for each $x \in M$, the linear isomorphism $f'(x)\colon T_xM \to T_{f(x)}N$ preserves orientation. In fact, this condition defines the correspondence $x \mapsto \mathcal{O}_x$. In order to obtain a positive coordinate system $\varphi\colon U \to \mathbb{R}^m$ around the point $x \in M$, we just have to take a positive coordinate system $\psi\colon V \to \mathbb{R}^m$ around the point $y = f(x) \in N$, an open set $U \ni x$ in M that it is mapped diffeomorphically by f onto a subset of V, and we set $\varphi = \psi \circ f$. The orientation $\mathcal{O}$ is said to be *induced* by f. In particular, if N is orientable and there exists a local diffeomorphism $f\colon M \to N$, then M is orientable. ◁

Soon we will present examples of nonorientable manifolds.

Two coordinate systems, $\varphi\colon U \to \mathbb{R}^m$ and $\psi\colon V \to \mathbb{R}^m$, in a manifold M are said to be *compatible* when $U \cap V = \varnothing$, or when $U \cap V \neq \varnothing$ and the change of coordinates $\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V)$ has a positive Jacobian determinant at every point of $\varphi(U \cap V)$. An atlas in M is said to be *coherent* when any two of its coordinate systems are compatible.

In an oriented manifold M, the set of positive coordinate systems is a coherent atlas. Since we are taking all of the positive systems, this is a *maximal* coherent atlas; that is, it is not a proper subset of any coherent atlas on M.

Conversely, if there exists a coherent atlas $\mathcal{A}$ in the manifold M, we define an orientation $\mathcal{O}_x$ in each tangent space T_xM by taking a coordinate system $\varphi\colon U \to \mathbb{R}^m$ that belongs to $\mathcal{A}$ and requiring that $\varphi'(x)\colon T_xM \to \mathbb{R}^m$ preserve orientation. The orientation of each tangent space T_xM is well defined because of the compatibility of the systems in $\mathcal{A}$. Note that the way that we defined $\mathcal{O}_x$ shows that we obtain an orientation of M.

Every coherent atlas in a manifold M is contained in a unique maximal coherent atlas. We could have defined, equivalently, an orientation of M as a maximal coherent atlas. The definition we gave has a better geometric flavor.

We say that an atlas $\mathcal{A}$, in an oriented manifold M, is *positive* when any coordinate systems belonging to $\mathcal{A}$ is positive relatively to the orientation of M. This means that $\mathcal{A}$ is contained in the maximal coherent atlas of M.

Proposition 8.1. *Let M, N be two oriented manifolds with the same dimension and $f\colon M \to N$ be a local diffeomorphism. The set of points $x \in M$ at which the derivative $f'(x)\colon T_xM \to T_{f(x)}N$ is positive is an open subset of M.*

Proof. Given $x \in M$ and $y = f(x) \in N$, let $\varphi\colon U \to \mathbb{R}^m$ and $\psi\colon V \to \mathbb{R}^m$ be coordinate systems in M and N respectively, with $x \in U$, $y \in V$ and $f(U) \subset V$. Then $f'(x)\colon T_xM \to T_yN$ is positive if, and only if, $(\psi \circ f \circ \varphi^{-1})'(\varphi(x))\colon \mathbb{R}^m \to \mathbb{R}^m$ has a positive Jacobian. Since this Jacobian is a continuous function of x, this concludes the proof of the proposition. □

Corollary 8.1. *Let M and N be oriented manifolds. If M is connected, then a local diffeomorphism $f\colon M \to N$ is either positive or negative.*

In fact, the set of all points $x \in M$ at which the derivative $f'(x)\colon T_xM \to T_yN$ reverses the orientation is also open. Since this set and the set of Proposition 8.1 are disjoint, one of them is empty by virtue of the connectedness of M.

Corollary 8.2. *Let $\varphi\colon U \to \mathbb{R}^m$ be a coordinate system in an oriented manifold M. If the domain U is connected, then φ is either positive or negative.*

Corollary 8.3. *In a connected oriented manifold, there are two possible orientations.*

In fact, consider the orientations $\mathcal{O}$ and $\mathcal{O}'$ in a connected manifold M. The identity map $f\colon (M, \mathcal{O}) \to (M, \mathcal{O}')$ is a local diffeomorphism. Hence, either f is positive (and in this case, $\mathcal{O} = \mathcal{O}'$) or f is negative (and then $\mathcal{O}' = -\mathcal{O}$).

Corollary 8.4. *Suppose that, in a manifold M, there exist coordinate systems $\varphi\colon U \to \mathbb{R}^m$, $\psi\colon V \to \mathbb{R}^m$, with connected domains U, V, such that at two points of $\varphi(U \cap V)$ the change of coordinates $\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V)$ has Jacobian determinants with distinct signs. Then M is nonorientable.*

Remark. In the case of Corollary 8.4, the intersection $U \cap V$ is necessarily disconnected.

Example 8.4. If M and N are orientable manifolds, the same happens with their product $M \times N$. In fact, let $\mathcal{A}$ and $\mathcal{B}$ be the coherent atlas, respectively, in M and N, defining the orientations of these manifolds. The atlas $\mathcal{A} \times \mathcal{B}$ is coherent because if $\varphi_1,\ \varphi_2 \in \mathcal{A}$ e $\psi_1,\ \psi_2 \in \mathcal{B}$, then $(\varphi_2 \times \psi_2) \circ (\varphi_1 \times \psi_1)^{-1} = (\varphi_2 \circ \varphi_1^{-1}) \times (\psi_2 \circ \psi_1^{-1})$ has as Jacobian determinant the product of the (positive) Jacobians of $\varphi_2 \circ \varphi_1^{-1}$ and $\psi_2 \circ \psi_1^{-1}$.

The orientation defined in $M \times N$ by the atlas $\mathcal{A} \times \mathcal{B}$ is called *product* of the orientations of M and N (in this order). If $(u_1, \ldots, u_m)$ and $(v_1, \ldots, v_m)$ are positive bases in T_xM and T_yN respectively, then $(u_1,\ldots,u_m,v_1,\ldots,v_m)$ is a positive basis in $T_{(x,y)}(M \times N)$.

Conversely, if the product $M \times N$ is an orientable manifold, then each of the manifolds M, N is orientable. In fact, fix an orientation in $M \times N$ and a coordinate system $\overline{\psi}\colon V \to \mathbb{R}^n$ in N, whose domain V is connected. For each $x \in M$, take a coordinate system $\varphi\colon U \to \mathbb{R}^m$ in M, with $x \in U$, such that $\varphi \times \overline{\psi}$ is positive in $M \times N$. The systems thus obtained form an atlas $\mathcal{A}$ in M. We claim that $\mathcal{A}$ is coherent. In fact, if $\varphi_1,\ \varphi_2 \in \mathcal{U}$ then the Jacobian of

$$(\varphi_2 \times \overline{\psi}) \circ (\varphi_1 \times \overline{\psi})^{-1} = (\varphi_2 \circ \varphi_1^{-1}) \times (\overline{\psi} \circ \overline{\psi}^{-1}) = (\varphi_2 \circ \varphi_1^{-1}) \times id$$

is positive at each point $(\varphi_1(x), \overline{\psi}(y))$. But this equals the Jacobian of $\varphi_2 \circ \varphi_1^{-1}$ at the point $\varphi_1(x)$. In a similar way, we prove that N is orientable. ◁

Consider a fixed point $b \in N$ and a basis $(v_1, \ldots, v_n)$ in T_bN. For each $x \in M$, a basis $(u_1, \ldots, u_m)$ in T_xM is positive if, and only if, the basis $(u_1, \ldots, u_m, v_1, \ldots, v_m)$ is positive in $T_{(x,b)}(M \times N)$.

