Lesson 24 - 28/4/2026

Th. $u \in L^p(\Omega)$ $\quad 1 < p \leq +\infty$

i) $u \in W^{1,p}(\Omega)$

ii) $\exists C>0: \ \forall \varphi \in C_0^\infty(\Omega) \ \forall j = 1 \dots d$

$$\left| \int_\Omega u \, \partial_j \varphi \right| \leq C \|\varphi\|_{L^{p'}(\Omega)}$$

iii) $\exists C>0: \ \forall \omega$ open, relatively compact set in $\Omega$
$\forall h \in \mathbb{R}^d: \quad |h| \leq \text{dist}(\omega, \partial\Omega)$

$$\|\tau_h u - u\|_{L^p(\omega)} \leq C|h|$$

rem. if $p = 1$

the proof we have seen gives

$$\underset{i)}{u \in W^{1,1}(\Omega)} \Rightarrow \exists C>0, \ \overset{ii)}{\left| \int u \, \partial_j \varphi \right| \leq C \|\varphi\|_{L^\infty}}$$

$$\Uparrow$$

$$\exists C>0: \quad \underset{iii)}{\|\tau_h u - u\|_{L^1(\omega)} \leq C|h|}$$

i) $\Rightarrow$ ii) and iii) $\Rightarrow$ ii)

ii) $\not\Rightarrow$ i) but ii) $\Rightarrow$ iii) (it is possible to see this)

def. let $u \in L^1(\Omega)$ and suppose

that $\exists C>0: \forall \varphi \in C_0^\infty(\Omega) \quad \left| \int_\Omega u \, \partial_j \varphi \right| \leq C \|\varphi\|_{L^\infty}$ ⊛
$j = 1, \dots d$

(or equivalently
$\exists C>0; \ \forall \omega$ rel comp in $\Omega$ $\ \forall h \ \ |h| \leq \text{dist}(\omega, \partial\Omega)$
$\|\tau_h u - u\|_{L^1(\omega)} \leq C|h|$)

$u$ is called $BV(\Omega)$ - function

bounded variation function

⊛ say that the derivative of $u$ in the sense of distribution is a (bounded) Radon measure

in particular (from Riesz) the derivative is an integral w.r.t. a bounded regular Borel measure

$$\int_\Omega u \, \partial_j \varphi = \int_\Omega \varphi \, d\nu$$

$\nu$ Borel measure

rem. It is possible to prove

Th. let $I = ]a,b[$ let $u \in L^1(a,b)$.
the following are equivalent.

i) $\exists \tilde{u} \in BV([a,b])$ ← old def. $\left( \sup_\Delta V(\tilde{u}, \Delta) = V_a^b(\tilde{u}) < +\infty \right.$

s.t. $\tilde{u} = u$ a.e. in $]a,b[$ $\quad V(\tilde{u}, \Delta) = \sum_{j=1}^N |\tilde{u}(t_j) - \tilde{u}(t_{j-1})|$

$\left. \Delta = \{a = t_0 < t_1 < \dots < t_N = b\} \right)$

ii) $\exists C>0: \ \forall \varphi \in C_0^\infty(]a,b[)$

$$\left| \int_I u \varphi' \right| \leq C \|\varphi\|_{L^\infty}$$

iii) $\forall \varepsilon > 0$

$$\int_a^{b-\varepsilon} |u(t+\varepsilon) - u(t)| \, dt \leq C \varepsilon$$

$$\left( \|\tau_\varepsilon u - u\|_{L^1[a, b-\varepsilon]} \leq C \varepsilon \right)$$

rem. Sobolev space via Fourier Transform.

rem. Sobolev space via Fourier Transform.

now $p=2$ (remember $\underbrace{W^{1,2}(\Omega) = H^1(\Omega)}$) — Hilbert space.

