Introduction to Space Forms

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Constant Sectional Curvature Manifolds

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Up to scaling, $K \in \{-1, 0, 1\}$ and then

- $K = 0$: $\mathbb{R}^n$ (with the Euclidean metric)
- $K = 1$: S^n (with the induced metric since $S^n \subset \mathbb{R}^{n+1}$)
- $K = -1$: $H^n = \{(x_1, \ldots, x_n) \ : \ x_n > 0\}$ (with the hyperbolic metric tensor $g_{ij} = \dfrac{\delta_{ij}}{x_n^2}$)

Conformal Metrics on Manifolds

Definition

We say that two metrics $\langle \; , \; \rangle$ and $\langle\langle \; , \; \rangle\rangle$ on a manifold M are *conformal* if there exists a positive differentiable function $\kappa : M \to \mathbb{R}$ such that

$$\langle u, v \rangle_p = \kappa(p) \langle\langle u, v \rangle\rangle_p$$

for all $p \in M$ and for all $u, v \in T_pM$.

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For example, the metric $\dfrac{\delta_{ij}}{x_n^2}$ of H^n is conformal to the Euclidean metric of $\mathbb{R}^n$ restricted to H^n. In H^n consider, more in general, the metric

$$g_{ij} = \frac{\delta_{ij}}{L^2} \quad (\kappa = 1/L^2)$$

conformal to the Euclidean metric of $\mathbb{R}^n$ restricted to H^n.

Conformal Metrics on Manifolds

Put $\ell = \log L$ and observe that $L^2\delta_{ij}$ is the inverse of g_{ij}, i.e. $L^2\delta_{ij} := g^{ij}$ and $\sum_m g^{im}g_{mj} = \delta_{ij}$.

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$$
\begin{aligned}
R_{ijij} &= \sum_m R^m_{iji} g_{mj} = R^j_{iji} g_{jj} = \frac{R^j_{iji}}{L^2} = \\
&= \frac{1}{L^2}\left(\sum_m \Gamma^m_{ii}\Gamma^j_{jm} - \sum_m \Gamma^m_{ji}\Gamma^j_{im} + \frac{\partial}{\partial x_j}\Gamma^j_{ii} - \frac{\partial}{\partial x_i}\Gamma^j_{ji}\right)
\end{aligned}
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Since

$$\Gamma^j_{ii} = \frac{\partial}{\partial x_j}\ell, \quad \Gamma^i_{ij} = -\frac{\partial}{\partial x_j}\ell, \quad \Gamma^j_{ij} = -\frac{\partial}{\partial x_i}\ell, \quad \Gamma^i_{ii} = -\frac{\partial}{\partial x_i}\ell$$

and $\Gamma^k_{ij} = 0$ we conclude that

$$L^2 R_{ijij} = -\sum_{m\ m\neq i,\ m\neq j}\left(\frac{\partial}{\partial x_m}\ell\right)^2 + \left(\frac{\partial}{\partial x_i}\ell\right)^2 - \left(\frac{\partial}{\partial x_j}\ell\right)^2 - \left(\frac{\partial}{\partial x_i}\ell\right)^2 + \left(\frac{\partial}{\partial x_j}\ell\right)^2 + \frac{\partial^2}{\partial x_j^2}\ell + \frac{\partial^2}{\partial x_i^2}\ell$$

Constant Negative Sectional Curvature Riemannian Manifolds

Therefore, if K_{ij} is the sectional curvature with respect to the 2-dimensional subspace spanned by $\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right\}$ (which are orthogonal), we have

$$K_{ij} = \frac{R_{ijij}}{g_{ii}g_{ij}} = \frac{R_{ijij}}{1/L^4} = \left(-\sum_m \left(\frac{\partial}{\partial x_m}\ell\right)^2 + \left(\frac{\partial}{\partial x_i}\ell\right)^2 + \left(\frac{\partial}{\partial x_j}\ell\right)^2 + \frac{\partial^2}{\partial x_j^2}\ell + \frac{\partial^2}{\partial x_i^2}\ell\right) L^2$$

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If $L^2 = x_n^2$, then $\ell = \log x_n$, hence, if $i \neq n$ and $j \neq n$,

$$K_{ij} = \left(\frac{\partial}{\partial x_n} \log x_n \right)^2 \cdot x_n^2 = -\frac{1}{x_n^2} \cdot x_n^2 = -1;$$

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if $i = n$ and $j \neq n$

$$K_{nj} = \left[-\left(\frac{\partial}{\partial x_n}\log x_n\right)^2 + \left(\frac{\partial}{\partial x_n}\log x_n\right)^2 + \left(\frac{\partial^2}{\partial x_n^2}\log x_n\right)\right] x_n^2 = -1$$

and, similarly, $K_{in} = -1$.