Example 8.5. The map $f\colon S^m \times \mathbb{R} \to \mathbb{R}^{m+1} - \{0\}$, defined by $f(x,t) = e^t \cdot x$, is a diffeomorphism of the product $S^m \times \mathbb{R}$ onto the open subset $\mathbb{R}^{m+1} - \{0\}$ of the Euclidean space. Hence, $S^m \times \mathbb{R}$ is orientable. It follows from the previous example that the sphere S^m is orientable. Given $x \in S^m$, a basis $(v_1, \ldots, v_m)$ of T_xS^m is positive if, and only if $\det[x, v_1, \ldots, v_m] > 0$. ◁

Example 8.6. Let $A = (0,5) \times (0,1)$ be the open rectangle with base 5 and height 1. Given two integers $i < j$ in the interval $[0,5]$, we let $A_{ij} = (i,j) \times (0,1)$ be a rectangle with base $j - i$ and height 1.

The *Moebius band* M is the quotient space of A by the equivalence relation that identifies each point $(s,t) \in A_{01}$ with $(s+4, 1-t) \in A_{45}$ (see Figure 8.1). Let $\pi\colon A \to M$ be the quotient map. The restrictions $\pi|A_{03}$ e $\pi|A_{25}$ are, respectively, homeomorphisms onto open sets U and V in M. We denote by $\varphi\colon U \to A_{03}$ and $\psi\colon V \to A_{25}$ their inverses. We see that

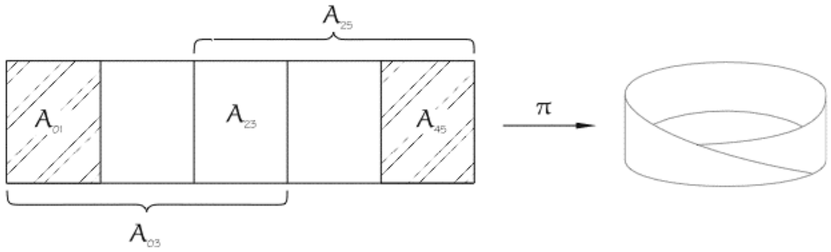


Figure 8.1. The Moebius band.

$\varphi(U \cap V) = A_{01} \cup A_{23}$ and $\psi(U \cap V) = A_{23} \cup A_{45}$. Moreover, the change of coordinates

$$\psi \circ \varphi^{-1} \colon A_{01} \cup A_{23} \to A_{23} \cup A_{45}$$

is the identity in A_{23} and it is given by $(s,t) \mapsto (s+4, 1-t)$ in A_{01}. It follows from Corollary 8.4 that the Moebius band is a nonorientable manifold. ◁

Example 8.7. The antipodal map $\alpha \colon S^m \to S^m$, $\alpha(x) = -x$ is a diffeomorphism, with $\alpha^{-1} = \alpha$. Let us check whether α preserves or reverses the orientation of S^m. Given $x \in S^m$, we set $E_x = T_x S^m$. We have $E_x = E_{-x}$. A basis $(v_1, \ldots, v_m)$ em E_x is positive if, and only if, $\det[x, v_1, \ldots, v_m] > 0$. It results from this that, although the nonoriented vector spaces E_x and E_{-x} are the same, we have $\mathcal{O}_{-x} = -\mathcal{O}_x$; that is, the orientations of E_x and E_{-x} do not coincide. The derivative $\alpha'(x) \colon E_x \to E_{-x}$ is given by the multiplication by -1. With respect to the orientations $\mathcal{O}_x$ and $\mathcal{O}_{-x}$ adopted in these spaces, $\alpha'(x)$ is a positive isomorphism if, and only if, m is odd. Thus, the antipodal map $\alpha \colon S^m \to S^m$ preserves orientation when m is odd and reverses it when m is even. ◁

Example 8.8. We prove now that the real projective space P^m is orientable when m is odd and that it is nonorientable when m is even. With this in mind, consider the canonical projection $\pi \colon S^m \to P^m$, which is a local diffeomorphism, and the antipodal map $\alpha \colon S^m \to S^m$. We have $\pi \circ \alpha = \pi$. If m is odd, we define an orientation in each tangent space $T_y P^m$, $y = \pi(x)$, by requiring that the linear isomorphism $\pi'(x) \colon T_x S^m \to T_y P^m$ be positive. It seems that there is an ambiguity, because we also have $y = \pi(-x)$. But, since $\pi'(-x) \circ \alpha'(x) = \pi'(x)$ and $\alpha'(x)$ is positive, the isomorphism $\pi'(-x)$ would induce the same orientation in $T_y P^m$. This defines an orientation in P^m.

Conversely, assume that P^m is orientable. Since S^m is connected, we can (see Corollary 8.1) choose the orientation of P^m in such a way that

$\pi\colon S^m \to P^m$ is positive. Now we fix $x \in S^m$. Since the isomorphisms $\pi'(-x)$ and $\pi'(x)$ are both positive, it follows that $\alpha'(x) = \pi'(-x)^{-1} \circ \pi'(x)$ is positive and therefore m is odd, according to the previous example. ◁

8.3 Properly Discontinuous Groups of Diffeomorphisms

Let $f\colon M \to N$ be a local diffeomorphism. When N is orientable, we know that f induces an orientation in M. Consider now the inverse situation: Supposing that M is orientable, is it possible to define, by using f, an orientation in N? The particular case $\pi\colon S^m \to P^m$ was solved in Example 8.8. The hypothesis that f is surjective is, evidently, necessary.

Proposition 8.2. *Let $f\colon M \to N$ be a surjective local diffeomorphism, defined on a connected oriented manifold. In order that N be orientable, it is necessary and sufficient that, for any x, $y \in M$ with $f(x) = f(y)$, the linear isomorphism $f'(y)^{-1} \circ f'(x)\colon T_xM \to T_yM$ be positive.*

Proof. If the condition holds, we define an orientation in N by taking, for each $b \in N$, a point $x \in f^{-1}(b)$ and imposing that the linear isomorphism $f'(x)\colon T_xM \to T_bN$ be positive. The admitted condition means that the orientation thus defined in each T_bN does not depend on the choice of the point x in $f^{-1}(b)$. Moreover, if we take in M a positive coordinate system $\varphi\colon U \to \mathbb{R}^m$, defined in an open set $U \ni x$ which is mapped diffeomorphically onto an open set $V \ni b$, the composite map $\psi = \varphi \circ f^{-1}\colon V \to \mathbb{R}^m$ is a positive coordinate system in N. This shows that we have indeed an orientation in N.

Conversely, let M, N be oriented manifolds. (Here we use the connectedness of M.) Then $f\colon M \to N$ is either positive or negative. By changing, if necessary, the orientation of N, we may suppose that f is positive. Then, for any x, $y \in M$ with $f(x) = f(y)$, the linear isomorphisms $f'(x)\colon T_xM \to T_{f(x)}N$, $f'(y) = T_yM \to T_{f(y)}N$ are positive and consequently the isomorphism $f'(y)^{-1} \circ f'(x)$ is also positive. □

A frequent situation where Proposition 8.2 applies is that of a properly discontinuous group of diffeomorphisms, as we explain now.