consider $W^{1,2}(\mathbb{R}^d) = H^1(\mathbb{R}^d)$

$$\| \quad \{u \in L^2(\mathbb{R}^d) : \nabla u \in L^2(\mathbb{R}^d)\}$$

↑ gradient in the sense of distribution

**Th** Let $u \in L^2(\mathbb{R}^d)$.

$u \in H^1(\mathbb{R}^d)$ if and only if $\hat{u}(\xi)(1+|\xi|^2)^{\frac{1}{2}} \in L^2(\mathbb{R}^d)$

and $\|u\|_{H^1} = \frac{1}{(2\pi)^{d/2}} \|\hat{u}(\xi)(1+|\xi|^2)^{\frac{1}{2}}\|_{L^2}$

**def.** let $s \in \mathbb{R}$

$$H^s(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) : \hat{u}(\xi)(1+|\xi|^2)^{s/2} \in L^2(\mathbb{R}^d)\}$$

$$\|u\|_{H^s(\mathbb{R}^d)} = \|\hat{u}(\xi)(1+|\xi|^2)^{s/2}\|_{L^2(\mathbb{R}^d)}$$

**Lemma** let $f, g \in L^2(\mathbb{R}^d)$

then i) $\int_{\mathbb{R}^d} f(x)\hat{g}(x)\,dx = \int_{\mathbb{R}^d} \hat{f}(\xi) g(\xi)\,d\xi$

ii) $\int_{\mathbb{R}^d} f(x) g(x)\,dx = \frac{1}{(2\pi)^d} \int \hat{f}(\xi)\hat{g}(\xi)\,d\xi$

proof. remark that we cannot write $\hat{g}(x) = \int_{\mathbb{R}^d} e^{-ix\xi} g(\xi)\,d\xi$

because $g \notin L^1$

to prove i) and ii) we have to use density of $\mathcal{S}(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$

$\mathcal{S} \ni f_n \to f$ in $L^2$ — $(f_n)_n$ is Cauchy

$(\hat{f}_n)_n$ is Cauchy

$\hat{f}_n \to \hat{f}$ in $L^2$

similarly for $g$. and we pass to the limit.

proof (theorem in 1d)

let $u \in H^1(\mathbb{R})$ i.e. $u \in L^2(\mathbb{R})$ and $u' \in L^2(\mathbb{R})$

we have $-\int_{\mathbb{R}} \underset{\in L^2}{u}\,\underset{\in L^2}{\varphi'} = \int_{\mathbb{R}} \underset{L^2}{u'}\,\underset{L^2}{\varphi} = \frac{1}{2\pi}\int \hat{u'}\,\hat{\varphi} = \frac{1}{2\pi}\int \hat{u'}(\xi)\hat{\varphi}(\xi)\,d\xi$

$$\| $$

$-\frac{1}{2\pi}\int \hat{u}\,\widehat{\varphi'}$ $\qquad \varphi' \in C_0^\infty \subseteq \mathcal{S} \Rightarrow \widehat{\varphi'}(\xi) = i\xi\hat{\varphi}(\xi)$

$$\| $$

$-\frac{1}{2\pi} i \int \hat{u}(\xi)\cdot\xi\cdot\hat{\varphi}(\xi)$

$-i\int \hat{u}(\xi)\cdot\xi\cdot\hat{\varphi}(\xi)\,d\xi = \int \widehat{u'}(\xi)\hat{\varphi}(\xi)\,d\xi \quad \forall \varphi \in \mathcal{S}(\mathbb{R})$

$\Rightarrow -i\,\hat{u}(\xi)\cdot\xi = \widehat{u'}(\xi) \in L^2 \Rightarrow \xi\hat{u}(\xi) \in L^2$

