A very short introduction to Riemann surfaces

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May 13, 2026

Riemann surface

Definition

A *Riemann surface* is a one-dimensional connected complex manifold.

A one-dimensional (connected) *complex* manifold is a connected, Hausdorff (or T^2) topological space X equipped with a maximal *complex atlas*, that is a family $\mathcal{U}$ of complex local coordinates $\{U_j, \varphi_j\}_{j \in J}$ where

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- if $j, k \in J$ are such that $U_j \cap U_k \neq \varnothing$, then $f_{kj} := \varphi_k \circ \varphi_j^{-1} : \varphi_j(U_j \cap U_k) \to \varphi_k(U_j \cap U_k)$ is a holomorphic function.

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Any such a function f_{kj} is called a *change of local coordinate* function or a *transition* function.

Holomorphic functions between Riemann surfaces

Definition

Given two Riemann surfaces X and Y a function $f : X \to Y$ is said to be *holomorphic* at $p \in X$ if, for any local coordinate $(U_j \varphi_j)$ of X such that $p \in U_j$ and for any local coordinate $(V_k \psi_k)$ of Y such that $f(p) \in V_k$. the function $\psi_k \circ f \circ \varphi_j^{-1}$ is holomorphic in p.

Rieman Uniformization Theorem

Theorem (Riemann Uniformization)

Every simply connected Riemann surface is biholomorphic either to the Riemann sphere $\mathbb{C} \cup \{\infty\}$ or to the complex plane $\mathbb{C}$ or to the unit disc $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$.

Existence of universal covering for Riemann surfaces.

Proposition

Given a Riemann surface X there exists a simply connected Riemann surface $\widetilde{X}$ and a holomorphic function $p : \widetilde{X} \to X$ such that $(\widetilde{X}, p)$ is a covering of X. Moreover, $\widetilde{X}$ is uniquely determined.

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$(\widetilde{X}, p)$ is the holomorphic universal covering of X.

Classification of Riemann surfaces

Definition

A Riemann surface is called *elliptic* (respectively *parabolic* or *hyperbolic*) if the universal covering space is the Riemann sphere (respectively the complex plane or the unit disk).

Deck transformations

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The set $\mathrm{Aut}(\widetilde{X})$ of all automorphisms of the universal covering $(\widetilde{X}, p)$ of X is a group with respect to standard composition $\circ$ of functions.

Major results on deck transformations.

Proposition

- *The group* $\mathrm{Aut}(\widetilde{X})$ *of all automorphisms of the universal covering* $(\widetilde{X}, p)$ *of* X *is isomorphic to the (first) fundamental group* $\Pi_1(X, x_0)$ *of* X*.*

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- *The group* $\mathrm{Aut}(\widetilde{X})$ *acts transitively on the fibers of the covering, i.e.* $\forall x \in X$ *and any pair* $y_1, y_2 \in p^{-1}(x)$ *there exists* $\varphi \in \mathrm{Aut}(\widetilde{X})$ *such that* $\varphi(y_1) = y_2$.

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- X *is biholomorphic to (the orbit space)* $\widetilde{X}/\mathrm{Aut}(\widetilde{X})$.

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We say that Γ is *properly discontinuous* at a point $x \in X$ if there exists a neighborhood U of x such that

$$\{\gamma \in \Gamma \ : \ \gamma(U) \cap U \neq \varnothing\}$$

is finite.

Remark

Remark

There is no subgroup of $\mathrm{Aut}(\widehat{\mathbb{C}})$ *acting freely besides the trivial subgroup* $\{Id\}$.

Important remark

Proposition

Let Γ be a group of automorphisms of the Riemann surface X which acts properly discontinuously at some point of X. Then Γ is discrete.

Fundamental theorem

Proposition

The group $\mathrm{Aut}(\widetilde{X})$ *of all automorphisms of the universal covering* $(\widetilde{X}, p)$ *of* X *is properly discontinuous and acts freely on* $\widetilde{X}$*. Conversely if* Γ *is a properly discontinuous subgroup of the group* $\mathrm{Aut}(\widetilde{X})$ *of deck transformations acting freely on* $\widetilde{X}$*, then* $\widetilde{X}/\Gamma$ *has a (natural) structure of Riemann surface and* $(\widetilde{X}, \pi)$*, where* $\pi : \widetilde{X} \to \widetilde{X}/\Gamma$ *is the canonical quotient–map, is its universal holomorphic covering.*

Consequences

Proposition

Two Riemann surfaces X_1 X_2 are biholomorphic if and only if they have the same universal covering surface $\widetilde{X}$ and their fundamental groups $\Pi_1(X_1, x_1)$ and $\Pi_1(X_2, x_2)$ are conjugated in the group of deck transformations $\mathrm{Aut}(\widetilde{X})$.

Classification result

Proposition

There is only one elliptic Riemann surface (up to biholomorphisms): the Riemann sphere.

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The properly discontinuous subgroups of Aut($\mathbb{C}$) acting freely on $\mathbb{C}$ are, up to conjugation,

$\{Id_{\mathbb{C}}\}$

$\{z \mapsto z + n, \ n \in \mathbb{Z}\}$

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Parabolic Riemann surfaces $\mathbb{C}/\Gamma_\tau$ are also called *complex tori*.