Constant Negative Sectional Curvature Riemannian Manifolds

Remark

Any isometry in $\mathbb{R}^n$ which involves only the variable $x_1, \ldots, x_{n-1}$ does not affect $g_{ij} = \delta_{ij}/x_n^2$ and hence is an isometry in (H^n, g_{ij}).

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Hopf-Killing Thm

Theorem

A complete, simply connected manifold with constant sectional curvature K is isometric (up to scaling) to one of the three models $\mathbb{R}^n, S^n$ or H^n.

Hopf-Killing (Sketch of the) proof

Proof.

Case $K \leq 0$ ($K = 0$ or $K = -1$):
If Ω is either $\mathbb{R}^n$ or H^n, take $p \in \Omega$ and $\tilde{p} \in \tilde{M}$ and a linear isometry $\varphi : T_p\Omega \to T_{\tilde{p}}\tilde{M}$.

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$$\Phi = \exp_{\tilde{p}} \circ \varphi \circ \exp_p^{-1}$$

is an isometry.

Case $K > 0$ ($K = 1$):
Take $p \in S^n$ and $\tilde{p} \in \tilde{M}$ and a linear isometry $\varphi : T_pS^n \to T_{\tilde{p}}\tilde{M}$. If $q \in S^n$ is antipodal to p, define

$$\Phi = \exp_{\tilde{p}} \circ \varphi \circ \exp_p^{-1} : S^n \setminus \{q\} \to \tilde{M}$$

by Cartan-Hadamard Theorems it follows that Φ is a local isometry.

□

Hopf-Killing (Sketch of the) proof

Proof.

Take now $p' \in S^n \setminus \{p, q\}$ and put $\tilde{p}' := \Phi(p')$ and define

$$\Phi' = \exp_{\tilde{p}'} \circ d\Phi_{p'} \circ \exp_{p'}^{-1} : S^n \setminus \{q'\} \to \tilde{M}$$

where $q' \in S^n$ is antipodal to p'.

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Φ' is a local isometry.
Furthermore we have $p' \in S^n \setminus \{q, q'\} = W$, W is connected and

$$\tilde{p}' = \Phi(p') = \Phi'(p') \quad d\Phi_{p'} = d\Phi'_p$$

hence $\Phi = \Phi'$ on W.

□

Liouville Thm

Definition

A map $\varphi : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is called *conformal* if the amplitude of the angle between any two vectors v_1 and v_2 at $p \in U$ is the same of the angle between $d\varphi_p(v_1)$ and $d\varphi_p(v_2)$ at $\varphi(p)$.

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Theorem (Liouville)

Any conformal map $\varphi : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ with $n \geq 3$ is the restriction to U of a compositions of isometries, dilations or inversions of $\mathbb{R}^n$.

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Corollary

The isometries of (H^n, g_{ij}) are the restrictions to H^n of the conformal maps of $\mathbb{R}^n$ that take H^n onto itself.

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Given a (left) action of the group $(G, \cdot)$ on the set M,

$$(g, x) \mapsto gx \in M$$

we say that the action is *free* (or G acts *freely* on M) if $gx = x$ implies $g = e_G$ (where $e_G \cdot g = g \cdot e_G$ for any $g \in G$).

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is called the *orbit* of x. We say that the action is *transitive* (or G acts *transitively* on M) if $G_x = M$.

The set of all orbits is generally denoted by M/G; the natural projection $\pi_M : M \to M/G$ is defined as follows $x \mapsto G_x := \pi_M(x)$.

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If M is a topological space and $(G, \circ)$ is the group of homeomorphisms of M, then there is a canonical action of G on M, namely $(g, x) \mapsto g(x)$ for any $g \in G$ and $x \in M$.

Definition

We say that the group of homeomorphisms of the topological space M acts in a *totally discontinuous manner* if every $x \in M$ has a neighborhood U such that $g(U) \cap U = \varnothing$ for any $g \in G$, $g \neq Id_M$.

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In this case, $\pi_M : M \to M/G$ is a regular covering map, where M/G is equipped with the quotient topology and G is the group of covering (or deck) transformations.

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- is complete if and only if M is complete;

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- is complete if and only if M is complete;
- has constant curvature if and only if M has constant curvature.

Theorem

Let M be a space form (i.e. a complete manifold with constant sectional curvature) of dimension n. Then M is isometric to S^n/Γ_S $(K = 1)$ H^n/Γ_H $(K = -1)$ or $\mathbb{R}^n/\Gamma_R$ $(K = 0)$, where Γ_S , Γ_H and Γ_R are subgroups of the groups of isometries of - respectively - S^n, H^n or $\mathbb{R}^n$ which act in a properly discontinuous manner.