Lesson 26 - 11/05/2026

Sobolev embeddings 1d

Th (Sobolev) let $p \in [1,+\infty]$, let I open interval in $\mathbb{R}$

Then $W^{1,p}(I) \subseteq L^\infty(I)$ and $\exists C = C(p, |I|)$ s.t.

$\forall u \in W^{1,p}(I)$, $\|u\|_{L^\infty(I)} \leq C \|u\|_{W^{1,p}(I)}$

proof. if $p = +\infty$ then $C = 1$ is ok

let $p \in [1,+\infty[$

1st step $I = \mathbb{R}$

consider $u \in \mathcal{C}^\infty_0(\mathbb{R})$

consider $G : \mathbb{R} \to \mathbb{R}$, $G(s) = s|s|^{p-1}$ ($\Rightarrow G \in \mathcal{C}^1(\mathbb{R})$)

and $G'(s) = p|s|^{p-1}$

consider $v(t) = G(u(t))$ $v \in \mathcal{C}^1_0(\mathbb{R})$

$$\Rightarrow v(t) = \int_{-\infty}^{t} v'(s)\,ds = \int_{-\infty}^{t} G'(u(s))\cdot u'(s)\,ds = \int_{-\infty}^{t} p|u(s)|^{p-1} u'(s)\,ds$$

$$\Rightarrow \sup_{t\in\mathbb{R}} |v(t)| \leq p \int_{-\infty}^{+\infty} |u(s)|^{p-1} |u'(s)|\,ds$$

but $|v(t)| = |G(u(t))| = |u(t)|^p$

$$\|u\|^p_{L^\infty} \leq p \int_{-\infty}^{+\infty} \underbrace{|u(s)|^{p-1}}_{\in L^{p'}} \underbrace{|u'(s)|}_{\in L^p} ds$$

$\frac{1}{p'} = 1 - \frac{1}{p}$

$\frac{1}{p'} = \frac{p-1}{p}$

$p' = \frac{p}{p-1}$

Hölder

$$\leq p \left(\int \left(|u(s)|^{p-1}\right)^{\frac{p}{p-1}} ds\right)^{\frac{p-1}{p}} \left(\int_{-\infty}^{+\infty} |u'(s)|^p\right)^{\frac{1}{p}}$$

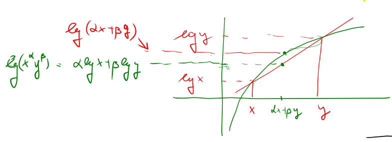
$$\|u\|^p_{L^\infty} \leq p \|u\|^{p-1}_{L^p} \|u'\|_{L^p}$$

finally $\forall u \in \mathcal{C}^\infty_0(\mathbb{R})$ $\|u\|_{L^\infty} \leq p^{\frac{1}{p}} \|u\|^{1-\frac{1}{p}}_{L^p} \|u'\|^{\frac{1}{p}}_{L^p}$

recall Young inequality $x, y > 0$

$\alpha, \beta \geq 0$ s.t. $\alpha + \beta = 1$

then $\boxed{x^\alpha y^\beta \leq \alpha x + \beta y} \leq x + y$



$$\Rightarrow \|u\|_{L^\infty(\mathbb{R})} \leq p^{\frac{1}{p}} \left(\|u\|_{L^p} + \|u'\|_{L^p}\right) \quad \forall u \in \mathcal{C}^\infty_0(\mathbb{R})$$

to pass to the case $u \in W^{1,p}(\mathbb{R})$ we use the density result $\mathcal{C}^\infty_0(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R})$ $(1 \leq p < +\infty)$

in particular let $u \in W^{1,p}(\mathbb{R})$ $\exists (u_n)_n \in \mathcal{C}^\infty_0(\mathbb{R})$

s.t. $u_n \to u$ in $W^{1,p}(\mathbb{R})$

and $u_n \to u$ a.e.

