Lesson 22 - 24/4/2026

Sobolev spaces

Ω open (connected) set in $\mathbb{R}^d$
$p \in [1, +\infty]$

$$W^{1,p}(\Omega) = \{ u \in L^p(\Omega) : \nabla u \in L^p(\Omega) \}$$

in the sense of distribution

rem. this means $\exists\, g_1, g_2, \dots g_d \in L^p(\Omega)$ s.t.

$$\forall \varphi \in \mathcal{C}_0^\infty(\Omega) \qquad \int_\Omega u\, \partial_j \varphi = -\int_\Omega g_j \varphi$$

the g_j's are unique and we denote $g_j = \partial_j u$

derivative in the sense of distr.

"weak derivative"

the same if we ask $\varphi \in \mathcal{C}_0^1(\Omega)$

Properties of $W^{1,p}(\Omega)$

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|\nabla u\|_{L^p} \quad \text{(or equivalent)}$$

$W^{1,p}(\Omega)$ Banach $p \in [1,+\infty]$, separable $p \in [1,+\infty[$, reflexive $p \in]1,+\infty[$, Hilbert $p = 2$

let $d = 1$

Th (continuous representative)

let I open interval in $\mathbb{R}$

let $u \in W^{1,p}(I)$ $\quad (p \in [1, +\infty])$

Then $\exists\, \bar{u} \in \mathcal{C}(I)$ s.t. for a.e. $x \in I$, $u(x) = \bar{u}(x)$.

and $\forall x, y \in I$, $\quad \bar{u}(y) - \bar{u}(x) = \int_x^y u'(t)\,dt$

$u' \in L^p(I) \subseteq L^1_{loc}(I)$

Lemma $T \in \mathcal{D}'(I)$

if $T' = 0$ then $\exists\, c \in \mathbb{C} : T = T_c$.

proof. consider $x_0 \in I$ and define

$$w(x) = \int_{x_0}^x u'(t)\,dt$$

w is the integral function of an L^1 function in $[a,b]$ for all $[a,b] \subseteq I$, $x_0 \in [a,b]$

We know that $w \in AC([a,b])$

moreover w is a.e. differentiable (in classical sense)

and $w'(x) = u'(x)$ a.e. (*)

and the integration by parts is possible

so that

$$\forall \varphi \in \mathcal{C}_0^\infty(I) \qquad \int_I w(x)\varphi'(x)\,dx = -\int_I w'(x)\varphi(x)\,dx \overset{(*)}{=} -\int u'(x)\varphi(x)\,dx \overset{u \in W^{1,p}(I)}{=} \int u\,\varphi'(x)\,dx$$

$$\int w(x)\varphi'(x) = \int u(x)\varphi'(x)$$

$$\forall \varphi \in \mathcal{C}_0^\infty(I) \qquad \int_I (w - u)\varphi' = 0 \iff T'_{w-u} = 0$$

from the lemma $T'_{w-u} = 0 \Rightarrow T_{w-u} = T_c$

constant

conclusion

$$T_{w-u-c} = 0$$

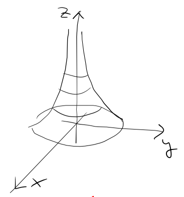
$$\int_I (w(x) - u(x) - c)\varphi = 0 \Rightarrow w(x) - u(x) - c = 0 \text{ a.e.}$$

$u(x) = w(x) - c$ a.e.

cont.

Rem (what happens if $d \geq 2$?

Ex. consider $\Omega = B(0,1) \subseteq \mathbb{R}^2$
$= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$



consider $u(x,y) = \left(x^2+y^2\right)^{-\frac{1}{4}}$

$\nexists \; \bar{u} \in \mathcal{C}(B(0,1))$ s.t. $\bar{u} = u$ a.e.

let's verify that $u \in W^{1,1}(\Omega)$

- $u \in L^1(\Omega)$ $\quad \displaystyle\int_0^{2\pi}\int_0^1 (r^2)^{-\frac{1}{4}} r\,dr\,d\theta = 2\pi \int_0^1 r^{\frac{1}{2}}dr = 2\pi \, \frac{2}{3} r^{\frac{3}{2}}\Big|_0^1 = 2\pi$