Very important remarks

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Theorem (Little Picard's Theorem)

Every entire function missing two values is constant.

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- $\Pi_1(X, x_0) \simeq \mathbb{Z}$ *and hence, up to biholomorphisms,* $X = \mathbb{C} \setminus \{0\}$ *or* $X = \mathbb{D} \setminus \{0\}$ *or* $X = A(1, r) = \{z \in \mathbb{C} \ : \ 1 < |z| < r, \ r > 1\}$;

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- $\Pi_1(X, x_0) \simeq \mathbb{Z} \oplus \mathbb{Z}$ *and X is a torus.*

Biholomorphic tori

Theorem

Two complex tori $\mathbb{C}/\Gamma_{\tau_1}$ and $\mathbb{C}/\Gamma_{\tau_2}$ are biholomorphic if and only if there exists a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$$

such that

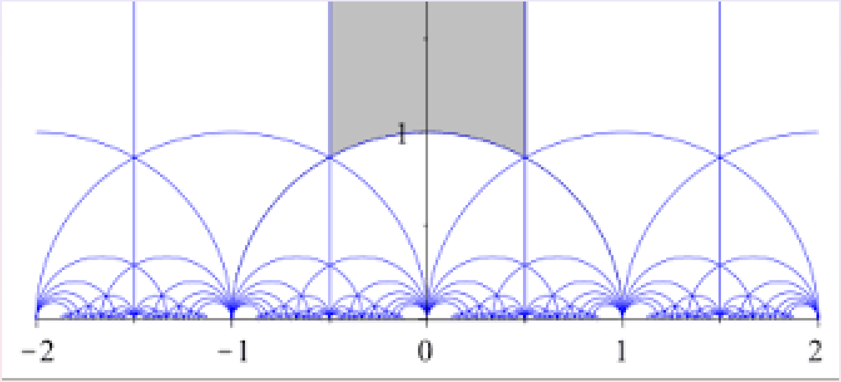
$$\tau_2 = \frac{a\tau_1 + b}{c\tau_1 + d}.$$

Biholomorphic tori

Theorem

Given a complex torus $\mathbb{C}/\Gamma_{\tau_1}$ there exists a complex torus $\mathbb{C}/\Gamma_\tau$ biholomorphic to $\mathbb{C}/\Gamma_{\tau_1}$ such that
1) Im$\tau > 0$
2) $-1/2 < \Re e\tau \leq 1/2$
3) $|\tau| > 1$
4) If $|\tau| = 1$, then $\Re e\tau_1 \geq 0$.
Furthermore τ is uniquely determined.

Complex tori



Projective plane curve

Definition

Let P be a homogeneous polynomial of degree d. Then the set

$$C_P := \{[x_0 : x_1 : x_2] \in \mathbb{CP}^2 \ : \ P(x_0, x_1, x_2) = 0\}$$

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The projective plane curve C_P is said to be *irreducible* if so is the polynomial P.

Projective plane curve

Furthermore, since $\frac{\partial P}{\partial x_j}$ is a homogeneous polynomial of degree $d-1$, the set

$$S_P := \{[x_0 : x_1 : x_2] \in C_P : \frac{\partial P}{\partial x_j}(x_0, x_1, x_2) = 0 \; j = 0, 1, 2\}$$

is well defined and is called *the singular–point set of the plane curve C_P*.

Theorem

Any non–singular projective plane curve C_P (i.e. any projective plane curve whose singular–point set is empty) is a compact Riemann surface.

Weierstrass elliptic function

Theorem

Let X be a compact Riemann surface. Then there exists an irreducible projective plane curve C_P, a holomorphic function $\Phi : X \to C_P$, and a finite set $A \subset X$ such that $\Phi : X \setminus A \to C_P \setminus S_P$ is a biholomorphism.

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In particular, using the *Weierstrass elliptic function* $\mathcal{P}_\Lambda$ associated with a $\mathbb{Z}$ lattice Λ in $\mathbb{C}$, it is possible to describe the projective plane curve associated with the complex torus $\mathbb{C}/\Lambda$.

Weierstrass elliptic function

If the complex torus is $\mathbb{C}/\Gamma_\tau$ then the associated Weierstrass elliptic function is

$$\mathcal{P}_{\Gamma_\tau}(z) := \frac{1}{z^2} + \sum_{w \in \Gamma_\tau} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

and the homogeneous polynomial P which defines C_P is

$$x_1^2 x_2 - 4x_0^3 + g_2 x_0 x_2^2 + g_3 x_2^3$$

where

$$g_2 := 60 S_4(\Gamma_\tau) \quad g_3 := 140 S_6(\Gamma_\tau)$$

and

$$S_k(\Gamma_\tau) = \sum_{w \in \Gamma_\tau \setminus \{0\}} \frac{1}{w^k}$$

Elliptic curves

The compact Riemann surfaces or projective plane curve obtained in this way are examples of *elliptic* curves (or algebraic curves of genus 1) which represent a very important example of *abelian variety*, i.e. any elliptic curve has a multiplication defined algebraically, with respect to which it is a (necessarily commutative) group.

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Elliptic curves are especially important in number theory, and constitute a major area of current research; for example, they were used in the proof, by Andrew Wiles (assisted by Richard Taylor), of Fermat's Last Theorem.

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a polynomial

$$y^2 = x(x-1)(x-\alpha)$$