$\Rightarrow (1+|\xi|^2)^{\frac{1}{2}}\hat{u}(\xi) \in L^2$

viceversa let $\hat{u} \in L^2$ and $(1+|\xi|^2)^{\frac{1}{2}}\hat{u}(\xi) \in L^2$

we have $\hat{u} \in L^2$ and $\xi\cdot\hat{u}(\xi) \in L^2$

let $\varphi \in C_0^\infty(\mathbb{R})$. consider $\int_{\mathbb{R}} \underset{L^2}{u}\,\underset{L^2}{\varphi'} = \frac{1}{2\pi}\int \hat{u}\,\widehat{\varphi'} = \frac{1}{2\pi}\int \underbrace{\hat{u}(\xi)\, i\xi}_{v(\xi)}\,\hat{\varphi}(\xi)\,d\xi$

viceversa let $\hat u \in L^2$ and $(1+|\xi|^2)^{\frac12}\hat u(\xi) \in L^2$

we have $\hat u \in L^2$ and $\xi \cdot \hat u(\xi) \in L^2$

let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. consider $\int_{\mathbb{R}} \underset{L^2}{u}\, \underset{L^2}{\varphi'} = \frac{1}{2\pi}\int \hat u\, \widehat{\varphi'} = \frac{1}{2\pi}\int \underbrace{\hat u(\xi)\, i\xi}_{\hat v(\xi)}\, \hat\varphi(\xi)\, d\xi$

$v \in L^2(\mathbb{R}) \Leftrightarrow \hat v \in L^2(\mathbb{R})$

$= \int v \cdot \varphi$

at the end $\int_{\mathbb{R}} u\varphi' = \int_{\mathbb{R}} v\varphi$ with $v \in L^2$ for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$

$\Rightarrow u \in H^1(\mathbb{R})$ $\quad (u' = -v)$

## The extension operator (1 d)

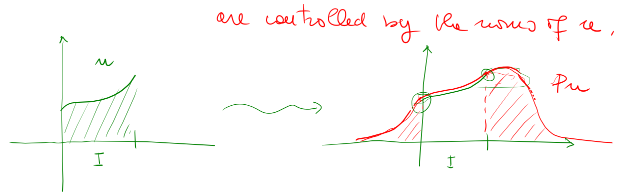
**Problem** let $I$ be an open interval in $\mathbb{R}$

consider $u \in W^{1,p}(I)$, $p \in [1,+\infty]$.

Find $Pu \in W^{1,p}(\mathbb{R})$ s.t.

i) $Pu_{|I} = u$

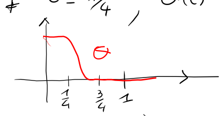
ii) the norms of $Pu$ in $L^p$, $W^{1,p}$ are controlled by the norms of $u$.




**Lemma** take $u \in W^{1,p}(0,1)$ $\quad I = ]0,1[$

consider $\theta \in \mathcal{C}^\infty(]0,+\infty[)$

$\theta(t) = 1$ if $t \le 1/4$, $\theta(t) = 0$ if $t \geqslant 3/4$




consider $\bar u(t) = \begin{cases} u(t) & \text{if } t \in ]0,1[ \\ 0 & \text{if } t \geqslant 1 \end{cases}$


$\overline{u'}(t) = \begin{cases} u'(t) & \text{if } t \in ]0,1[ \\ 0 & \text{if } t \geqslant 1 \end{cases}$


then $\bar u\theta \in W^{1,p}(]0,+\infty[)$ and $(\bar u\theta)' = \overline{u'}\theta + \bar u\theta'$

proof. $\bar u \in L^p(]0,+\infty[) \Rightarrow \underset{L^p}{\bar u}\,\underset{L^\infty}{\theta} \in L^p(]0,+\infty[)$

similarly

$\overline{u'}\theta + \bar u\theta' \in L^p(]0,+\infty[)$

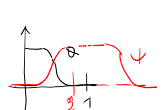
it remains to prove that $(\bar u\theta)' = (\overline{u'}\theta + \bar u\theta')$

in the sense of distributions

so take $\psi \in \mathcal{C}_0^\infty(]0,+\infty[)$

consider $\int_0^{+\infty} (\bar u\theta)\psi' = \int_0^{+\infty} \bar u(t)\, \underset{\mathcal{C}^\infty(]0,+\infty[)}{\theta(t)}\, \underset{\mathcal{C}_0^\infty(]0,+\infty[)}{\psi'(t)}\, dt$

