Lezione 25 - 5/5/2026

Extension operator in 1 d

Th Let I open interval in $\mathbb{R}$. Let $p \in [1,+\infty]$

Then there exist

$P : W^{1,p}(I) \longrightarrow W^{1,p}(\mathbb{R})$ s.t.

$$\left.\begin{array}{l} i) \ Pu|_I = u \\ ii) \ \exists C_0 > 0 \text{ s.t. } \|Pu\|_{L^p(\mathbb{R})} \le C_0 \|u\|_{L^p(I)} \\ iii) \ \exists C_1 > 0 \text{ s.t. } \|Pu\|_{W^{1,p}(\mathbb{R})} \le C_1 \|u\|_{W^{1,p}(I)} \end{array}\right\} \forall u \in W^{1,p}(I)$$

rem. C_0, C_1 depend only on p and $|I|$

rem. Extension in 5 steps. Suppose $I =]0,1[$

1) $W^{1,p}(I) \ni u \longrightarrow \theta \bar{u} \in W^{1,p}(]0,+\infty[)$ — by "truncation"

$\theta \in C^\infty(]0,+\infty[)$ $\quad \theta(x) = \begin{cases} 1 & x < \frac{1}{4} \\ 0 & x > \frac{3}{4} \end{cases}$

$\bar{u}(x) = \begin{cases} u(x) & x \in]0,1[\\ 0 & x \geq 1 \end{cases}$

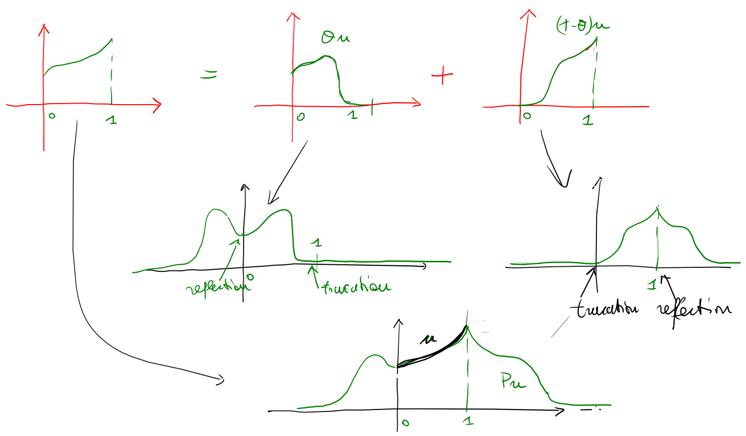
2) $\theta \bar{u} \in W^{1,p}(]0,+\infty[) \longmapsto \widetilde{\theta \bar{u}} = \begin{cases} \theta \bar{u}(x) & x \geq 0 \\ \theta \bar{u}(-x) & x < 0 \end{cases}$ — by reflection

3) $W^{1,p}(]0,1[) \ni u \longmapsto \underset{\underset{W^{1,p}]-\infty,1[}{\cap}}{(1-\theta)\bar{\bar{u}}}$ $\quad \bar{\bar{u}}(x) = \begin{cases} u(x) & x \in]0,1[\\ 0 & x \le 0 \end{cases}$

$(1-\theta)(x) = \begin{cases} 1-\theta(x) & x \in]0,1[\\ 0 & x \le 0 \end{cases}$

4) $W^{1,p}(]-\infty,1[) \ni (1-\theta)\bar{\bar{u}} \longmapsto \widetilde{(1-\theta)\bar{\bar{u}}} = \begin{cases} (1-\theta)\bar{\bar{u}} & x \le 1 \\ (1-\theta)\bar{\bar{u}}(2-x) & x > 1 \end{cases}$

5) sum up $\quad u \longmapsto \underset{\underset{W^{1,p}(\mathbb{R})}{\cap}}{\widetilde{\theta\bar{u}} + \widetilde{(1-\theta)\bar{\bar{u}}}}$



Extension operator in $d \geq 2$

It is possible to prove

Th (change of variables)

Let Ω and Ω' two open (connected) sets in $\mathbb{R}^d$

Let $\Phi : \Omega' \to \Omega$ s.t.

