Introduction to Complete Manifolds

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Definition

A Riemannian manifold M is (geodesically) *complete* if for all $p \in M$ the exponential map $\exp_p$ is defined for all $v \in T_pM$, i.e. any geodesic $\gamma(t)$ starting at p is defined for all $t \in \mathbb{R}$.

Riemannian Manifolds as metric spaces

Proposition

If M is a Riemannian manifold and $p, q \in M$, then define $d_M(p, q)$ as the infimum of the lengths of all piece-wise differentiable curves in M joining p to q. Then (M, d_M) is a metric space.

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Corollary

The (metric) topology induced by d_M on M coincides with the original topology on M.

Hopf Rinow Thm

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Hopf Rinow Thm

Theorem (Hopf-Rinow part II)

Let M be a Riemannian manifold. Then M is complete if and only if there exists an exhaustion *by compact sets of M (i.e. an (ascending) chain of compacta in M, that is a sequence of compact sets in M $\{K_n\}_{n\in\mathbb{N}}$ such that $K_n \subset\subset K_{n+1}$, whose union is M) such that, if one picks $q_n \in M \setminus K_n$ for any $n \in \mathbb{N}$, then, for the sequence of points $\{q_n\}_{n\in\mathbb{N}}$ in M it turns out that for any $p \in M$ $d_M(p, q_n) \to +\infty$ as $n \to +\infty$.*

Hopf Rinow - consequences

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If M is a complete manifold, then for any $p \in M$ there exists a geodesic γ joining p to q such that $L(\gamma) = d_M(p, q)$.

Ambrose Thm

Proposition

Let (M, g) be a complete Riemannian manifold, and $\pi : \tilde{M} \to M$ be a smooth covering map. Then $\tilde{M}$ (with the metric induced by g, i.e. the pull-back of g via π) is a complete Riemannian manifold.

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Theorem (Ambrose)

Let (M, g_M) and (N, g_N) be two (connected) Riemannian manifolds. If $\phi : M \to N$ is a local isometry and M is complete, then ϕ is a smooth covering map and (N, g_N) is a complete manifold.

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Corollary

Let M be a (connected) Riemannian manifold. If for $p \in M$ $\exp_p : T_pM \to M$ is a local diffeomorphism, then $\exp_p$ is a covering map.

Hadamard Thm

Theorem (Hadamard)

Let M be a complete Riemannian manifold of dimension n, simply connected and with sectional curvature $K_p(\sigma) \leq 0$ for all $p \in M$ and for all two-dimensional subspaces σ of T_pM. Then $\exp_p : T_pM \to M$ is a diffeomorphism, or – equivalently – M is diffeomorphic to $\mathbb{R}^n$.

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Lemma

Let M be a complete Riemannian manifold and let $\psi : M \to N$ be a local diffeomorphism onto a Riemannian manifold N. If for all $p \in M$ and for all $v \in T_pM$ it turns out that $d\psi_p(v)| \geq |v|$, then ψ is a covering map.

Bonnet-Myers Thm

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Let M be a complete Riemannian manifold. Suppose that the Ricci curvature of M satisfies

$$Ric_p(v) \geq \frac{1}{r^2} > 0$$

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Corollary

Let M be a complete Riemannian manifold with $Ric_p(v) \geq \delta > 0$ for all $p \in M$ and for all $v \in T_pM$. Then the universal cover of M is compact and hence the fundamental group $\pi_1(M)$ of M is finite.

Corollary

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Corollary (Synge)

Let M be a compact Riemannian manifold with positive sectional curvature.
If M is orientable and $n = \dim(M)$ is even, then M is simply connected.
If $n = \dim(M)$ is odd, then M is orientable.

Comparison Theorems

Theorem (Rauch)

Let $\gamma : [0, a] \to M$ *and* $\tilde{\gamma} : [0, a] \to \tilde{M}$ *be geodesics where* $\dim \tilde{M} = \dim M + k$ *(*$k \geq 0$*) such that* $|\gamma'(t)| = |\tilde{\gamma}'(t)|$ *for any* $t \in [0, a]$*. Let* J *and* $\tilde{J}$ *be Jacobi fields along* γ *and* $\tilde{\gamma}$ *respectively, such that:*

$$J(0) = \tilde{J}(0) = 0, \quad \langle J'(0), \gamma'(0)\rangle = \langle \tilde{J}'(0), \tilde{\gamma}'(0)\rangle \quad |\tilde{J}'(0)| = |J'(0)|.$$

Assume that $\tilde{\gamma}$ *doesnot have conjugate points on* $(0, a]$ *and that for all* $x \in T_{\gamma(t)}M$*,* $x \in T_{\gamma(t)}M$ $\tilde{x} \in T_{\tilde{\gamma}(t)}\tilde{M}$ $\tilde{K}_{\tilde{\gamma}(t)}(\tilde{\sigma}) \geq K_{\gamma(t)}(\sigma)$ *where* $\sigma =$*span*$\{x, \gamma'(t)\}$ *and* $\tilde{\sigma} =$*span*$\{\tilde{x}, \tilde{\gamma}'(t)\}$*. Then*