$\theta(t)\psi'(t) = (\theta(t)\psi(t))' - \theta'(t)\psi(t)$




$= \int_0^{+\infty} \bar u(t)(\theta(t)\psi(t))'\, dt - \int_0^{+\infty} \bar u(t)\,\theta'(t)\,\psi(t)\, dt$

$= \int_0^1 u(t)\, \underset{\mathcal{C}_0^\infty(I)}{(\theta(t)\psi(t))'}\, dt - \int_0^{+\infty} \bar u(t)\,\theta'(t)\,\psi(t)\, dt$

$$= \int_0^1 \underset{W^{1,p}(I)}{u(t)}\,\underset{\mathcal{C}_0^\infty(I)}{(\Theta(t)\psi(t))'}\,dt - \int_0^{+\infty} \bar u(t)\,\Theta'(t)\,\psi(t)\,dt$$

$$= -\int_0^1 u'(t)\,\Theta(t)\,\psi(t)\,dt - \int_0^{+\infty} \bar u(t)\,\Theta'(t)\,\psi(t)\,dt$$

$$= -\int_0^{+\infty} \overline{u'}(t)\,\Theta(t)\,\psi(t)\,dt - \int_0^{+\infty} \bar u(t)\,\Theta'(t)\,\psi(t)\,dt$$

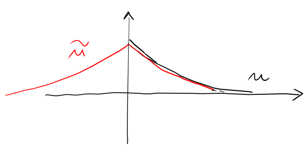
$$= -\int_0^{+\infty} \left(\bar u'\Theta + \bar u\Theta'\right)\psi$$

QED

**Lemma** let $u \in W^{1,p}(]0,+\infty[)$

consider $\tilde u(x) = \begin{cases} u(x) & \text{if } x>0 \\ u(-x) & \text{if } x<0 \end{cases}$ ũ is the reflection of

we have
- $\tilde u \in W^{1,p}(\mathbb{R})$
- $\|\tilde u\|_{L^p(\mathbb{R})} = 2^{\frac{1}{p}}\,\|u\|_{L^p(]0,+\infty[)}$
- $\|\tilde u\|_{W^{1,p}(\mathbb{R})} = 2^{\frac{1}{p}}\,\|u\|_{W^{1,p}(]0,+\infty[)}$




proof

consider $u$ as the continuous representative

define $\tilde u(x) = \begin{cases} u(x) & x \geq 0 \\ u(-x) & x \leq 0 \end{cases}$

consider $w(x) = \begin{cases} u'(x) & \text{if } x>0 \\ -u'(-x) & \text{if } x<0 \end{cases}$

I show that $w = \tilde u'$ (in the sense of distribution)

immediately $\tilde u \in L^p(\mathbb{R})$ $\left( \int_{-\infty}^{0} |\tilde u(x)|^p dx = \int_0^{+\infty} |u(x)|^p dx \right)$

(and $\|\tilde u\|^p_{L^p(\mathbb{R})} = 2\,\|u\|^p_{L^p(\mathbb{R})}$ )

similarly $w \in L^p(\mathbb{R})$ and $\|w\|^p_{L^p} = 2\,\|u'\|_{L^p}$

it remain to prove that

$$\forall \psi \in \mathcal{C}_0^\infty(\mathbb{R}) \qquad \int_{-\infty}^{+\infty} \tilde u(x)\,\psi'(x) = -\int_{-\infty}^{+\infty} w(x)\,\psi(x)\,dx$$