Φ bijective, $\Phi \in C^1(\overline{\Omega'})$, $\Phi^{-1} \in C^1(\overline{\Omega})$, $\operatorname{Jac}\Phi \in L^\infty$, $\operatorname{Jac}\Phi^{-1} \in L^\infty$ — Jacobian matrix

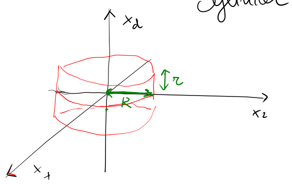
Let $u \in W^{1,p}(\Omega)$ consider $v = u \circ \Phi$

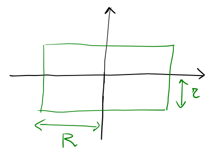
Then $v \in W^{1,p}(\Omega')$ and $\dfrac{\partial v}{\partial y_j}(y) = \displaystyle\sum_{i=1}^{d} \frac{\partial \phi_i}{\partial y_j}(y) \cdot \frac{\partial u}{\partial x_i}(\phi(y))$

def. Let $x_0 \in \mathbb{R}^d$. I denote $x = (x', x_d)$, $x' \in \mathbb{R}^{d-1}$, $x_d \in \mathbb{R}$

$$B(x_0, R, r) = \{ x \in \mathbb{R}^d : |x_0' - x'| < R, \ |x_{0_d} - x_d| < r \}$$

cylinder



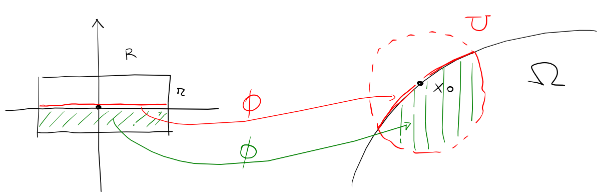


def. Let Ω be an open set in $\mathbb{R}^d$.
I say that Ω is of class $\mathcal{C}^1$ if
$\forall x_0 \in \partial\Omega$ $\exists R, r > 0$, $\exists U$ nbhd of x_0
$\exists \ \Phi : B(0; R, r) \longrightarrow U$ Φ bijective, $\mathcal{C}^1$ with Φ^{-1} $\mathcal{C}^1$

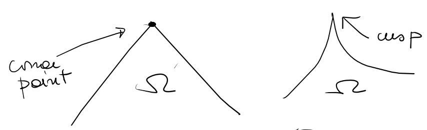
s.t.
i) $\Phi(0) = x_0$

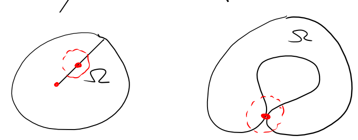
ii) $\Phi(B(0; R, r) \cap \{x_d < 0\}) = \Omega \cap U$

iii) $\Phi(B(0; R, r) \cap \{x_d = 0\}) = \partial\Omega \cap U$



not permitted





Lemma. Let $B(0; R, r)$ a cylinder in $\mathbb{R}^d$
Let $p \in [1, +\infty]$
Let $w \in W^{1,p}(B \cap \{x_d < 0\})$

define
$$\tilde{w}(x', x_d) = \begin{cases} w(x', x_d) & \text{if } (x', x_d) \in B \text{ and } x_d < 0 \\ w(x', -x_d) & \text{if } (x', x_d) \in B \text{ and } x_d > 0 \end{cases}$$

then $\tilde{w} \in W^{1,p}(B)$

and
$$\|\tilde{w}\|_{L^p(B)} \leq 2^{\frac{1}{p}} \|w\|_{L^p(B \cap \{x_d < 0\})}$$

$$\|\tilde{w}\|_{W^{1,p}(B)} \leq C \|w\|_{W^{1,p}(B \cap \{x_d < 0\})}$$

proof on Brezis' book

Theorem. Let Ω be an open (connected) set of class $\mathcal{C}^1$ in $\mathbb{R}^d$

suppose Ω is an halfspace
or
$\partial\Omega$ is bounded

let $p \in [1, +\infty]$
Then $\exists \ P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ s.t.