- $\partial_x u \in L^1(\Omega)$

$\partial_x u(x,y) = -\frac{1}{4}(x^2+y^2)^{-\frac{5}{4}} \cdot 2x = -\frac{1}{2}(x^2+y^2)^{-\frac{5}{4}} \cdot x$

$$\iint_{x^2+y^2<1} \frac{|x|}{(x^2+y^2)^{\frac{5}{4}}} = \int_0^{2\pi}\int_0^1 \frac{|r\cos\theta|}{r^{\frac{5}{2}}} r\,dr\,d\theta \sim \int_0^1 r^{-\frac{1}{2}}dr < +\infty$$

Th (characterization of $W^{1,p}(\Omega)$ for $1 < p \leq +\infty$)

let Ω as before (open connected set in $\mathbb{R}^d$)
let $p \in]1,+\infty]$
let $u \in L^p(\Omega)$

the following are equivalent

1) $u \in W^{1,p}(\Omega)$

2) $\exists C > 0 :$ $\forall \varphi \in \mathcal{C}_0^\infty(\Omega)$, $\forall j = 1,\dots,d$, $\left|\int_\Omega u\,\partial_{x_j}\varphi\right| \leq C \|\varphi\|_{L^{p'}(\Omega)}$ $\quad \left(\frac{1}{p}+\frac{1}{p'}=1\right)$

3) $\exists C > 0 :$ $\forall \omega \subseteq \Omega$, $\bar{\omega}$ compact set in Ω, $\forall h \in \mathbb{R}^d$: $|h| < \text{dist}(\omega, \partial\Omega)$

$\|\tau_h u - u\|_{L^p(\omega)} \leq C|h|$.

proof. 1) $\Rightarrow$ 2)

$u \in W^{1,p}(\Omega)$

by definition $\forall j = 1,\dots,d$ $\forall \varphi \in \mathcal{C}_0^\infty(\Omega)$

$$\int_\Omega u\,\partial_{x_j}\varphi = -\int_\Omega \partial_{x_j}u \cdot \varphi$$

then $\left|\int_\Omega u\,\partial_{x_j}\varphi\right| = \left|\int_\Omega \partial_{x_j}u \cdot \varphi\right| \overset{\text{Hölder}}{\leq} \|\partial_{x_j}u\|_{L^p}\|\varphi\|_{L^{p'}} \leq \underbrace{\|\nabla u\|_{L^p}}_{C}\|\varphi\|_{L^{p'}}$

I prove 2) $\Rightarrow$ 1)

we know that $\exists C > 0 :$ $\underbrace{\left|\int_\Omega u\,\partial_{x_j}\varphi\right| \leq C\|\varphi\|_{L^{p'}(\Omega)}}_{\circledast}$

we consider $\underbrace{\mathcal{C}_0^\infty(\Omega)}_{\text{subspace}} \subseteq L^{p'}(\Omega)$

we consider $\Phi : \mathcal{C}_0^\infty(\Omega) \longrightarrow \mathbb{C}$, $\varphi \longmapsto \int_\Omega u\,\partial_{x_j}\varphi$

linear

$\circledast$ says that Φ is continuous w.r.t. the top. of $L^{p'}(\Omega)$

we consider $\Phi : \mathcal{C}_0^\infty(\Omega) \longrightarrow \mathbb{C}$ $\quad (p' < +\infty)$

linear

$$\varphi \longmapsto \int_\Omega u\, \partial_{x_j} \varphi$$

⊛ says that Φ is continuous w.r.t. the top. of $L^{p'}(\Omega)$

by density & using Hahn-Banach we extend Φ to $L^{p'}(\Omega)$

$$\tilde{\Phi} : L^{p'}(\Omega) \to \mathbb{C}$$

$$\tilde{\Phi} \in (L^{p'})' \underset{\uparrow \text{ Riesz}}{=} L^p(\Omega) \qquad (p' < +\infty)$$

$\exists g_j \in L^p(\Omega)$ s.t.

$$\tilde{\Phi}(f) = \int_\Omega f g_j \quad \forall f \in L^{p'}$$

in particular

$$\int_\Omega u\, \partial_{x_j} \varphi = \Phi(\varphi) = \int_\Omega g \varphi \qquad \forall \varphi \in \mathcal{C}_0^\infty(\Omega)$$

$$\Rightarrow \quad g = -\partial_{x_j} u \quad \text{and} \quad u \in W^{1,p}(\Omega)$$

(if $p = 1$ the proof does not work)

$$\|\tau_h u - u\|_{L^p(\omega)} \leq C|h|$$

$$\tau_h u(x) = u(x-h)$$

1) $\Rightarrow$ 3) I do the proof for $d = 1$

$u \in W^{1,p}(I)$ $\quad$ u is the continuous representative

$$u(x-h) - u(x) = \int_x^{x-h} u'(t)\,dt = \int_0^1 u'(x-sh)\cdot(-h)\,ds = -h \int_0^1 u'(x-sh)\,ds$$