Proposition 8.3. *Let M be a connected manifold of class C^k and G be a properly discontinuous group of diffeomorphisms of class C^k in M. If the quotient space M/G is Hausdorff then there exists a unique manifold structure of class C^k in M/G such that the quotient map $\pi\colon M \to M/G$ is a local diffeomorphism of class C^k. Suppose that M is oriented. In order that M/G be orientable, it is necessary and sufficient that each diffeomorphism belonging to G preserve orientation.*

Proof. Consider an open covering of M such that each open set contains at most one point of each orbit of G. Let U, V be two of these open sets. Then $\pi|U$ and $\pi|V$ are homeomorphisms onto open sets $U_0, V_0 \subset M/G$. Suppose that $U_0 \cap V_0 \neq \varnothing$. Let $A = (\pi|U)^{-1}(U_0 \cap V_0)$ and $B = (\pi|V)^{-1}(U_0 \cap V_0)$. We claim that the homeomorphism $\xi = (\pi|V)^{-1} \circ (\pi|U)\colon A \to B$ is of class C^k. In order to prove this, we remark that if $x \in A$, then $\xi(x) = y \Rightarrow \pi(x) = \pi(y) \Rightarrow y = \alpha(x)$ for some $\alpha \in G$. That is, for all $x \in A$, there exists $\alpha \in G$ such that $\xi(x) = \alpha(x)$. By fixing $x \in A$, let Z be an open set such that $x \in Z \subset A$ and $\xi(Z) \subset \alpha(A)$. We know that $\beta \neq \alpha \Rightarrow \beta(A) \cap \alpha(A) = \varnothing$. Thus, $\xi(Z) \cap \beta(Z) = \varnothing$ for all $\beta \neq \alpha$. This shows that $\xi(y) \neq \beta(y)$ for all $y \in Z$ and therefore, $\xi|Z = \alpha|Z$. It follows that $\xi \in C^k$ in Z. Since Z is a neighborhood of an arbitrary point $x \in A$, we conclude that $\xi\colon A \to B$ is of class C^k. Thus, the topological space M/G is covered by open domains of homeomorphisms $\varphi = (\pi|U)^{-1}\colon U_0 \to U$, which take values in open subsets of M, in such a way that, when the domain of $\psi\colon V_0 \to V$ intersects that of φ, then the change of coordinates $\xi = \psi \circ \varphi^{-1}\colon \varphi(U_0 \cap V_0) \to \psi(U_0 \cap V_0)$ is of class C^k. Since φ is open, M/G inherits from M a countable basis. Thus the homeomorphisms φ define a manifold structure of class C^k in M/G. Evidently, the quotient map $\pi\colon M \to M/G$ is a local diffeomorphism. The uniqueness of the structure in M/G follows easily from this property. Suppose now that M is oriented and each $\alpha \in G$ is a positive diffeomorphism of M. Then the local diffeomorphism $\pi\colon M \to M/G$ satisfies $\pi(x) = \pi(y) \Rightarrow y = \alpha(x)$, with $\alpha \in G$. Since $\pi \circ \alpha = \pi$, we conclude that $\pi'(y) \circ \alpha'(x) = \pi'(x)$; that is, $\pi'(y)^{-1} \circ \pi'(x) = \alpha'(x)$, which is a positive linear isomorphism. It follows from Proposition 8.2 that M/G is orientable.

Conversely, if M/G is orientable, we take arbitrarily $\alpha \in G$ and $x \in M$. Let $y = \alpha(x)$. Then $\pi(x) = \alpha(y)$. By Proposition 8.2, the isomorphism $\pi'(y)^{-1} \circ \pi'(x)$ is positive. But this isomorphism coincides with $\alpha'(x)$. It follows that α is positive, which completes the proof. □

Example 8.9. Now we will see that the orientability of the projective space (P^m is orientable if, and only if, m is odd) is explained more generally by Proposition 8.3. In Example 7.8, where we have the group of translations of $\mathbb{R}^n$ by vectors with integral coordinates, the quotient space is the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n = S^1 \times \ldots \times S^1$ (n factors). Since each translation $x \mapsto x + v$ is a positive diffeomorphism of $\mathbb{R}^n$, we conclude that the n-dimensional torus is orientable, a fact that we already knew, because it is the product of n orientable manifolds, or because it is a Lie group. ◁

Example 8.10. Let $M = S^1 \times \mathbb{R}$. The diffeomorphism $h\colon M \to M$, defined by $h(x, y, z) = (x, -y, z + 1)$, generates a cyclic group $G = \{h^n; n \in \mathbb{Z}\}$ of

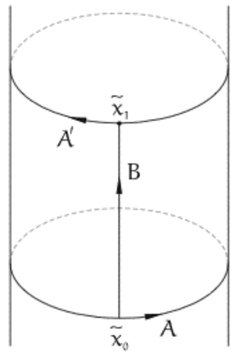


Figure 8.2.

diffeomorphisms of M, which is properly discontinuous. We can imagine M as a vertical cylinder in $\mathbb{R}^3$. The diffeomorphism h maps each horizontal circle of M onto the circle located one unity above it but, when doing this, it also reflects the circle around a diameter. The n-th iterate h^n is a positive diffeomorphism of M if n is even and it is negative when n is odd. It follows that the quotient space M/G is a nonorientable manifold. It is called the *Klein bottle*. ◁

Example 8.11. In the example above, let $p\colon M \to M/G$ be the canonical covering map of the cylinder M onto the Klein bottle M/G. The fundamental group of the cylinder is cyclic infinite, generated by the homotopy class of the path $A(s) = (\cos 2\pi s, \sin 2\pi s, 0)$. Consider now the paths $A'(s) = (\cos 2\pi s, -\sin 2\pi s, 1)$, $B(s) = (1, 0, s)$, in the cylinder, and their images $a = p \circ A = p \circ A'$ and $b = p \circ B$ in the Klein bottle (see Figure 8.2). Both a and b are closed paths, with base at the point $x_0 = p(\tilde{x}_0) = p(\tilde{x}_1)$, where $\tilde{x}_0 = (1, 0, 0)$ and $\tilde{x}_1 = (1, 0, 1)$. We claim that we do not have $ab \cong ba$ in the Klein bottle. In fact, by taking liftings starting at the point $\tilde{x}_0$, we have $\widetilde{ab} = AB \simeq A$ and $\widetilde{ba} = BA' \simeq A' \simeq A^{-1}$ (free homotopies). Since A and A^{-1} are not freely homotopic in the cylinder, it follows that $\widetilde{ab}$ is not homotopic to $\widetilde{ba}$ in this cylinder. Consequently, we do not have $ab \cong ba$ in the Klein bottle. (See Proposition 6.10.) Thus, we have proved, once more, that the fundamental group of the Klein bottle is not commutative. ◁

8.4 Oriented Double Covering

An *oriented double covering* is a map $p\colon \widetilde{M} \to M$, of class C^k, with the following properties:

1. M is a connected manifold, $\widetilde{M}$ is an oriented manifold, and p is a local diffeomorphism;
2. For each $y \in M$, the inverse image $p^{-1}(y)$ contains exactly two points;
3. If $p(x_1) = p(x_2)$ with $x_1 \neq x_2$, then the linear isomorphism $p'(x_2)^{-1} \colon p'(x_1) \colon T_{x_1}\widetilde{M} \to T_{x_2}\widetilde{M}$ is negative.