Theorem. Let Ω be an open (connected) set of class $\mathcal{C}^1$ in $\mathbb{R}^d$

suppose Ω is an halfspace or $\partial\Omega$ is bounded

let $p \in [1,+\infty]$

Then $\exists\, P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ s.t.

1) $Pu_{|\Omega} = u$

2) $\exists C_0$ s.t. $\|Pu\|_{L^p(\mathbb{R}^d)} \le C_0 \|u\|_{L^p(\Omega)}$

3) $\exists C_1$ s.t. $\|Pu\|_{W^{1,p}(\mathbb{R}^d)} \le C_1 \|u\|_{W^{1,p}(\Omega)}$

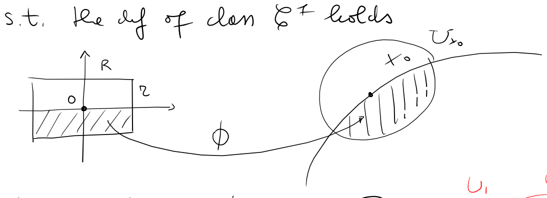
with C_0, C_1 depending only on p and Ω.

proof (idea) if Ω halfspace $\to$ reflection

if $\partial\Omega$ is bounded.

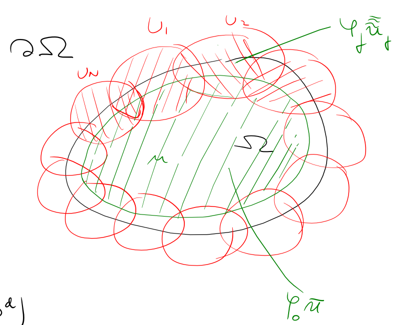
$\forall x_0 \in \partial\Omega \quad \exists R, r > 0, \ \exists\, U_{x_0}$ nbh of x_0, $\exists \phi : B(0,R,r) \to U_{x_0}$

s.t. the def of class $\mathcal{C}^1$ holds



Extract a finite covering of $\partial\Omega$

$U_1, U_2, \ldots U_N$



modifying the proof of the th on partition of unity

$\exists\, \varphi_0, \varphi_1, \ldots \varphi_N \in \mathcal{C}^\infty(\mathbb{R}^d)$

s.t $\forall j = 1, \ldots, N \quad \varphi_j \in \mathcal{C}^\infty_0(U_j)$

supp $\varphi_0 \subseteq \mathbb{R}^d \setminus \partial\Omega$

and $\forall x \in \mathbb{R}^d \quad \sum_{j=0}^N \varphi_j(x) = 1$

now: take $u \in W^{1,p}(\Omega)$

consider $u_{|\Omega \cap U_j} \in W^{1,p}(\Omega \cap U_j)$

$$\downarrow \phi^{-1}$$

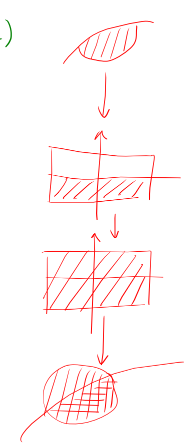
$\tilde{u}_{|B \cap \{x_d < 0\}}$

$\downarrow$ extend by reflection

$\tilde{\tilde{u}} \in W^{1,p}(B)$

$$\downarrow \phi$$

$\tilde{\tilde{\tilde{u}}} \in W^{1,p}(U_j)$



consider

$$Pu = \sum_{j=1}^N \varphi_j \cdot \tilde{\tilde{\tilde{u}}}_j + \varphi_0 \bar{u}$$

$$\bar{u} = \begin{cases} u & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$

A density result

Th. $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{1,p}(\mathbb{R}^d)$ for $1 \leq p < +\infty$.

