Feynman rules

Enrico Morgante

April 24, 2026

This note follows very closely the derivation of App. B of the QFT book by Cheng & Li.

1 General Feynman rules

Consider the Lagrangian

$$\mathcal{L}(x) = \frac{1}{2}\phi_i(x)P_{ij}(x)\phi_j(x) + \chi_i^*(x)V_{ij}(x)\chi_j(x)\bar{\psi}_i(x)X_{ij}(x)\psi_j(x) + \mathcal{L}_I[\phi,\chi,\chi^*,\psi,\bar{\psi}](x) \tag{1.1}$$

where ϕ_i (χ_i) denotes a set of real (complex) boson fields that may be scalar or vector fields and ϕ_i the set of fermion fields. The indices i, j stand for any spinor, Lorentz, gauge, etc. index. P, V, and X are matrix operators that may contain derivatives and have an inverse. X and V are taken to be hermitian while P is real and symmetric:

$$V^\dagger = V\,, \qquad X^\dagger = X\,, \qquad P^{\mathrm{T}} = P \tag{1.2}$$

The $i\epsilon$ is assumed to be included in those operator.

1.1 Propagators

Define the inverses of P, X, V as

$$\sum_j V_{ij}(x)V^{-1}_{jl}(y) = \delta_{il}\delta^{(4)}(x-y) \tag{1.3}$$

$$\sum_j X_{ij}(x)X^{-1}_{jl}(y) = \delta_{il}\delta^{(4)}(x-y) \tag{1.4}$$

$$\sum_j P_{ij}(x)P^{-1}_{jl}(y) = \delta_{il}\delta^{(4)}(x-y) \tag{1.5}$$

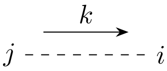
and the Fourier transform

$$V^{-1}_{ij}(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-ik\cdot x}\widetilde{V}^{-1}_{ij}(k) \tag{1.6}$$

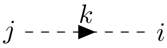
$$X^{-1}_{ij}(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-ik\cdot x}\widetilde{X}^{-1}_{ij}(k) \tag{1.7}$$

$$P^{-1}_{ij}(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-ik\cdot x}\widetilde{P}^{-1}_{ij}(k) \tag{1.8}$$

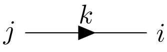
The momentum space Feynman propagators are given by $\widetilde{V}^{-1}, \widetilde{X}^{-1}, \widetilde{P}^{-1}$:



$$\Delta_F^{\phi}(k)_{ij} = \int d^4x \, e^{ik\cdot x} \langle 0|T[\phi_i(x)\phi_j(0)]|0\rangle = i[\widetilde{P}^{-1}(k)]_{ij}$$



$$\Delta_F^{\chi}(k)_{ij} = \int d^4x \, e^{ik\cdot x} \langle 0|T[\chi_i^*(x)\chi_j(0)]|0\rangle = i[\widetilde{V}^{-1}(k)]_{ij}$$



$$S_F(k)_{ij} = \int d^4x \, e^{ik\cdot x} \langle 0|T[\bar{\psi}_i(x)\psi_j(0)]|0\rangle = i[\widetilde{X}^{-1}(k)]_{ij} \tag{1.9}$$

1.2 Vertices

Consider a general term in $\mathcal{L}_I$

$$\begin{aligned}\mathcal{L}_I = \int \mathrm{d}^4 x_1 \, \mathrm{d}^4 x_2 \ldots \alpha_{i_1 \ldots i_m \ldots i_n \ldots i_p \ldots i_q \ldots}(x; x_1 \ldots x_m \ldots x_n \ldots x_p \ldots x_q \ldots) \\ \bar{\psi}_{i_1}(x_1) \ldots \psi_{i_m}(x_m) \ldots \phi_{i_n}(x_n) \ldots \chi_{i_p}^*(x_p) \ldots \chi_{i_q}(x_q) \ldots\end{aligned} \tag{1.10}$$

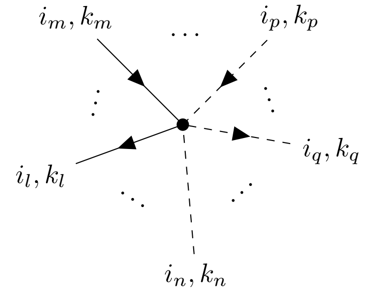
We define the Fourier transform as

$$\alpha_{i_1,i_2\ldots}(x; x_1, x_2 \ldots) = \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \ldots e^{ik_1(x-x_1)+ik_2(x-x_2)+\ldots} \, \widetilde{\alpha}_{i_1,i_2\ldots}(k_1, k_2 \ldots) \tag{1.11}$$