By virtue of Proposition 6.5, an oriented double covering $p \colon \widetilde{M} \to M$ is a proper covering map.

Sometimes we say, rather incorrectly, that $\widetilde{M}$ (not p) is an oriented double covering of M.

Example 8.12. When m is even, the quotient map $\pi \colon S^m \to P^m$ is an oriented double covering of the projective space P^m. When m is odd, π does not satisfy the Condition 3. above. ◁

Example 8.13. Let $\alpha \colon \widetilde{M} \to \widetilde{M}$ be a negative involution ($\alpha \circ \alpha = id$), of class C^k, without fixed points, on a connected oriented manifold. Then $\{\alpha, id\}$ is a properly discontinuous group of diffeomorphisms of $\widetilde{M}$. We indicate by $\widetilde{M}/\alpha$ the quotient manifold. (See Proposition 8.3.) The quotient map $\pi \colon \widetilde{M} \to \widetilde{M}/\alpha$ is an oriented double covering. An example of this situation is $\widetilde{M} = S^1 \times S^1 =$ the two-dimensional torus. We define $\alpha \colon \widetilde{M} \to \widetilde{M}$ by setting $\alpha(z, w) = (\bar{z}, -w)$, where $\bar{z}$ is the complex conjugate of z. Then α is a negative involution without fixed points in the torus $\widetilde{M}$. The quotient manifold $\widetilde{M}/\alpha$ is diffeomorphic to the Klein bottle. The canonical projection $\pi \colon \widetilde{M} \to \widetilde{M}/\alpha$ shows that the torus is an oriented double covering of the Klein bottle. Later, we will show that every oriented double covering is essentially obtained in this way. ◁

Example 8.14. **(Product covering)** Let M be a connected oriented manifold. Take $M_1 = M \times \{1\}$ and $M_2 = M \times \{2\}$. Then $\widetilde{M} = M_1 \cup M_2$ is a manifold, the disjoint union of two diffeomorphic copies of M. Now define $p \colon \widetilde{M} \to M$ by setting $p(x, 1) = p(x, 2) = x$. Let's choose an orientation of $\widetilde{M}$ by requiring that $p|M_1$ be positive and $p|M_2$ negative. Then $p \colon \widetilde{M} \to M$ is an oriented double covering, called the *product covering*. ◁

More generally, we say that an oriented double covering $p \colon \widetilde{M} \to M$ is *trivial* when $\widetilde{M} = \widetilde{M}_1 \cup \widetilde{M}_2$ is the disjoint union of two open subsets, such that each one of them is mapped by p diffeomorphically onto M. We now show that this is essentially the only possible oriented double covering when the base space M is orientable.

It is important to note that if $\widetilde{M}$ is connected, then the base M of the oriented double covering $p\colon \widetilde{M} \to M$ must be a nonorientable manifold, according to Proposition 8.2. Observe also that $p\colon S^1 \to S^1$, $p(z) = z^2$ is a covering with two leaves, but it is not an oriented double covering.

Proposition 8.4. *Let $p\colon \widetilde{M} \to M$ be an oriented double covering. If $U \subset M$ is an oriented open set, then $p^{-1}(U) = \widetilde{U}_1 \cup \widetilde{U}_2$, is the disjoint union of two open sets, such that p maps each one of them diffeomorphically onto U. In $\widetilde{U}_1$, p is positive and in $\widetilde{U}_2$, it is negative.*

Proof. Let $\widetilde{U}_1 = \{x \in p^{-1}(U); p'(x) < 0\}$ and $\widetilde{U}_2 = \{x \in p^{-1}(U); p'(x) > 0\}$. Evidently, $p^{-1}(U) = \widetilde{U}_1 \cup \widetilde{U}_2$, the union of two disjoint open sets. In each $\widetilde{U}_i$, p is injective because $p(x_1) = p(x_2)$ with $x_1 \neq x_2$ would force $p'(x_1)$ and $p'(x_2)$ to have opposite signs. Moreover, each point $y \in U$ is the image of two points x_1, $x_2 \in p^{-1}(U)$; in one of them, the derivative of p is positive and in the other, it is negative. Hence, $p(\widetilde{U}_1) = p(\widetilde{U}_2) = U$. We conclude that $p|\widetilde{U}_1$ and $p|\widetilde{U}_2$ are bijections (and therefore, diffeomorphisms) onto U. □

Proposition 8.5. *Let $p\colon \widetilde{M} \to M$ be an oriented double covering. The following statements are equivalent:*

1. *M is orientable;*
2. *$\widetilde{M}$ is disconnected;*
3. *The covering $p\colon \widetilde{M} \to M$ is trivial.*

Proof. $1 \Rightarrow 2$: This follows from Proposition 8.2.

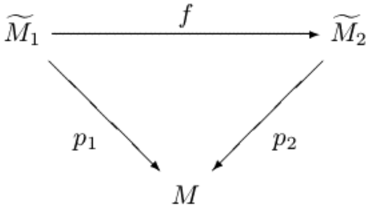
$2 \Rightarrow 3$: Suppose that $\widetilde{M}$ is disconnected and take a connected component C of $\widetilde{M}$. Since p is a proper local diffeomorphism, the image $p(C)$ of the open-closed set C is open and closed in the connected manifold M. Hence, $p(C) = M$. Thus, p maps each connected component of $\widetilde{M}$ onto M. Since the inverse image by p of each point of M has two points, we conclude that $\widetilde{M}$ cannot have more than two components. As a consequence, $\widetilde{M} = \widetilde{M}_1 \cup \widetilde{M}_2$ has precisely two connected components and p is injective in each one of them. Thus, $p|M_1$ and $p|M_2$ are diffeomorphisms onto M; that is, p is trivial.

$3 \Rightarrow 1$: Obvious. □

Corollary 8.5. *Let $p\colon \widetilde{M} \to M$ be an oriented double covering. $\widetilde{M}$ is connected if, and only if, M is nonorientable.*

Now we prove the uniqueness of the oriented double covering.

Proposition 8.6. *Let $p_1 \colon \widetilde{M}_1 \to M$ and $p_2 \colon \widetilde{M}_2 \to M$ be oriented double coverings of the same manifold M. There exists a unique positive diffeomorphism $f \colon \widetilde{M}_1 \to \widetilde{M}_2$ such that $p_2 \circ f = p_1$. This is illustrated by the commutative diagram below.*

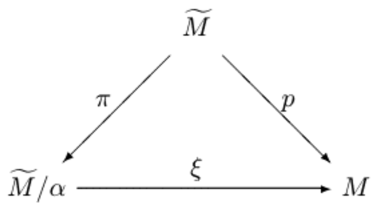


Proof. The conditions that f be positive and satisfy $p_2 \circ f = p_1$ already define a bijection $f \colon \widetilde{M}_1 \to \widetilde{M}_2$: For each point $x \in \widetilde{M}_1$, $f(x) = y$ is the point of $\widetilde{M}_2$, which is mapped by p_2 onto the point $z = p_1(x)$ in such a way that $p_2'(y)^{-1} \circ p_1'(x)$ becomes a positive linear isomorphism. (There exist two points of M_2 which are mapped by p_2 onto the point z, but only one of them satisfies the last condition.) Now we just have to prove that $f \in C^k$ if p_1 and p_2 are of class C^k. We fix $x \in \widetilde{M}_1$. Let $U \ni z = p_1(x)$ be a domain of a coordinate system in M; we orient U in such a way that $p_1^{-1}(U) = \widetilde{U}_1 \cup \widetilde{V}_1$ with $x \in \widetilde{U}_1$ and $p_1|\widetilde{U}_1$ be a positive diffeomorphism onto U. Then $p_2^{-1}(U) = \widetilde{U}_2 \cup \widetilde{V}_2$, where $p_2|\widetilde{U}_2$ is a positive diffeomorphism onto U. It follows that $y = f(x) \in \widetilde{U}_2$ and $f|\widetilde{U}_1 = (p_2|\widetilde{U}_2)^{-1} \circ (p_1|\widetilde{U}_1)$. □

The corollary below shows that every oriented double covering is essentially obtained by taking the quotient space of a manifold by a negative involution without fixed points.