proof. let $(\rho_n)_n$ a family of mollifiers

$\left(\rho_n(x) = n^d \rho(nx), \quad \rho \in C_0^\infty(\mathbb{R}^d), \ \rho(x) \geq 0, \ \text{supp}\,\rho \subseteq B(0,1), \ \int_{\mathbb{R}^d} \rho(x)\,dx = 1 \right)$

let $\chi \in C_0^\infty(\mathbb{R}^d)$ cut off function

$$\chi(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| \geq 2 \end{cases}$$

$$\chi_n(x) = \chi\left(\frac{x}{n}\right)$$

let $u \in W^{1,p}(\mathbb{R}^d)$

consider $u_n(x) = \chi_n(x) \cdot ((\rho_n * u)(x))$

1) $u_n \in C_0^\infty(\mathbb{R}^d)$ $\to$ $\rho_n * u \in C^\infty(\mathbb{R}^d)$, $\to$ $\chi_n \in C_0^\infty(\mathbb{R}^d)$

try to prove this lemma

2) $\partial_{x_j} u_n = \partial_{x_j}(\chi_n(x)) \cdot ((\rho_n * u)(x)) + \chi_n(x)\, \partial_{x_j}((\rho_n * u)(x))$

$= \frac{1}{n} \partial_j \chi\left(\frac{x}{n}\right) \cdot ((\rho_n * u)(x)) + \chi_n(x) ((\rho_n * \partial_j u)(x))$

(lemma $u \in W^{1,p}(\mathbb{R}^d)$, $f \in L^1(\mathbb{R}^d)$
then $f * u \in W^{1,p}(\mathbb{R}^d)$
and $\partial_{x_j}(f * u) = f * \partial_j u$)

$$\| u_n - u \|_{L^p(\mathbb{R}^d)} = \| \chi_n (\rho_n * u) - u \|_{L^p(\mathbb{R}^d)}$$

$$\leq \| \chi_n (\rho_n * u) - \chi_n u \|_{L^p} + \| \chi_n u - u \|_{L^p}$$

$$\leq \underbrace{\|\chi_n\|_{L^\infty}}_{\leq 1} \underbrace{\| \rho_n * u - u \|_{L^p}}_{\xrightarrow{n \to +\infty} 0} + \underbrace{\| \chi_n u - u \|_{L^p}}_{\xrightarrow{n \to +\infty} 0}$$

we have seen this

by dom. conv.

$$\| \partial_{x_j} u_n - \partial_{x_j} u \|_{L^p} = \left\| \frac{1}{n} \partial_{x_j}\chi\left(\frac{x}{n}\right)(\rho_n * u) + \chi_n (\rho_n * \partial_j u) - \partial_j u \right\|_{L^p}$$

$$\leq \left\| \frac{1}{n} \partial_j \chi\left(\frac{x}{n}\right)(\rho_n * u) \right\|_{L^p} + \underbrace{\| \chi_n (\rho_n * \partial_j u) - \partial_j u \|_{L^p}}_{\to 0 \text{ as before}}$$

$$\leq \frac{1}{n} \underbrace{\| \partial_j \chi \|_{L^\infty}}_{\leq C} \underbrace{\| \rho_n * u \|_{L^p}}_{\leq \|u\|_{L^p}}$$

QED

Corollary let $u \in W^{1,p}(I)$ (in 1d) or $u \in W^{1,p}(\Omega)$

Ω half space
or Ω of class C^1
with bounded $\partial\Omega$.
(in $d \geq 2$)

$p \neq +\infty$

Corollary. let $u \in W^{1,p}(I)$ (in 1d) or $u \in W^{1,p}(\Omega)$

Ω half space or Ω of class C^1 with bounded $\partial\Omega$. (in $d \geq 2$)

then $\exists (u_n)_n$ in $C_0^\infty(\mathbb{R})$ $(\text{or } C_0^\infty(\mathbb{R}^d))$

s.t. $u_n|_I \longrightarrow u$ in $W^{1,p}(I)$

or $u_n|_\Omega \longrightarrow u$ in $W^{1,p}(\Omega)$

proof. take $u \in W^{1,p}(I)$ $(\text{or } W^{1,p}(\Omega))$

Extend u to $Pu \in W^{1,p}(\mathbb{R})$ $(\text{or } W^{1,p}(\mathbb{R}^d))$

apply the density theorem

we have $\exists (u_n)_n$ in $C_0^\infty(\mathbb{R})$ $(\text{or } C_0^\infty(\mathbb{R}^d))$

s.t.
$$\|u_n - Pu\|_{W^{1,p}(\mathbb{R}^d)} \xrightarrow{n} 0$$
$$\vee|$$
$$\|u_n|_\Omega - Pu|_\Omega\|_{W^{1,p}(\mathbb{R}^d)} \xrightarrow{n} 0$$
$$\|$$
$$\|u_n|_\Omega - u\|_{W^{1,p}(\Omega)} \longrightarrow 0$$