from $\|u_n\|_{L^\infty} \leq p^{\frac{1}{p}} \|u_n\|_{W^{1,p}}$

$\Rightarrow$ $(u_n)_n$ is Cauchy in $W^{1,p}$ $\Rightarrow$ $(u_n)_n$ is Cauchy in L^∞

$\Rightarrow \exists v \in L^\infty$ s.t. $u_n \to v$ in $L^\infty(\mathbb{R})$

from $\|u_n\|_{L^\infty} \le p^{\frac{1}{p}} \|u_n\|_{W^{1,p}}$

$\Rightarrow$ $(u_n)_n$ is Cauchy in $W^{1,p}$ $\Rightarrow$ $(u_n)_n$ is Cauchy in L^∞

$\Rightarrow \exists v \in L^\infty$ s.t. $u_n \to v$ in $L^\infty(\mathbb{R})$

but (possibly passing to a subsequence) $u_n \to v$ a.e.

$$\begin{array}{c} u_n \\ \downarrow \\ u \text{ a.e.} \end{array}$$

$\Rightarrow v = u \Rightarrow u \in L^\infty$

and $\|u\|_{L^\infty(\mathbb{R})} \le p^{\frac{1}{p}} \|u\|_{W^{1,p}(\mathbb{R})}$

2nd Step let $I \subseteq \mathbb{R}$ open interval

consider $u \in W^{1,p}(I)$

use the extension operator $u \longmapsto Pu \in W^{1,p}(\mathbb{R})$

then by step 1 $Pu \in L^\infty$ then $\underset{\substack{\| \\ u}}{Pu|_I} \in L^\infty(I)$

and

$$\|u\|_{L^\infty(I)} \le \|Pu\|_{L^\infty(\mathbb{R})} \le p^{\frac{1}{p}} \|Pu\|_{W^{1,p}(\mathbb{R})} \le p^{\frac{1}{p}} \underset{\substack{\uparrow \\ \text{from the ext. th.}}}{C_{p,|I|}} \|u\|_{W^{1,p}(I)}$$

QED

Corollary 1

let $u \in W^{1,p}(\mathbb{R})$, $p < +\infty$

then $\lim_{|x| \mapsto +\infty} u(x) = 0$ ($\uparrow$ u is the continuous representative)

proof. • From Sobolev we know that $\exists C_0 > 0$ s.t.

$$\|u\|_{L^\infty(\mathbb{R})} \le C_0 \|u\|_{W^{1,p}(\mathbb{R})}$$

• we know that $C_0^\infty(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R})$ ($p < +\infty$)

Given $u \in W^{1,p}(\mathbb{R})$, fixed $\varepsilon > 0$, $\exists w \in C_0^\infty(\mathbb{R})$

s.t. $\|u - w\|_{W^{1,p}(\mathbb{R})} \le \frac{\varepsilon}{C_0}$

from Sobolev $\|u - w\|_{L^\infty(\mathbb{R})} \le \varepsilon$

and $w \in C_0^\infty(\mathbb{R}) \Rightarrow \exists R > 0$ s.t. $\text{Supp } w \subseteq B(0,R)$

take $|x| > R$ $\Rightarrow$ $w(x) = 0$ $\forall x$ s.t. $|x| > R$.

now $\forall$ $|u(x)| = |u(x) - w(x)| \le \|u - w\|_{L^\infty} \le \varepsilon$

I have $\forall \varepsilon > 0, \exists R > 0 : |x| > R \Rightarrow |u(x)| < \varepsilon$