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in $(0, a]$*. In addition, if for some* $t_0 \in (0, a]$ *we have* $|\tilde{J}(t_0)| \leq |J(t_0)|$*, then* $\tilde{K}_{\tilde{\gamma}(t)}(\tilde{S}) = K_{\gamma(t)}(S)$ *for all* $t \in [0, t_0]$*, where* $S =$*span*$\{J(t), \gamma'(t)\}$ *and* $\tilde{S} =$*span*$\{\tilde{J}(t), \tilde{\gamma}'(t)\}$*.*

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For any $q \in V$ there exists a unique (normalized) geodesic γ joining p to q. Consider the (normalized) geodesic $\tilde{\gamma}$ in $\tilde{M}$ uniquely determined by $\tilde{\gamma}(0) = \tilde{p}$ and $\tilde{\gamma}'(0) = \varphi(\gamma'(0))$.

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Let P_t and $\tilde{P}_t$ be the parallel transports along γ and $\tilde{\gamma}$ respectively and define

$$\psi_t(v) := \tilde{P}_t \circ \varphi \circ P_t^{-1}(v)$$

for $v \in T_q M$.

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$$\psi_t : T_qM \to T_{\Phi(q)}\tilde{M}$$

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Theorem (Cartan)

Denote by $\mathcal{R}$ and $\tilde{\mathcal{R}}$ the curvatures in M and $\tilde{M}$ respectively; if for all $q \in V$ and all $x, y, z, w \in T_qM$ we have

$$\mathcal{R}(x, y, z, w) = \tilde{\mathcal{R}}(\psi_t(x), \psi_t(y), \psi_t(z), \psi_t(w))$$

then $\Phi : V \subseteq M \to \Phi(V) \subseteq \tilde{M}$ is a local isometry and $d\Phi_p = \varphi$.

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Proof.

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Take $w \in T_pM$ and let J be the Jacobi field along γ with $J(0) = 0$, $J(\ell) = w$.
Write

$$J(t) = \sum_i v_i(t) e_i(t)$$

where $\{e_i(t)\}$ is a parallel orthonormal frame with $e_n(0) = \gamma'(0)$.
Then

$$v_j''(t) + \sum_i \langle R(e_n, e_i)e_n, e_j\rangle v_i(t) = 0 \quad j = 1, \ldots, n.$$

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Let $\tilde{\gamma}[0, \ell] \to \tilde{M}$ be a normalized geodesic in $\tilde{M}$ such that $\tilde{\gamma}(0) = \tilde{p}$ and $\tilde{\gamma}'(0) = \varphi(\gamma'(0))$ and define along $\tilde{\gamma}$ the vector field

$$\tilde{J}(t) = \psi_t(J(t)).$$

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Proof.

In coordinates, if $\tilde{e}_j(t) = \varphi_t(e_j(t))$, it turns out that, from the linearity of φ_t

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that is $\tilde{J}$ is a Jacobi field along $\tilde{\gamma}$ with $\tilde{J}(0) = 0$. □

Cartan Thm - proof

Proof.

Furthermore since the parallel transport ìs an isometry

$$|J(\ell)| = |\tilde{J}(\ell)|$$

and, from the definition, $\tilde{J}'(0) = \varphi(J'(0))$ since $\varphi : T_pM \to T_p\tilde{M}$ a linear isometry.

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$$J(t) = (d\exp_p)_{t\gamma'(0)}(tJ'(0))$$

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hence

$$\begin{aligned} \tilde{J}(t) &= (d\exp_{\tilde{p}})_{t\tilde{\gamma}'(0)}(t\varphi(J'(0))) = \\ &= (d\exp_{\tilde{p}})_{t\tilde{\gamma}'(0)} \circ \varphi \circ ((d\exp_p)_{t\gamma'(0)})^{-1}(J(t)) = d\Phi_{\gamma(t)}(J(t)). \end{aligned}$$

Constant Sectional Curvature

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Up to scaling, $K \in \{-1, 0, 1\}$ and then

- $K = 0$: $\mathbb{R}^n$ (with the Euclidean metric)
- $K = 1$: S^n (with the induced metric since $S^n \subset \mathbb{R}^{n+1}$)
- $K = -1$: $H^n = \{(x_1, \ldots, x_n) \ : \ x_n > 0\}$ (with the hyperbolic metric $ds^2 = \frac{\sum_{j=1}^n dx_j^2}{x_n^2}$)

Hopf-Killing Thm

Theorem

A complete, simply connected manifold with constant sectional curvature K is isometric (up to scaling) to one of the three models

$\mathbb{R}^n, \quad S^n, \quad or \quad H^n.$