Rem. It is possible to prove that

$\Omega \subseteq \mathbb{R}^d$, Ω open (connected)

let $p \in [1, +\infty[$ let $v \in W^{1,p}(\Omega)$

Then $\exists (v_n)_n \in C^\infty(\mathbb{R}^d)$

s.t. $v_n|_\Omega \longrightarrow v$ in $W^{1,p}(\Omega)$

(theorem of Meyers and Serrin)

$\boxed{\text{Sobolev embeddings}}$ 1d

Th let I be an interval in $\mathbb{R}$

let $p \in [1, +\infty]$

let $u \in W^{1,p}(I)$.

then $u \in L^\infty(I)$.

moreover $\exists C$ (depending only on p and $|I|$)

s.t. $\|u\|_{L^\infty(I)} \leq C \|u\|_{W^{1,p}(I)}$

($W^{1,p}(I) \hookrightarrow L^\infty(I)$ with continuous ~~injection~~ embedding)

proof. Step 1 let $I = \mathbb{R}$

if $p = +\infty$ there is nothing to prove

let $1 \leq p < +\infty$

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if $p = +\infty$ there is nothing to prove

let $1 \leq p < +\infty$

consider $v \in \mathcal{C}^1_0(\mathbb{R})$

$p \in [1, +\infty[$, consider $G(s) = s \cdot |s|^{p-1}$ this function is $\mathcal{C}^1$

$$G'(s) = p\,|s|^{p-1}$$

consider $G(v(t))$ this is a $\mathcal{C}^1_0(\mathbb{R})$

$$G(v(t)) = \int_{-\infty}^{t} \big(G(v(t))\big)'\,dt \qquad |G(s)| = |s|^p$$

$$G(v(t)) = \int_{-\infty}^{t} p\,|v(t)|^{p-1}\,v'(t)\,dt$$

$$\sup_{t\in\mathbb{R}} |G(v(t))| \leq \int_{-\infty}^{+\infty} p\,|v(t)|^{p-1}\,|v'(t)|\,dt$$

so that

$$\sup_{t\in\mathbb{R}} |v(t)|^p \leq \int_{-\infty}^{+\infty} p\,\underbrace{|v(t)|^{p-1}}_{L^{p'}}\,\underbrace{|v'(t)|}_{L^p}\,dt$$

$$\frac{1}{p} + \frac{1}{p'} = 1 \qquad p' = \frac{p}{p-1}$$

Hölder

$$\|v\|^p_{L^\infty} \leq p\,\big\||v|^{p-1}\big\|_{L^{p'}}\,\|v'\|_{L^p}$$

$$\big\||v|^{p-1}\big\|_{L^{p'}} = \left(\int \left(|v|^{p-1}\right)^{p'}\right)^{\frac{1}{p'}} = \left(\int \left(|v|^{\cancel{p-1}}\right)^{\frac{p}{\cancel{p-1}}}\right)^{\frac{p-1}{p}}$$

finally

$$\|v\|^p_{L^\infty} \leq p\,\|v\|^{p-1}_{L^p}\,\|v'\|_{L^p}$$

$$\boxed{\forall v \in \mathcal{C}^1_0(\mathbb{R})} \; \boxed{\|v\|_{L^\infty(\mathbb{R})} \leq p^{\frac{1}{p}}\,\|v\|^{1-\frac{1}{p}}_{L^p(\mathbb{R})}\,\|v'\|^{\frac{1}{p}}_{L^p(\mathbb{R})}}$$