QED

Corollary let $u, v \in W^{1,p}(I)$ ($p \in [1, +\infty]$, I open interval of $\mathbb{R}$)

then $uv \in W^{1,p}(I)$ ($W^{1,p}(I)$ is an algebra)

and $\underbrace{(uv)' = u'v + uv'}_{\text{in weak sense}}$

proof. $u, v \in W^{1,p}(I) \underset{\substack{\uparrow \\ \text{Sobolev}}}{\Rightarrow} u, v \in L^\infty(I)$

then $\underset{L^\infty}{u}\,\underset{L^p}{v} \in L^p(I)$ similarly $\underset{L^p}{u'}\,\underset{L^\infty}{v} + \underset{L^\infty}{u}\,\underset{L^p}{v'} \in L^p(I)$

proof. $u, v \in W^{1,p}(I) \xRightarrow[\text{Sobolev}]{} u, v \in L^\infty(I)$

then $\underset{L^\infty}{u}\,\underset{L^p}{v} \in L^p(I)$ similarly $\underset{L^p}{u'}\underset{L^\infty}{v} + \underset{L^\infty}{u}\underset{L^p}{v'} \in L^p(I)$

the only thing to prove is that, $\forall \varphi \in C_0^\infty(I)$

$$\int_I (uv)\varphi' = -\int_I (u'v + uv')\varphi$$

1^{st} step, $1 \le p < +\infty$

let $u \in W^{1,p}(I)$. We know that $\exists (u_n)_n \in C_0^\infty(\mathbb{R})$

s.t. $u_{n|_I} \to u$ in $W^{1,p}(I)$, in L^∞ and a.e.

similarly $\exists (v_n)_n$ in $C_0^\infty(\mathbb{R})$ s.t. $v_{n|_I} \to v$ in $W^{1,p}(I)$, in L^∞, a.e.

take $\varphi \in C_0^\infty(I)$

$$\left| \int_I uv\varphi' - \int_I (u_{n|_I} v_{n|_I})\varphi' \right| \le \| uv - u_{n|_I} v_{n|_I} \|_{L^p} \|\varphi'\|_{L^{p'}}$$

and $\| uv - u_{n|_I} v_{n|_I} \|_{L^p} \le \| uv - u_{n|_I} v \|_{L^p} + \| u_{n|_I} v - u_{n|_I} v_{n|_I} \|_{L^p}$

$$\le \underbrace{\| u - u_{n|_I} \|_{L^\infty}}_{\downarrow\, 0} \|v\|_{L^p} + \underbrace{\| u_{n|_I} \|_{L^\infty}}_{\text{bounded}} \underbrace{\| v - v_{n|_I} \|_{L^p}}_{\downarrow\, 0}$$

we know that $\int_I (u_{n|_I} v_{n|_I})\varphi' = -\int_I (u'_{n|_I} v_{n|_I} - u_{n|_I} v'_{n|_I})\varphi$

It remain to verify that

$$\int [(u'_{n|_I} v_{n|_I} - u_{n|_I} v'_{n|_I}) - (u'v - uv')]\varphi \longrightarrow 0$$

similar to previous $\| u'_{n|_I} v_{n|_I} - u'v \|_{L^p(I)}$

2^{nd} step $p = +\infty$

recall that we have to prove only

$$\int_I (uv)\varphi' = -\int_I (u'v + uv')\varphi \quad \forall \varphi \in C_0^\infty(I)$$

fix $\varphi \in C_0^\infty(I)$ consider $[\alpha, \beta] \subseteq I$ s.t.

$\operatorname{supp} \varphi \subseteq]\alpha, \beta[$

then $u_{|[\alpha,\beta]}, v_{|[\alpha,\beta]} \in W^{1,p}([\alpha,\beta]) \quad \forall p \in [1, +\infty]$

so the formula of int. by parts is valid by step 1.

QED

Corollary let $G : \mathbb{R} \to \mathbb{R}$, G of class C^1 and $G(0) = 0$

let $u \in W^{1,p}(I)$

then $\boxed{v = G \circ u}$ $\quad (v(t) = G(u(t)))$

$v \in W^{1,p}(I)$ and $v'(t) = G'(u(t)) \cdot u'(t)$.

Corollary: let $G: \mathbb{R} \to \mathbb{R}$, G of class $\mathcal{C}^1$ and $G(0)=0$
let $u \in W^{1,p}(I)$
then $\boxed{v = G \circ u}$ $\quad \big(v(t) = G(u(t))\big)$
$v \in W^{1,p}(I)$ and $v'(t) = G'(u(t)) \cdot u'(t)$.