$t = x - sh \quad 0 < s < 1$

$dt = -h\,ds$

$$|u(x-h) - u(x)|^p = |h|^p \left| \int_0^1 u'(x-sh)\,ds \right|^p \leftarrow \text{I use Hölder}$$

$$\leq |h|^p \int_0^1 |u'(x-sh)|^p\,ds$$

$$\left| \int_0^1 u'(x-sh)\,ds \right| = \left| \int 1 \cdot u'(x-sh)\,ds \right|$$

$1 \in L^{p'}$, $u' \in L^p$

$$\left(\int_0^1 1^{p'}\right)^{\frac{1}{p'}} = 1 \qquad \left(\int_0^1 |u'(x-h)|^p\right)^{\frac{1}{p}}$$

$$\left(\int_\omega |u(x-h) - u(x)|^p\,dx \right)^{\frac{1}{p}} \leq \left(|h|^p \int_\omega \int_0^1 |u'(x-sh)|^p\,ds\,dx \right)^{\frac{1}{p}}$$

exchange

$$\|\tau_h u - u\|_{L^p(\omega)} \leq |h| \left(\int_0^1 \underbrace{\int_\omega |u'(x-sh)|^p\,dx}_{= \int_{\omega + sh} |u'(y)|^p\,dx \;\leq\; \int_\Omega |u'|^p} ds \right)^{\frac{1}{p}}$$

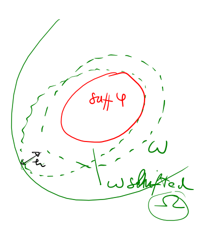
conclusion

$$\|\tau_h u - u\|_{L^p(\omega)} \leq |h| \underbrace{\|u'\|_{L^p(\Omega)}}_{C}$$

before ~~p~~roving 1) $\Rightarrow$ 3) in $d \geq 2$

I prove 3) $\Rightarrow$ 2)

I know that $\|\tau_h u - u\|_{L^p(\omega)} \leq C|h|$



consider $\varphi \in \mathcal{C}_0^\infty(\Omega)$

I take ω (s.t. $\overline{\omega}$ is compact and $\overline{\omega} \subseteq \Omega$)

such that $\operatorname{supp} \varphi \subseteq \omega$

I take h s.t. $|h| < \operatorname{dist}(\omega, \partial\Omega)$

$$\left| \int_\omega (u(x-h) - u(x))\varphi(x)\,dx \right| \underset{\uparrow \text{ Hölder}}{\leq} \underbrace{\|\tau_h u - u\|_{L^p(\omega)}}_{\text{by 3)}} \cdot \|\varphi\|_{L^{p'}(\Omega)} \leq C \cdot |h| \cdot \|\varphi\|_{L^{p'}(\Omega)}$$

$$\left| \int_\omega (u(x-h) - u(x))\varphi(x)\,dx \right| = \left| \int_\omega u(x-h)\varphi(x)\,dx - \int_\omega u(x)\varphi(x)\,dx \right|$$

I take h s.t. $|h| <$ dist$(\omega, \partial\Omega)$

$$\left| \int_\omega (u(x-h) - u(x))\varphi(x)\,dx \right| \underset{\text{Hölder}}{\leq} \underbrace{\|\tau_h u - u\|_{L^p(\omega)}}_{\text{by 3)}} \cdot \|\varphi\|_{L^{p'}(\Omega)} \leq C_1 \cdot |h| \cdot \|\varphi\|_{L^{p'}(\Omega)}$$

$$= \left| \int_\omega u(x-h)\varphi(x)\,dx - \int_\omega u(x)\varphi(x)\,dx \right|$$

$x - h = y \quad x = y + h$

$$= \left| \int_{\omega - h} u(y)\varphi(y+h)\,dy - \int_{\omega} u(y)\varphi(y)\,dy \right|$$

$$= \left| \int_\Omega u(y)\left(\varphi(y+h) - \varphi(y)\right) dy \right| \leq C\,|h|\,\|\varphi\|_{L^{p'}(\Omega)}$$

$$\left| \int_\Omega u(y) \left(\frac{\varphi(y+h) - \varphi(y)}{h} \right) dy \right| \leq C \|\varphi\|_{L^{p'}(\Omega)}$$

I "pass to the limit" for $h \to 0$

in 1 d for example $\left| \int_I u\,\varphi' \right| \leq C \|\varphi\|_{L^{p'}(I)}$

condition 2

similarly in several variables.

in 1 d we have proved 1) $\Leftrightarrow$ 2) and 1) $\Rightarrow$ 3)
and 3) $\Rightarrow$ 2)

It remains to see 1) $\Rightarrow$ 3) in $d \geq 2$

I need the following density result.