$\|$

$$= \int_{-\infty}^{0} \underset{AC}{u(-x)}\,\underset{L^1}{\psi'(x)}\,dx + \int_0^{+\infty} \underset{AC}{u(x)}\,\underset{L^1}{\psi'(x)}\,dx$$ we can ~~use~~ int. by parts

$$= u(-x)\psi(x)\Big|_{-\infty}^{0} - \int_{-\infty}^{0} -u'(-x)\,\psi(x)\,dx + u(x)\psi(x)\Big|_0^{+\infty} - \int_0^{+\infty} u'(x)\,\psi(x)\,dx$$

$$= \cancel{u(0)\psi(0)} - \int_{-\infty}^{0} -u'(x)\,\psi(x) - \cancel{u(0)\psi(0)} - \int_0^{+\infty} u'(x)\,\psi(x)\,dx$$

$$= -\int_{-\infty}^{+\infty} w(x)\,\psi(x)\,dx$$

QED

**Theorem** (Extension result in 1d)

let $I$ be an open interval in $\mathbb{R}$. let $p \in [1,+\infty]$

then $\exists$ $P : W^{1,p}(I) \to W^{1,p}(\mathbb{R})$ s.t.

i) $\forall u \in W^{1,p}(I)$, $Pu|_I = u$

**Theorem** (Extension result in 1d)

Let $I$ be an open interval in $\mathbb{R}$. Let $p \in [1,+\infty]$

then $\exists\ P : W^{1,p}(I) \to W^{1,p}(\mathbb{R})$ s.t.

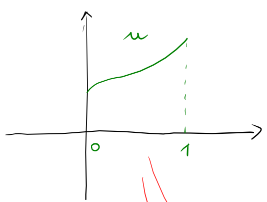
i) $\forall u \in W^{1,p}(I)$, $\quad Pu_{|_I} = u$

ii) $\exists C_0 > 0$ s.t. $\|Pu\|_{L^p(\mathbb{R})} \le C_0 \|u\|_{L^p(I)}$

iii) $\exists C_1 > 0$ s.t. $\|Pu\|_{W^{1,p}(\mathbb{R})} \le C_1 \|u\|_{W^{1,p}(I)}$

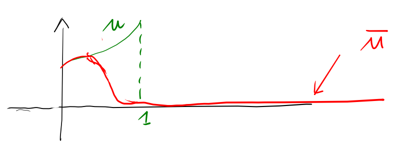
$C_0, C_1$ depend only on $p$ and $|I|$

dim  suppose $I = ]0,1[$



consider $\Theta(t) = \begin{cases} 1 & t \le \frac{1}{4} \\ 0 & t \ge 3/4 \end{cases}$

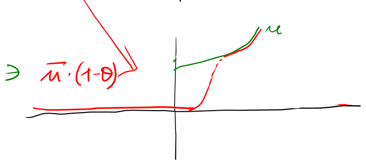
$\Theta \in \mathcal{C}_c^\infty(\mathbb{R})$

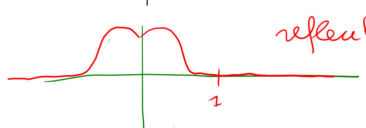


$\bar{u} \cdot \Theta \in W^{1,p}(]0,+\infty[)$ (lemma 1)

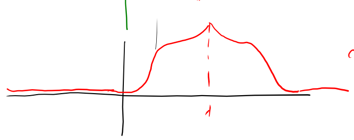
$W^{1,p}(]-\infty,1[) \ni \bar{u} \cdot (1-\Theta)$

~ lemma 1



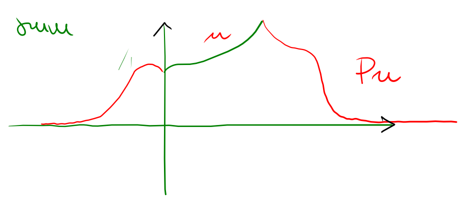


reflection of $\bar{u}\Theta$ in 0 (lemma 2)



reflection of $\bar{u}(1-\Theta)$ in 1 (~lemma 2)

take the sum



in $d \ge 2$ – we will use reflection

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x_d < 0 \\ u(x') & \text{if } x_d > 0 \end{cases}$$

$x' = (x_1, \dots, x_{d-1}, -x_d)$

$x = (x_1, \dots, x_{d-1}, x_d)$