Lesson 27 - 14/05/2026

Lemma (Gagliardo)

let $\tilde{x}_j = (x_1, x_2, \dots x_{j-1}, x_{j+1}, \dots x_d)$ $\quad j = 1, 2, \dots d$

let $f_j \in L^{d-1}(\mathbb{R}^{d-1})$

define $f(x) = \prod_{j=1}^{d} f_j(\tilde{x}_j)$

Then $f \in L^1(\mathbb{R}^d)$ $\quad \|f\|_{L^1(\mathbb{R}^d)} \le \prod_{j=1}^{d} \|f_j\|_{L^{d-1}(\mathbb{R}^{d-1})}$

proof $d = 2$ $\quad \underset{L^1(\mathbb{R}^1)}{\underset{\cap}{f_1(x_2)}}, \underset{L^1(\mathbb{R}^1)}{\underset{\cap}{f_2(x_1)}}$ $\quad \underset{L^1(\mathbb{R}^2)}{\underset{\cap}{f(x_1,x_2)}} = f_1(x_2) f_2(x_1)$

$$\|f\|_{L^1(\mathbb{R}^2)} = \int |f_1 f_2| \, dx_1 dx_2 = \int |f_1| dx_2 \int |f_2| dx_1 = \|f_1\| \|f_2\|$$

$d = 3$

$f_1, f_2, f_3 \in L^2(\mathbb{R}^2)$

$f(x_1, x_2, x_3) = f_1(x_2, x_3) f_2(x_1, x_3) f_3(x_1, x_2)$

to be proved $\quad f \in L^1(\mathbb{R}^3)$

$\|f\|_{L^1(\mathbb{R}^3)} \le \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2}$

$(x_1, x_2) \longmapsto |f_3(x_1, x_2)|^2 \in L^1(\mathbb{R}^2)$

$\Rightarrow$ for a.e. x_1

$x_2 \longmapsto |f_3(x_1, x_2)|^2 \in L^1(\mathbb{R}_{x_2})$

similarly

for a.e. x_1

$x_3 \longmapsto |f_2(x_1, x_3)|^2 \in L^1(\mathbb{R}_{x_3})$

consequently for a.e. x_1

$(x_2, x_3) \longmapsto |f_2(x_1, x_3)|^2 |f_3(x_1, x_2)|^2 \in L^1(\mathbb{R}^2_{x_2, x_3})$

then

for a.e. x_1 $\quad (x_2, x_3) \longmapsto |f_2(x_1, x_3)| |f_3(x_1, x_2)| \in L^2(\mathbb{R}^2_{x_2, x_3})$

but $\quad (x_2, x_3) \longmapsto |f_1(x_2, x_3)| \in L^2(\mathbb{R}^2_{x_2, x_3})$ by hyp

so, for a.e. x_1,

$|f_1(x_2, x_3)| |f_2(x_1, x_3)| |f_3(x_1, x_2)| \in L^1(\mathbb{R}^2_{x_2, x_3})$

(*) $\displaystyle \iint |f_1(x_2, x_3)| |f_2(x_1, x_3)| |f_3(x_1, x_2)| \, dx_2 dx_3 \le \|f_1\|_{L^2} \left(\iint |f_2(x_1, x_3)|^2 |f_3(x_1, x_2)|^2 dx_2 dx_3 \right)^{\frac{1}{2}}$

but $\displaystyle \left(\iint |f_2(x_1, x_3)|^2 |f_3(x_1, x_2)|^2 dx_2 dx_3 \right)^{\frac{1}{2}} = \left(\int |f_2(x_1, x_3)|^2 dx_3 \right)^{\frac{1}{2}} \left(\int |f_3(x_1, x_2)|^2 dx_2 \right)^{\frac{1}{2}}$

and $\displaystyle \int_{x_1} \underbrace{\left(\int |f_2(x_1, x_3)|^2 dx_3 \right)^{\frac{1}{2}}}_{\in L^2(\mathbb{R}_{x_1})} \cdot \underbrace{\left(\int |f_3(x_1, x_2)|^2 dx_2 \right)^{\frac{1}{2}}}_{\in L^2(\mathbb{R}_{x_1})} dx_1$

$\le \|f_2\|_{L^2} \|f_3\|_{L^2}$

conclusion integrating (*) w.r.t. x_1

we obtain

$$\int |f_1(x_2, x_3)| |f_2(x_1, x_3)| |f_3(x_1, x_2)| \, dx_1 dx_2 dx_3 \le \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2}$$

QED

Th (Sobolev - Gagliardo - Nirenberg)

let $1 \leq p < d \quad (d \geqslant 2)$

let p^* s.t. $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d} \quad \left(p^* = \frac{pd}{d-p} \right)$

Then $W^{1,p}(\mathbb{R}^d) \subseteq L^{p^*}(\mathbb{R}^d)$ and

$\exists C = C(p,d)$ s.t. $\forall u \in W^{1,p}(\mathbb{R}^d)$

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq C \, \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

proof let $u \in \mathcal{C}_0^\infty(\mathbb{R}^d)$

then $u(x) = \displaystyle\int_{-\infty}^{x_j} \frac{\partial u}{\partial x_j}(x_1, \dots x_{j-1}, t, x_{j+1} \dots x_d) \, dt$

then $\forall x \in \mathbb{R}^d$ $|u(x)| \leq \underbrace{\displaystyle\int_{-\infty}^{+\infty} \left| \frac{\partial u}{\partial x_j}(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots x_d) \right| dt}_{= g_j(\tilde{x}_j)}$ $\iff |u(x)| \leq \left(f_j(\tilde{x}_j) \right)^{d-1}$