Th (Friedrichs)

let $\Omega \subseteq \mathbb{R}^d$ $(d \geq 2)$ let $p \in [1, +\infty[$

let $u \in W^{1,p}(\Omega)$

then there exists $(u_n)_n$ in $C_0^\infty(\mathbb{R}^d)$ ← not in $C_0^\infty(\Omega)$

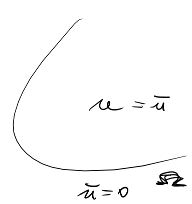
s.t. 1) $u_n|_\Omega \xrightarrow{n} u$ in $L^p(\Omega)$

2) $\forall \omega^{\text{open}}$ s.t. $\bar{\omega}$ is a compact set in Ω

$\nabla u_n|_\omega \to \nabla u|_\omega$ in $L^p(\omega)$

sketch

consider $\bar{u}(x) = \begin{cases} u(x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$



$u \in L^p(\Omega) \Rightarrow \bar{u} \in L^p(\mathbb{R}^d)$

take $(\rho_n)_n$ usual mollifier

consider $\rho_n * \bar{u} \in L^p(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$

and since $1 \leq p < +\infty$, $\rho_n * \bar{u} \to \bar{u}$ in L^p

consider $\chi \in C_0^\infty(\mathbb{R}^d)$, $\chi(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| \geq 2 \end{cases}$

$\chi_n(x) = \chi\left(\frac{x}{n}\right)$ ← "truncation sequence"

define $u_n(x) = \chi_n(x) \cdot (\rho_n * \bar{u})(x)$ convolution + truncation

prove that $(u_n)_n$ has the wanted properties

$$\|u_n|_\Omega - u\|_{L^p}^p = \int_\Omega |u_n(x) - u(x)|^p dx \leq \int_{\mathbb{R}^d} |u_n(x) - \bar{u}(x)|^p dx$$

$$\left(\int_{\mathbb{R}^d} |u_n(x) - \bar u(x)|^p dx\right)^{\frac1p} = \| u_n - \bar u\|_{L^p} = \| \chi_n(\bar u * \rho_n) - \bar u\|_{L^p}$$

$$= \| \chi_n(\bar u * \rho_n) - \chi_n \bar u + \chi_n \bar u - \bar u\|_{L^p}$$

$$\leq \| \chi_n(\bar u * \rho_n) - \chi_n \bar u\|_{L^p} + \|\chi_n \bar u - \bar u\|_{L^p}$$

$$\leq \underbrace{\|\chi_n\|_{L^\infty}}_{=1} \underbrace{\| \rho_n * \bar u - \bar u\|_{L^p}}_{\xrightarrow{n} 0} + \|\chi_n \bar u - \bar u\|_{L^p}$$

this goes to 0 by dominated convergence

for the gradient is more complicated

Ex. let $f \in L^1(\mathbb{R}^d)$
let $u \in W^{1,p}(\mathbb{R}^d)$ $(p < +\infty)$

then $f * u \in W^{1,p}(\mathbb{R}^d)$

and $\partial_j (f * u) = f * \partial_j u$

Sketch of 1) $\Rightarrow$ 3) for $d \geq 2$,

$u \in W^{1,p}(\Omega) \Rightarrow \exists C : \forall \omega \subset\subset \Omega \ \forall e_i \ |e_i| < \text{dist}(\ ) \ \|\tau_{e_i} u - u\|_{L^p(\omega)} \leq C |e_i|$

start with $1 < p < +\infty$

take $(u_n)_n$ in $C_0^\infty(\mathbb{R}^d)$ with the properties of Friedrichs

prove 3) for u_n and pass to the limit

end

if $p = \infty$, It is necessary another approximation...

new $p \in]1, +\infty]$ 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 3)

¿ $p = 1$? $1) \not\Rightarrow \underbrace{2) \Leftrightarrow 3)}_{p=1.}$ (we have seen only 3) $\Rightarrow$ 2) 2) $\Rightarrow$ 3 to see for $p = 1$)

2) $\Leftrightarrow$ 3) defined different space bigger than $W^{1,1}$