Sketch of the proof: we have to show
a) $v \in L^p(I)$
b) $G'(u(t))\, u'(t) \in L^p(I)$
c) the formula of int. by parts holds
i.e. $\forall \varphi \in \mathcal{C}^\infty_0(I)$

$$\int_I G(u(t))\, \varphi'(t) = - \int_I G'(u(t))\, u'(t)\, \varphi(t)\, dt$$

Ex ¿ $G(u(t)) \in L^p$?

$u \in W^{1,p}(I) \Rightarrow u \in L^\infty$

$G(0) = 0 \Rightarrow \underset{s \in [-M,M]}{\wedge} |G(s)| \le \sup_{s \in [-M,M]} |G'(s)| \cdot s$

$|u(t)| \le [-\|u\|_{L^\infty}, \|u\|_{L^\infty}]$

$$|G(u(t))| \le \underbrace{\sup_{s \in [-\|u\|_{L^\infty}, \|u\|_{L^\infty}]} |G'(s)|}_{=C} \cdot \underset{L^p}{|u(t)|} \Rightarrow G(u(t)) \in L^p$$

similarly $G'(u(t))\, u'(t) \in L^p$

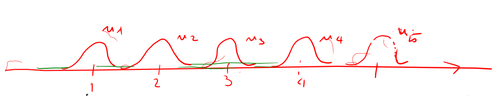
$d = 1$ compact embeddings.

Th (Rellich) $(d=1)$
let $|I| \le \infty$ $\quad$ I open interval.
let $p \in \,]1, +\infty]$
then $W^{1,p}(I) \hookrightarrow \mathcal{C}(\bar{I})$ with compact embedding
(topology of sup norm)

let $p = 1$
then $\forall q \in [1, +\infty[$, $W^{1,p}(I) \hookrightarrow L^q(I)$
with compact embedding

Rem. if $|I| = +\infty$ the embedding is not a compact embedding

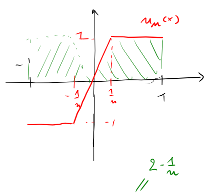
suppose $I = \mathbb{R}$,
take $\bar{u} \in \mathcal{C}^\infty_0(\mathbb{R})$ with $\|\bar{u}\|_{W^{1,p}(\mathbb{R})} \neq 0$
consider $u_n(x) = \bar{u}(x - n)$



$\forall x$, $\lim_n u_n(x) = 0$ so $u_n \to 0$ pointwise
if $\exists (u_{n_k})_k$ and $w \in \mathcal{C}^\infty_0(\mathbb{R})$ s.t. $u_{n_k} \to w$ in sup norm ($\|\cdot\|_{L^\infty}$)
then $w = 0$ but $\|u_{n_k}\|_{L^\infty} = \text{const} \neq 0$

rem, let $p = 1$ let $I = [-1, 1]$

I want to show that $W^{1,1}(I) \hookrightarrow \mathcal{C}(I)$ is not a compact embedding



$$u_n(x) = \begin{cases} 1 & \text{if } x > \frac{1}{n} \\ nx & \text{if } x \in [-\frac{1}{n}, \frac{1}{n}] \\ -1 & x < -\frac{1}{n} \end{cases}$$

$u_n \in W^{1,1}$

$$u_n'(x) = \begin{cases} 0 & x > \frac{1}{n} \\ n & x \in [-\frac{1}{n}, \frac{1}{n}] \\ 0 & x < -\frac{1}{n} \end{cases}$$

$\forall n$, $\|u_n\|_{L^1([-1,1])} \leq 2$

$\forall n$, $\|u_n'\|_{L^1([-1,1])} = 2$ $\boxed{\|u_n\|_{W^{1,1}} \leq 4 \quad \forall n}$

$(u_n)_n$ does not have a converging subsequence in $\mathcal{C}([-1,1])$

$u_n(x) \to 1$ $x > 0$ pointwise

$u_n(x) \to -1$ $x < 0$ pointwise.