$|u(x)|^{\frac{1}{d-1}} \leq |f_j(\tilde{x}_j)|$

I call $f_j(\tilde{x}_j) = \left(g_j(\tilde{x}_j) \right)^{\frac{1}{d-1}}$ $\quad f_j \in \mathcal{C}_0(\mathbb{R}^{d-1}) \cap L^{d-1}(\mathbb{R}^{d-1})$

I define $f(x) = \displaystyle\prod_{j=1}^{d} f_j(\tilde{x}_j)$

I apply the lemma of Gagliardo

$f \in L^1(\mathbb{R}^d) \qquad \displaystyle\int_{\mathbb{R}^d} |f(x)| \, dx \leq \prod_{j=1}^{d} \|f_j\|_{L^{d-1}(\mathbb{R}^{d-1})}$

$$|f(x)| \leq \prod_{j=1}^{d} |f_j(\tilde{x}_j)| \leq \prod_{j=1}^{d} \left(g_j(\tilde{x}_j) \right)^{d-1}$$

$$\left(\int |f_j(\tilde{x}_j)|^{d-1} d\tilde{x}_j \right)^{\frac{1}{d-1}} \leq \left(\int g_j(\tilde{x}_j) \, d\tilde{x}_j \right)^{\frac{1}{d-1}} \leq \|\nabla u\|_{L^1}^{\frac{1}{d-1}}$$

$$\int_{\mathbb{R}^{d-1}} g_j(\tilde{x}_j) \, d\tilde{x}_j = \underbrace{\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \left| \frac{\partial u}{\partial x_j}(x_1, \dots x_{j-1}, t, x_{j+1} \dots x_d) \right| dt \, d\tilde{x}_j}_{\left\| \frac{\partial u}{\partial x_j} \right\|_{L^1(\mathbb{R}^d)} \leq \|\nabla u\|_{L^1}}$$

at the end $\quad \forall j \quad \|f_j\|_{L^{d-1}(\mathbb{R}^{d-1})} \leq \|\nabla u\|_{L^1}^{\frac{1}{d-1}}$

$$\|f\|_{L^1} \leq \prod_{j=1}^{d} \|f_j\|_{L^{d-1}} \leq \|\nabla u\|_{L^1}^{\frac{d}{d-1}}$$

$f(x) = \displaystyle\prod_{j=1}^{d} |f_j(\tilde{x}_j)| \implies |u(x)|^{\frac{d}{d-1}} \leq f(x)$

condemi $\quad \left\| |u|^{\frac{d}{d-1}} \right\|_{L^1} \leq \displaystyle\prod_{j=1}^{d} \|f_j\|_{L^{d-1}} \leq \|\nabla u\|_{L^1}^{\frac{d}{d-1}}$

Gagliardo lemma + $u(x) \leq \int \left| \frac{\partial u}{\partial x_j}(\tilde{x}_j, t) \right| dt$

Consequently

$$\boxed{\|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \|\nabla u\|_{L^1(\mathbb{R}^d)} \text{ for all } u \in \mathcal{C}_0^\infty(\mathbb{R}^d)}$$

Gagliardo lemma + $u(x) \leq \int \left|\frac{\partial u}{\partial x_j}(\tilde{x}_j, t)\right| dt$

Consequently

$$\|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \|Du\|_{L^1(\mathbb{R}^d)} \quad \text{for all } u \in C_0^\infty(\mathbb{R}^d) \qquad (*)$$

remark SGN $\qquad \|u\|_{L^{p^*}} \leq C \, \|Du\|_{L^p}$

$1 < p < d$

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$$

if $p = 1$? $\quad \frac{1}{p^*} = 1 - \frac{1}{d} = \frac{d-1}{d}$

SGN for $p=1$ $\qquad \|u\|_{L^{\frac{d}{d-1}}} \leq C \, \|Du\|_{L^1}$

consider $t > 1$

consider $v(x) = |u(x)|^{t-1} \cdot u(x) \qquad v \in C_0^1(\mathbb{R}^d)$

we can apply $(*)$ to v

$$\|v\|_{L^{\frac{d}{d-1}}} \leq \|\nabla v\|_{L^1}$$

$$\|v\|_{L^{\frac{d}{d-1}}} = \left(\int |v(x)|^{\frac{d}{d-1}} dx\right)^{\frac{d-1}{d}} = \left(\int |u(x)|^{\frac{td}{d-1}} dx\right)^{\frac{d-1}{dt} \cdot t}$$

$$= \|u\|^t_{L^{\frac{td}{d-1}}}$$

$$\nabla v(x) = t \, |u(x)|^{t-1} \cdot \nabla u(x)$$

$$\|\nabla v\|_{L^1} = t \int_{\mathbb{R}^d} \underbrace{|u(x)|^{t-1}}_{\in L^{p'}} \cdot \underbrace{|\nabla u(x)|}_{\in L^p} dx \qquad \text{Hölder}$$

$$\leq t \, \|u\|^{t-1}_{L^{p'(t-1)}} \cdot \|Du\|_{L^p}$$

we obtain, $\forall u \in C_0^\infty(\mathbb{R}^d)$

$$\|u\|^t_{L^{\frac{td}{d-1}}} \leq t \, \|u\|^{t-1}_{L^{p'(t-1)}} \|Du\|_{L^p} \qquad (**)$$

$$\frac{1}{p} + \frac{1}{p'} = 1$$