$\widetilde{\alpha}$ contains a factor $ik_{j\mu}$ for every derivative $\partial/\partial x_j^{\mu}$ acting on a field at x_j. The vertex is finally given by

$$I(k_1, k_2, \ldots) = i \sum_{\{1\ldots m-1\}} \sum_{\{m\ldots n-1\}} \sum_{\{n\ldots p-1\}} \sum_{\{p\ldots q-1\}} \sum_{\{q\ldots\}} (-1)^P \, \widetilde{\alpha}_{i_1,i_2\ldots}(k_1, k_2 \ldots) \tag{1.12}$$

The sums are on all possible permutations of indices and momenta among identical particles. A factor (-1) appears for every permutation of two identical fermion fields Momenta are taken to flow *inward*. Anti-particle fields $\bar{\psi}$ and χ^* correspond to lines with an arrow pointing *outwards*, as in the figure below.



2 Examples

2.1 ϕ^4 theory

This is a simple example. We write the interaction term as

$$-\frac{\lambda}{4!}\phi^4(x) = -\frac{\lambda}{4!}\int \mathrm{d}^4\,x_1 \dots \mathrm{d}^4\,x_4 \delta^{(4)}(x-x_1)\delta^{(4)}(x-x_2)\delta^{(4)}(x-x_3)\delta^{(4)}(x-x_4) \\ \times\, \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \tag{2.1}$$

Thus

$$\alpha(x;x_1\dots x_4) = -\frac{\lambda}{4!}\delta^{(4)}(x-x_1)\dots\delta^{(4)}(x-x_4) \tag{2.2}$$

and

$$\widetilde{\alpha}(k_1\dots k_4) = -\frac{\lambda}{4!}\,. \tag{2.3}$$

The vertex is

$$I(k_1\dots k_4) = i\sum_{1,2,3,4}\widetilde{\alpha}(k_1,k_2,k_3,k_4) \tag{2.4}$$

$$i\sum_{1,2,3,4}(-\lambda/4!) = -i\lambda\,. \tag{2.5}$$

2.2 Derivatively coupled scalars

Consider a Lagrangian with two scalar particles s and ϕ, with coupling

$$\mathcal{L}_I = \frac{s}{\Lambda}(\partial_\mu\phi)^2 \tag{2.6}$$

$$= \int \mathrm{d}^4\,x_1\,\mathrm{d}^4\,x_2\,\mathrm{d}^4\,x_3 \frac{g^{\mu\nu}}{\Lambda}\delta^{(4)}(x-x_1)\delta^{(4)}(x-x_2)\delta^{(4)}(x-x_3) \\ \times \frac{\partial}{\partial x_1^\mu}\frac{\partial}{\partial x_2^\mu}\phi(x_1)\phi(x_2)s(x_3) \tag{2.7}$$

Then

$$\alpha = \int \frac{\mathrm{d}^4\,x_1}{(2\pi)^4}\frac{\mathrm{d}^4\,x_2}{(2\pi)^4}\frac{\mathrm{d}^4\,x_3}{(2\pi)^4} e^{i[k_1(x-x_1)+k_2(x-x_2)+k_3(x-x_3)]}\frac{g^{\mu\nu}}{\Lambda} ik_{1\mu}ik_{2\nu} \tag{2.8}$$

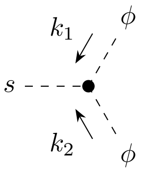
and

$$\widetilde{\alpha}(k_1,k_2,k_3) = \frac{g^{\mu\nu}}{\Lambda}ik_{1\mu}ik_{2\nu} = -\frac{k_1\cdot k_2}{\Lambda}\,. \tag{2.9}$$

The vertex is therefore

$$I = i\sum_{1,2}\widetilde{\alpha}(k_1,k_2,k_3) = -2i\frac{k_1\cdot k_2}{\Lambda}\,, \tag{2.10}$$

where the momenta are inward, as in the figure:



If the momenta flow outwards, just put a minus sign in front.