$$v(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \qquad v \notin \mathcal{C}([-1,1])$$

proof (of Rellich)

1st part we have to use Ascoli-Arzelà

let $B \subseteq W^{1,p}(I)$ $(p > 1)$

let B bounded in $W^{1,p}(I)$

From Sobolev B is bounded in L^∞

and $B \subseteq \mathcal{C}(I)$ from the th. on cont. representative

It remains to prove the B is "equicontinuous"

$u \in B$, $|u(x) - u(y)| = \left|\int_y^x u'(t)\,dt\right|$

$\leq \left|\int_x^y \underset{L^{p'}}{1} \cdot \underset{L^p}{|u'(t)|}\,dt\right|$ Hölder

$\leq |y - x|^{\frac{1}{p'}} \|u'\|_{L^p}$

$\leq C\, |x - y|^{1 - \frac{1}{p}}$ $(p > 1)$

($\|u'\|_{L^p}$ is bounded) $\Rightarrow$ equicontinuous

2nd part $W^{1,1}(I) \underset{\text{compact}}{\hookrightarrow} L^q(I)$ $(+\infty > q \geq 1)$

idea use Riesz-Fréchet-Kolmogoroff theorem

Sobolev embeddings $(d \geq 2)$

Lemma (Gagliardo)

let $x \in \mathbb{R}^d$, denote $x = (x_1, x_2, \dots, x_d)$

Let $j = 1, \dots, d$, denote $\tilde{x}_j = (x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_d)$

let $f_j \in L^{d-1}(\mathbb{R}^{d-1})$ $j = 1, \dots, d$

define $f(x) = \prod_{j=1}^{d} f_j(\tilde{x}_j)$

Then $f \in L^1(\mathbb{R}^d)$ and $\|f\|_{L^1(\mathbb{R}^d)} \leq \prod_{j=1}^{d} \|f_j\|_{L^{d-1}(\mathbb{R}^{d-1})}$

Lemma (Gagliardo)

Let $x \in \mathbb{R}^d$, denote $x = (x_1, x_2, \dots, x_d)$

Let $j = 1, \dots, d$, denote $\tilde{x}_j = (x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_d)$

Let $f_j \in L^{d-1}(\mathbb{R}^{d-1})$ $\quad j = 1, \dots, d$

define $f(x) = \prod_{j=1}^{d} f_j(\tilde{x}_j)$

Then $f \in L^1(\mathbb{R}^d)$ and $\|f\|_{L^1(\mathbb{R}^d)} \leq \prod_{j=1}^{d} \|f_j\|_{L^{d-1}(\mathbb{R}^{d-1})}$

proof. $d = 2$

$f_1, f_2 \in L^1(\mathbb{R})$

$$f(x_1, x_2) = f_1(\tilde{x}_1) f_2(\tilde{x}_2) = f_1(x_2) f_2(x_1)$$

$$\|f\|_{L^1(\mathbb{R}^2)} = \|f_1 f_2\|_{L^1(\mathbb{R}^2)} = \|f_1\|_{L^1(\mathbb{R})} \|f_2\|_{L^1(\mathbb{R})}$$ trivial

$$\iint |f_1(x_2) f_2(x_1)| \, dx_1 dx_2 = \int |f_1(x_2)| \, dx_2 \int |f_2(x_1)| \, dx_1$$

$d=3$. $\quad f_1, f_2, f_3 \in L^2(\mathbb{R}^2)$

$$f(x) = f(x_1, x_2, x_3) = f_1(x_2, x_3) f_2(x_1, x_3) f_3(x_1, x_2)$$

¿ $\int |f_1(x_2,x_3) f_2(x_1,x_3) f_3(x_1,x_2)| \, dx_1 dx_2 dx_3 \leq \|f_1\|_{L^2(\mathbb{R}^2)} \|f_2\|_{L^2(\mathbb{R}^2)} \|f_3\|_{L^2(\mathbb{R}^2)}$?