Corollary 8.6. *Let $p \colon \widetilde{M} \to M$ be an oriented double covering. There exists a unique negative involution $\alpha \colon \widetilde{M} \to \widetilde{M}$, of class C^k, such that $p \circ \alpha = p$. The involution α does not have fixed points. There exists a unique diffeomorphism $\xi \colon \widetilde{M}/\alpha \to M$ such that $p = \xi \circ \pi$, where $\pi \colon \widetilde{M} \to \widetilde{M}/\alpha$ is the quotient map.*

In fact, let $\widetilde{M}_1$ and $\widetilde{M}_2$ be the same manifold $\widetilde{M}$ with two opposite orientations. The map p determines two oriented double coverings $p_i \colon \widetilde{M}_i \to M$ $(i = 1, 2)$, that differ from the original one only by the orientation of their domains. By Proposition 8.6, there exists a unique positive diffeomorphism $\alpha \colon \widetilde{M}_1 \to \widetilde{M}_2$ such that $p_2 \circ \alpha = p_1$. Going back to the original notation, $\alpha\colon \widetilde{M} \to \widetilde{M}$ is a diffeomorphism that reverses the orientation of $\widetilde{M}$; this, along with the equality $p \circ \alpha = p$, shows that, for each $x \in \widetilde{M}$, $\alpha(x) = y$ is the other point of $\widetilde{M}$ such that $p(x) = p(y)$. Thus, α does not have fixed points and $\alpha \circ \alpha = id$; that is, α is an involution. The fact that $\alpha \in C^k$ was proved in Proposition 8.6. Finally, considering the quotient space $\widetilde{M}/\alpha$, since the equivalence relation defined by p in $\widetilde{M}$ is the same defined by $\pi\colon \widetilde{M} \to \widetilde{M}/\alpha$, it follows that there exists a continuous bijection $\xi\colon \widetilde{M}/\alpha \to M$ such that $\xi \circ \pi = p$, as illustrated by the commutative diagram below.



Since p is a local diffeomorphism of class C^k, it follows that ξ is a diffeomorphism of class C^k.

Proposition 8.7. *Every connected manifold M of class C^k has an oriented double covering.*

Proof. Let $\widetilde{M}$ be the set of ordered pairs $(x, \mathcal{O}_x)$ where $x \in M$ and $\mathcal{O}_x$ is an orientation in the tangent space T_xM. We define a map $p\colon \widetilde{M} \to M$ by setting $p(x, \mathcal{O}_x) = x$. Clearly, for each $x \in M$, the inverse image $p^{-1}(x)$ contains exactly two points: $(x, \mathcal{O}_x)$ and $(x, -\mathcal{O}_x)$. We introduce now a manifold structure of class C^k in $\widetilde{M}$ in such a way that p becomes an oriented double covering of class C^k. For each oriented open set $U \subset M$, let $\widetilde{U}$ be the set of pairs $(x, \mathcal{O}_x)$ such that $x \in U$, and $\mathcal{O}_x$ is the orientation of U at the point x. The map $\varphi_U : \widetilde{U} \to U$, defined by $\varphi_U = p|\widetilde{U}$, is a bijection. The domains $\widetilde{U}$ of these bijections φ_U cover the set $\widetilde{M}$. Given $\varphi_U : \widetilde{U} \to U$ and $\varphi_V : \widetilde{V} \to V$, if $\widetilde{U} \cap \widetilde{V} \neq \varnothing$ then $\varphi_U|(\widetilde{U} \cap \widetilde{V}) = \varphi_V|(\widetilde{U} \cap \widetilde{V})$; hence the "change of coordinates" $\varphi_V \circ \varphi_U^{-1}\colon \varphi_U(\widetilde{U} \cap \widetilde{V}) \to \varphi_V(\widetilde{U} \cap \widetilde{V})$ is simply the identity map. Therefore, the atlas constituted by the bijections φ_U determines in $\widetilde{M}$ a manifold structure of class C^k, and $p\colon \widetilde{M} \to M$ is a local diffeomorphism with respect to this structure. Such structure is defined by the condition that each φ_U be a diffeomorphism. In order to show that $\widetilde{M}$ is orientable, we remark that $\widetilde{M}$ has a natural orientation, imposed by its definition: in each point $\tilde{x} = (x, \mathcal{O}_x) \in \widetilde{M}$, consider the orientation $\mathcal{O}_{\tilde{x}}$ which makes the linear isomorphism $p'(\tilde{x})\colon (T_{\tilde{x}}\widetilde{M}, \mathcal{O}_{\tilde{x}}) \to (T_xM, \mathcal{O}_x)$ positive. The map $p\colon \widetilde{M} \to M$ is an oriented double covering. □

Corollary 8.7. *Every simply connected manifold is orientable.*

In fact, let M be a simply connected manifold. Every covering of M with connected domain is a homeomorphism. Hence, the oriented double covering of M is disconnected; therefore, M is orientable.

8.5 Relations with the Fundamental Group

Let $p\colon \widetilde{M} \to M$ be an oriented double covering. Given a path $a\colon I \to M$ and a point $\tilde{x} \in \widetilde{M}$ such that $p(\tilde{x}) = a(0)$, there exists a unique path $\tilde{a}\colon I \to \widetilde{M}$ such that $p \circ \tilde{a} = a$ and $\tilde{a}(0) = \tilde{x}$. If we adopt for $\widetilde{M}$ the model presented in the proof of Proposition 8.7, we have $\tilde{x} = (x, \mathcal{O}_x)$ and the path $\tilde{a}$ can be interpreted as the continuation of the orientation $\mathcal{O}_x$, by continuity, along the path a. In fact, we have $\tilde{a}(s) = (a(s), \mathcal{O}_{a(s)})$ for all $s \in I$. Since $\tilde{a}(s)$ depends continuously on s, it is natural to say that the orientation $\mathcal{O}_{x(s)}$ also depends continuously on the parameter s.

Let $b\colon I \to M$ be another path in M with the same endpoints as a. If $a \cong b$ (that is, if a and b are homotopic with the endpoints fixed in M), then their liftings $\tilde{a}$ and $\tilde{b}$ starting from the same point $\tilde{x}$ are also homotopic with fixed endpoints in $\widetilde{M}$. In particular, $\tilde{a}(1) = \tilde{b}(1)$. This means that, starting from an orientation $\mathcal{O}_x$ in T_xM and extending it by continuity along the two homotopic paths with fixed endpoints, we obtain in the final the same orientation. In particular, if the manifold M is simply connected, it is possible to orient it by choosing an orientation $\mathcal{O}_{x_0}$ at a fixed point $x_0 \in M$ and, given any point $x \in M$, we connect x to x_0 using a path in M and we extend $\mathcal{O}_{x_0}$ continuously along this path. The orientation $\mathcal{O}_x$ thus obtained does not depend on the path chosen in order to connect x_0 to x because, since M is simply connected, any two of these paths are homotopic with fixed endpoints.

Consider now closed paths in M, with base at a point x_0. Given the oriented double covering $p\colon \widetilde{M} \to M$, the lifting of a closed path may be closed or open. In terms of the model of Proposition 8.7: extending continuously an orientation $\mathcal{O}_{x_0}$ along the path a, with $a(0) = a(1) = x_0$, it is possible to obtain, in the final, the orientation $\mathcal{O}_{x_0}$ or the opposite orientation $-\mathcal{O}_{x_0}$. This fact depends only on the path a, but not on the orientation $\mathcal{O}_{x_0}$. In the first case, we say that a is an *orienting path.* In the second case (in which, by extending $\mathcal{O}_{x_0}$ along a, we obtain in the end the opposite orientation $-\mathcal{O}_{x_0}$) we say that a is a *disorienting path.*

A manifold M is orientable if, and only if, every closed path in M is an orienting path. In the projective plane P^2, a projective line (image of half a great circle by the projection $\pi\colon S^2 \to P^2$) is a disorienting path.

Every closed path homotopic to a constant is an orienting path. In particular, if $\varphi\colon U \to \mathbb{R}^m$ is a coordinate system in M, and $\varphi(U)$ is the Euclidean ball, then every closed path contained in U is an orienting path. (We say then: Every sufficiently small closed path is an orienting path.)

The central circle of a Möbius ban is a disorienting path.

An interesting conjecture in cosmology states that the universe is an orientable manifold. Otherwise, a person who took a trip along a disorienting path would return mirrored: with the heart on the right side, writing everything in the opposite order and with the other hand. The arrows of his clock would move in the opposite sense and any books he carried with him would be illegible for us. On the other hand, he would think that everything here had changed while he was travelling.

Consider a nonorientable connected manifold M. Let a, b be two closed paths in M, with base in x_0. If $a \cong b$, then a is an orienting path if, and only if, b is also an orienting path. If two closed paths are orienting paths, their product is also an orienting path and so are their inverses. Thus, the homotopy classes of the orienting paths constitute a subgroup $H \subset \pi_1(M, x_0)$. H is the image of the fundamental group of $\widetilde{M}$ by the induced homomorphism $p_{\#}\colon \pi_1(\widetilde{M}, \tilde{x}_0) \to \pi_1(M, x_0)$. (It does not matter which is the chosen point $\tilde{x}_0$ over x_0, because every covering with two leaves is regular.)

In particular, we conclude that the fundamental group of a nonorientable manifold always has a subgroup of index 2.

8.6 Exercises

1. Let $\alpha\colon S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ be defined by $\alpha(z, t) = (-z, -t)$. Show that α is a negative diffeomorphism, with $\alpha \circ \alpha = id$, that $G = \{id, \alpha\}$ is a properly discontinuous group and that $M = (S^1 \times \mathbb{R})/G$ is the Möbius band. Conclude again that M is nonorientable.

2. In a differentiable manifold M, the domain of a coordinate system is an orientable submanifold, even when M is nonorientable.

3. Every complex analytic manifold is orientable.

4. If the fundamental group of a connected manifold has seven or nine elements, the manifold is orientable.

5. If, for each point of a spherical surface, there exists a straight line that varies continuously with the point, prove that at least one of these lines passes through the center of the sphere.

In the following exercises we use the notation below:

- $M \subset \mathbb{R}^m$ denotes a surface of dimension m and class C^∞.
- $TM = \{(x, v) \in \mathbb{R}^m \times \mathbb{R}^n; x \in M, v \in T_x M\}$ = *tangent bundle* of M.
- $T^1M = \{(x, u) \in TM; |u| = 1\}$ = *unit tangent bundle* of M.
- $\Delta M = \{(x, [u]); (x, u) \in T^1M, [u] = \{u, -u\}\}$ = *tangent direction bundle* of M.

6. Prove that TM and T^1M are surfaces of class C^∞ in $\mathbb{R}^n \times \mathbb{R}^n$, with dimensions $2m$ and $2m - 1$, respectively, both of them orientable.

7. Prove that the maps $\pi\colon TM \to M$ and $\pi^1 = \pi|T^1M\colon T^1M \to M$ are locally trivial fibrations with typical fiber $\mathbb{R}^m$ and S^{m-1} respectively.

8. Prove that the fibrations $\pi^1\colon T^1S^2 \to S^2$ and $\pi\colon SO(3) \to S^2$ (this last one was considered in Chapter 3) are equivalent; that is, there exists a diffeomorphism $\varphi\colon T^1S^2 \to SO(3)$ such that $\pi \circ \varphi = \pi^1$. In particular, the fundamental group of T^1S^2 is $\mathbb{Z}_2$.

9. Prove that ΔM is the quotient space of T^1M by the involution $(x, u) \mapsto (x, -u)$; hence, it is a differentiable manifold and the map $(x, u) \mapsto (x, [u])$ establishes T^1M as a covering of ΔM with two leaves. Conclude that the map $\overline{\pi}\colon \Delta M \to M$, where $\overline{\pi}(x, [u]) = x$, is a locally trivial fibration, with typical fiber P^{m-1}.

10. If M is connected, the same happens with TM and ΔM, and also with T^1M when $m \geq 2$. If M is compact, T^1M and ΔM are also compact.

11. TM and T^1M are orientable, independent of the orientability of M. ΔM is orientable if, and only if, m is odd.

12. A *continuous direction field* in the surface M is a correspondence $\delta\colon x \mapsto \delta(x)$ such that the map $\overline{\delta}\colon M \to \Delta M$, given by $\overline{\delta}(x) = (x, \delta(x))$, is continuous (a section). The direction field δ is said to be *orientable* when there exists a continuous unit tangent vector field $x \mapsto u(x)$ (that is, a continuous section $x \mapsto (x, u(x))$ of the unit tangent bundle $T^1M \to M$) such that $\delta(x) = \{u(x), -u(x)\}$ for all $x \in M$. If the surface M is simply connected, every continuous direction field is orientable. Conclude from this that the sphere S^2 (and, more generally, any sphere of even dimension) does not admit a continuous direction field.

13. Give an example of a continuous nonorientable direction field in $\mathbb{R}^2 - \{0\}$.

14. Consider $G = \{\pm 1, \pm i\}$ as a properly discontinuous group of homeomorphisms in S^3. Prove that the orbit space S^3/G is homeomorphic to ΔS^2. Conclude that the fundamental group of ΔS^2 is $\mathbb{Z}_4$.

15. Let $\pi\colon E \to B$ be a locally trivial fibration whose typical fiber is simply connected. Prove that $\pi_{\#}\colon \pi_1(E, x) \to \pi_1(B, y)$, $y = \pi(x)$, is an isomorphism. Conclude that, if $\dim M \geq 3$, T^1M and M have isomorphic fundamental groups. If, moreover, M is simply connected, then $\pi_1(\Delta M) = \mathbb{Z}_2$.

16. The homomorphism of the fundamental groups induced by the fibration $\overline{\pi}\colon \Delta M \to M$ is surjective. If $\dim M \geq 3$, its kernel is $\mathbb{Z}_2$.

17. Consider a continuous direction field δ, tangent to the surface M, and let $\widetilde{M} = \{(x, u) \in T^1M; [u] = \delta(x)\}$. Show that $p\colon \widetilde{M} \to M$, defined by $p(x, u) = u$, is a covering with two leaves and that there exists a continuous unit vector field $\widetilde{u}$ tangent to $\widetilde{M}$ such that $[p'(\widetilde{x}) \cdot \widetilde{u}(\widetilde{x})] = \delta(x)$, where $x = p(\widetilde{x})$.

18. With the notation of Exercise 17, show that the field of directions δ is orientable if, and only if, $\widetilde{M}$ is the union of two disjoint open sets, and the restriction of p to each one of them is a diffeomorphism onto M.

19. Use Exercise 17 from Chapter 7 in order to conclude that a compact surface admits a continuous tangent field of directions if, and only if, it admits a continuous, non-null tangent vector field.

Appendix
Proper Maps

A map $f\colon X \to Y$ between two topological spaces is said to be *closed* when, for every closed subset $F \subset X$, its image $f(F)$ is closed in Y. The following proposition characterizes closed maps. Note that it somehow expresses the continuity of the "inverse map" $y \mapsto f^{-1}(y)$, whose values are sets.

Proposition A.1. *In order that $f\colon X \to Y$ be closed, it is necessary and sufficient that, given arbitrarily $y \in Y$ and an open set $U \supset f^{-1}(y)$ in X, there exists an open set $V \subset Y$ such that $y \in V$ and $f^{-1}(y) \subset f^{-1}(V) \subset U$.*

Proof. (Necessary.) If f is closed then $f(X-U)$ is closed in Y and, since it does not contains y, there exists $V \ni y$ open set in Y, with $V \cap f(X-U) = \varnothing$. This means that $f^{-1}(y) \subset f^{-1}(V) \subset U$.

(Sufficient.) Suppose that the condition is satisfied, and take the closed set $F \subset X$. If $y \notin f(F)$ then $F \cap f^{-1}(y) = \varnothing$; hence the open set $U = X - F$ contains $f^{-1}(y)$. Then there exists an open set $V \ni y$ in Y, with $f^{-1}(V) \subset U$, which means that $V \cap f(F) = \varnothing$. Therefore $f(F)$ is closed in Y. □

A map $f\colon X \to Y$, between two topological spaces, is called *proper* when it is continuous, closed and the inverse image $f^{-1}(y)$ of each point $y \in Y$ is a compact subset of X.

For example, if X is compact and Y is Hausdorff, every continuous map $f\colon X \to Y$ is proper.

The inverse image $f^{-1}(y)$ by the inclusion map $f\colon (0,1) \to \mathbb{R}$ is a compact set, for each $y \in \mathbb{R}$, but f is not closed, therefore it is not a proper map. A constant map $f\colon X \to c \in Y$, defined on a non-compact space

X and taking values in a Hausdorff space Y, is closed but it is not proper because $f^{-1}(c) = X$ is not compact.

Proposition A.2. *Let $f: X \to Y$ be a proper map. If $K \subset Y$ is compact then $f^{-1}(K)$ is also compact.*

Proof. Let $\mathcal{U}$ a covering of $f^{-1}(K)$ by open sets $U \subset X$. For every $y \in K$, we can find a finite subcollection $\{U_1^y, U_2^y, \ldots, U_{n_y}^y\}$ of $\mathcal{U}$ covering the compact set $f^{-1}(y)$ and, by Proposition A.1, an open set $V_y \ni y$ in Y such that

$$f^{-1}(V_y) \subset U_1^y \cup U_2^y \cup \ldots \cup U_{n_y}^y.$$

We can also find points $y_1, \ldots, y_k \in K$ such that $K \subset V_{y_1} \cup \ldots \cup V_{y_k}$. Then

$$f^{-1}(K) \subset \bigcup_{j=1}^{k} \bigcup_{i=1}^{n_{y_j}} U_i^y,$$

which proves the proposition. □

Without imposing some restrictions, the converse of Proposition A.2 is false. But it is valid for most of the reasonable spaces. For example, we have the

Proposition A.3. *Let Y be a space whose topology has the following property: if $A \subset Y$ is such that $A \cap K$ is compact for every compact set $K \subset Y$ then A is closed in Y. Let $f: X \to Y$ a continuous map such that the inverse image $f^{-1}(K)$ of each compact set $K \subset Y$ is compact. Then f is closed. If Y is Hausdorff, then f is proper.*

Proof. Let $F \subset X$ be a closed set. For every compact set $K \subset Y$, $F \cap f^{-1}(K)$ is compact and therefore $f(F \cap f^{-1}(K)) = f(F) \cap K$ is compact. Thus $f(F)$ intersects each compact set $K \subset Y$ in a compact, hence $f(F)$ is closed in Y. □

Corollary A.1. *Let Y be a metrizable, or a Hausdorff locally compact space. A continuous map $f: X \to Y$ is proper if, and only if, for all compact set $K \subset Y$ the inverse image $f^{-1}(K)$ is compact.*

In fact, the topology of a Hausdorff locally compact space or of a metrizable space satisfies the condition of Proposition A.3.

Intuitively, the fact that a map $f: X \to Y$ is proper means that *if x approaches the boundary of the set X then $f(x)$ approaches the boundary* *of* Y. The precise formulation of this statement is given by the proposition below.

Proposition A.4. *Let X, Y be metrizable spaces. A continuous map $f\colon X \to Y$ is proper if, and only if, the image $(f(x_n))$ of every sequence in X without convergent subsequences is a sequence in Y that does not have convergent subsequences.*

Proof. Let f be a proper map. If $(f(x_n))$ has a convergent subsequence, say $f(x'_n) \to y \in Y$, then the set K formed by the elements $f(x'_n)$ and the limit y is compact. All of the elements x'_n belong to the compact set $f^{-1}(K)$ and thus they have a convergent subsequence, which is a subsequence of (x_n).

Conversely, suppose that the condition is satisfied and let $K \subset Y$ be a compact set. Then $f^{-1}(K)$ is closed; if it is not compact, there exists a sequence (x_n) in $f^{-1}(K)$ without convergent subsequences in X. From the condition, $(f(x_n))$ does not have convergent subsequences in Y, and this violates the compactness of K, because $f(x_n) \in K$ for every n. □

Proposition A.5. *Let X, Y be Hausdorff locally compact, but non-compact spaces and denote by $\widehat{X} = X \cup \{\alpha\}$, $\widehat{Y} = Y \cup \{\beta\}$ their Alexandrov compactifications. A continuous map $f\colon X \to Y$ is proper if, and only if, the map $\hat{f}\colon \widehat{X} \to \widehat{Y}$, given by $\hat{f}(x) = f(x)$ if $x \in X$ and $\hat{f}(\alpha) = \beta$, is continuous.*

Proof. We leave the proof as an exercise. □

Proposition A.6. *Let X, Y be two metric spaces without isolated points. If a continuous and locally injective map $f\colon X \to Y$ is closed then the inverse image $f^{-1}(y)$ of each point $y \in Y$ is finite and, as a consequence, f is proper.*

Proof. Suppose, by contradiction, that the inverse image $f^{-1}(y)$ of some point $y \in Y$ is infinite. Since $f^{-1}(y)$ is a discrete subset of the metric space X, we can find for each of its points x, an open set U_x containing x, such that these open sets are pairwise disjoint. Moreover, we may suppose that f is injective in each one of these open sets. Now we select a countable infinite family $U_1, \dots, U_n, \dots$ among the open sets U_x and we set $T_n = f(U_n)$. For each n, y is an accumulation point of T_n. Hence there exist $x_n \in U_n$ and $y_n \in T_n$ such that $0 < d(y_n, y) < 1/n$ and $f(x_n) = y_n$. Thus the set $F = \{x_n; n \in \mathbb{N}\}$ is closed in X but $f(F)$ is not closed in Y, which is a contradiction. □

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Index

SO(n), 73, 74, 77
SU(n), 82, 84
Sp(n), 91
U(n), 82

action
  of a group, 154
  orbit of an, 154
  transitive, 154
analytic continuation, 124
angle element, 109

base point, 20, 33

change of variable formula, 110
circle eversion, 104
coherent atlas, 186
complex projective space, 67
  as a locally trivial fibration, 69
  fundamental group of, 71
convenient neighborhood, 130
covering, 125
  automorphism, 162
  base of a, 125
  distinguished neighborhood, 125
  endomorphism, 162
  fiber of a, 125
  homomorphic, 160
  homomorphism, 161
  isomorphism, 162
  number of leaves of a, 126
  oriented double, 192
  product, 193
  regular, 153
  transformation, 162
  translation, 162
  universal, 67, 165
curve
  eversion of a, 103, 104
  normal vector field to a, 103, 104
  orientation of a, 103
  Peano, 42
  rotation number of a, 99, 103
  simple, 103
  Space filling, 42
  winding number of a, 95

deformation retraction, 22, 23
degree
  local, 111
  of a closed path, 98
  of a map $S^1 \to S^1$, 58
differential form, 105
  closed, 107
  exact, 106
differential of a function, 106
direction field, 199

distinguished neighborhood, 63, 68, 125

equivalence relation, 28
Euler characteristic, 180
eversion
  circle, 104
  curve, 103
  sphere, 104

fibration, *see* locally trivial fibration
fundamental group, 25, 33
  of SO(n), 74, 77
  of SU(n), 84
  of U(n), 84
  abelian, 34, 37, 39, 49
  induced homomorphism, 34
  of a polyhedron, 49
  of a surface, 176
  of the circle, 53, 57
  of the complex projective space, 71
  of the cylinder, 58
  of the real projective space, 66
  of the torus, 57
fundamental grupoid, 32

graph, 17
  circuit of a, 18
  connected, 18
  edge of a, 17
  edge path, 18
  finite, 18
  free vertex of a, 18
  maximal tree of a, 23
  subgraph, 18
  tree, 17
  vertex of a, 17
group, 28
  action of a, 154
  conjugation of a, 151
  isotropy, 155
  monodromy, 156
  of homeomorphisms, 129
  of spinors, 173
  properly discontinuous, 130
  rotation, 73
  simplectic, 91
  special unitary, 82
  topological, 76
  unitary, 82, 84

hermitian inner product, 82
homotopic
  maps, 3
  paths, 26
homotopy, 3
  class, 6, 28
  equivalence, 10
  free, 28
  inverse, 10
  linear, 4
  map extension and, 13
  of pairs, 20
  one-parameter family and, 6
  path, 25, 26
  regular, 99, 100
  relative, 26
  relative to a subspace, 20
  type, 10
hopf fibration, 78

immersion, 99
isomorphic coverings, 162
isomorphism
  orientation preserving, 184
  positive, 184
isotopy, 21

Klein bottle, 169, 192

lifting, 55, 64, 122
local
  homeomorphism, 119
  homomorphism, 143
locally trivial fibration, 68
  complex projective space, 69
  distinguished neighborhood, 68
  global trivialization, 69
  hopf fibration, 78
  local trivialization, 68
  path lifting, 70

product fibration, 69
trivial fibration, 69

manifold
Banach, 147
coherent atlas, 186
compatible coordinate systems, 186
differentiable, 185
induced orientation, 186
maximal atlas, 186
opposite orientation of a, 185
orientable, 185
orientation of a, 185
oriented, 185
Riemannian, 146
map
k-th root of a, 159
analytic continuation of a, 124
closed, 127, 201
covering, 125
exponential, 53
extension, 13
inclusion, 10
lifting of a, 122
logarithm of a, 158
odd, 61, 72
passing to the quotient, 62
proper, 127, 201
quotient, 119
retraction, 14
section of a, 123
stereographic projection, 41
Moebius band, 188

normalizer, 167

one-parameter family, 3
opposite points, 65
orbit of a point, 130
orientation
induced, 186
of a manifold, 185
of a plane curve, 103
product, 188
orthogonal couple, 114

pair of topological spaces, 20
parametrization
negative, 31
positive, 31
trivial, 31
path, 25
angle function of a, 55
closed, 26
decomposition, 32
degree of a, 53, 55
disorienting, 197
inverse of a, 29
lifting, 64, 70
lifting of a, 63
lifting property, 55
orienting, 197
parametrization, 25
partial, 32
piecewise C^1, 106
product of, 28
rectilinear, 145
regular, 99
reparametrization, 25
winding number of a, 96
path lifting property, 134
paths
homotopic, 26
operation with, 28
Peano curve, 42

quaternion, 75
conjugate of a, 75
modulus of a, 75
orthogonal, 91
unit, 75
quotient space
fundamental property, 62, 68

radial projection, 7, 8, 10, 96
real projective space
fundamental group of, 66
metric on the, 64
orientation of, 189
regular
curve, 99
homotopy, 99
path, 99

regular homotopy, 100
reparametrization
 of a regular curve, 99
retract, 14
 Euclidean neighborhood, 15
rotation number, 99, 103

second homotopy group, 92
simplectic group, 91
simply connected space, 40
solid torus, 57
space
 contractible, 11
 convex, 12
 covering, 125
 covering of a, 125
 figure 8, 99, 103, 114, 139, 153, 154, 170, 179, 180
 lens, 170
 of type ENR, 15
 real projective, 62, 189
 semi-locally simply connected, 142
 star, 12, 47
Space filling curve, 42
special unitary group, 82
sphere eversion, 104
stabilizer, 155
stereographic projection, 41
subgroup
 conjugate, 151
 normal, 151

theorem
 Borsuk, 16
 Borsuk-Ulam, 61
 Brower fixed point, 60
 Fundamental theorem of Algebra, 98
 Gram-Schmidt orthonormalization, 23, 81, 83
 Graustein-Whitney, 98, 101, 104
 Hahn-Mazurkiewickz, 42
 Hopf, 103
 Implicit function, 73
 Koebe uniformization, 178
 Mean Value Theorem, 146
 Poincaré, 80
 Poincaré-Bendixon, 61
 Poincaré-Bohl, 4
 rank, 77
 Rouché, 96, 98
 Sard's, 43, 86
 Seifert-Van Kampen, 46
 Tietze-Urisohn, 13
 Umlaufsatz, 103
topological group, 48
topological space
 base point of a, 33
 multiplication in a, 47
 simply connected, 40
topology
 compact-open, 6
 quotient, 62, 67
 uniform convergence, 6
torus, 10, 28
tree, *see* graph
typical fiber, 68

unique path lifting property, 55, 135
unitary group, 82, 84
Urysohn function, 17

vector field
 index of a, 115
 on spheres, 8
 singularity of a, 115
vector space
 basis of a, 183
 equally oriented bases, 183
 negative basis of a, 184
 negative isomorphism of a, 184
 opposite orientation, 184
 orientation of a, 183
 oriented, 184
 positive basis of a, 184
 positive isomorphism of a, 184
 transition matrix, 183

winding number, 105

Zorn Lemma, 23