Sunyaev-Zel'dovich Effect

(based on Birkinshaw 98)

• Radiation Basics

The state of a radiation field can be described by the distribution functions $f_\alpha(\vec{r},\vec{p},t) \rightarrow$ photon phase-space distribution function

α ↓ 1,2 polarization

Such that the # of particles in a infinitesimal volume of the phase space is:

$$dN_\alpha = f_\alpha(\vec{r},\vec{p},t)\, d^3r\, d^3p$$

The photon occupation number in polarization state α is:
↳ i.e. # of photons per quantum state

$$n_\alpha = \frac{dN_\alpha}{\underbrace{d^3r\, d^3p/h^3}_{\substack{\downarrow \\ \#\text{ of states in the} \\ \text{phase space volume } d^3r\, d^3p}}} = f_\alpha(\vec{r},\vec{p},t)\, h^3$$

$$h^3 \rightarrow = (\Delta x \Delta p)^3 = \text{Volume occupied by 1 state}$$

For a Planck form:

$$n_\alpha = \left(e^{h\nu/k_BT_{rad}} - 1\right)^{-1} \quad \text{for } \alpha = 1,2$$

And the # density of γ is:

$$n_\gamma = \sum_\alpha \int f_\alpha(\vec{p})\, d^3p = 16\pi \underset{\substack{\simeq \\ 1,202}}{\zeta(3)} \left(\frac{k_BT_{rad}}{hc}\right)^3$$

· The specific intensity of the radiation is:

$$I_\nu(\hat{k},\vec{r},t) = \sum_\alpha \left(\frac{h^4\nu^3}{c^2}\right) f_\alpha(\vec{r},\vec{p},t) = 2\frac{h\nu^3}{c^2}m$$

unit vector in the direction of radiation wavevector

from sum over α

∴

→ Remember

$$I_\nu = \frac{dE}{dA\,d\Omega\,dt\,d\nu} = \frac{\left(\sum_{\alpha=1}^{2} dN_\alpha\right)\cdot h\nu}{dA\,d\Omega\,dt\,d\nu} = *$$

# of γ in $d^3p\,d^3r = f_\alpha d^3r\,d^3p$

energy of a γ

Area

Solid angle

Now:

· $d^3p = p^2\,dp\,d\Omega = \left(\frac{h\nu}{c}\right)^2\left(\frac{h}{c}d\nu\right)d\Omega = \frac{h^3\nu^2}{c^3}d\nu\,d\Omega$

· $d^3r = c\,dA\,dt$ ≡ Volume swept by a γ in dt

$$* = \frac{h\nu\sum_\alpha f_\alpha\left(c\cancel{dA}\cancel{dt}\right)\left(\frac{h^3\nu^2}{c^3}\cancel{d\nu}\cancel{d\Omega}\right)}{\cancel{dA}\cancel{d\Omega}\cancel{dt}\cancel{d\nu}}$$ c.v.d.

Thus the energy crossing the area element dA in time dt from within the solid angle dΩ about $\hat{k}$, in the range ν, ν+dν is: $I_\nu(\hat{k}\cdot d\vec{A})\,d\Omega\,d\nu\,dt$

⇝ In the presence of absorption/emission/scattering processes, I_ν obey the transport equation:

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \hat{k}\cdot\vec{\nabla}I_\nu = j_\nu - \alpha_{\nu,abs}I_\nu - \alpha_{\nu,scat}I_\nu + \alpha_{\nu,scat}\int d\Omega'\,\phi_\nu(\hat{k},\hat{k}')\,I_\nu(\vec{k}')$$

absorption coefficient per unit length

emissivity along the path

scattering coefficient

Scattering Redistribution function

probabil of scatteri from $\hat{k}'$ to $\hat{k}$

* Single γ-e scattering

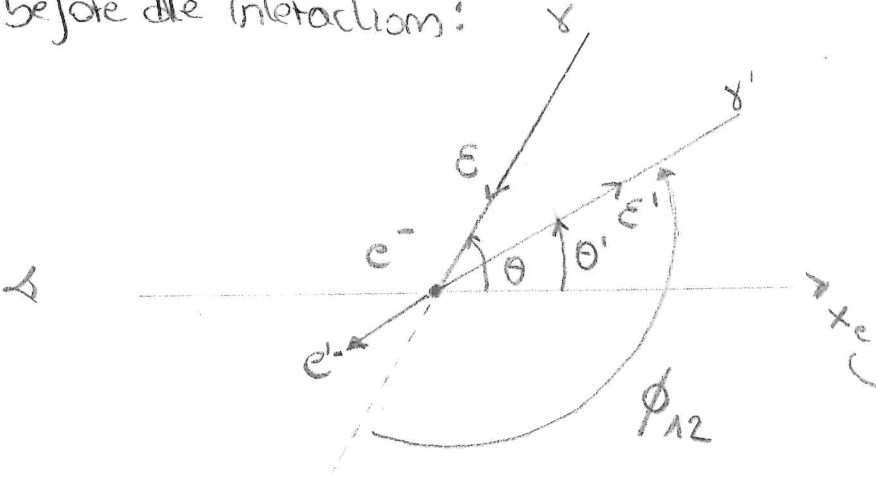
When a γ ~~and~~ is scattered by an e^-, The energy and direction of both are usually altered.

The change of properties of The γ is given by the Compton formula:

γ energy before and after

$$\varepsilon' = \frac{\varepsilon}{1+\left(\varepsilon/m_e c^2\right)\left(1-\cos\phi_{12}\right)}$$

e^- rest frame before the interaction:



↳ In the observer's rest frame the e^- is moving along x_e with speed $v_e = \beta c$

→ For low-energy γs and mildly relativistic e^-, $\varepsilon \ll m_e c^2$ and The scattering is almost elastic, i.e. $\varepsilon = \varepsilon'$ (Thomson scattering in This limit)

↓

Appropriate for SZ effect of CMB with ICM

↓

$\varepsilon_{CMB} \sim 10^{-6} m_e c^2$

↳ e^- rest frame

→ In this limit ~~the~~ the differential cross-section (Klein-Nishima formula) reduces to:

$$\frac{d\sigma}{d\Omega} = \frac{r_e^2}{2}\left(\frac{\varepsilon'}{\varepsilon}\right)^2\left(\frac{\varepsilon}{\varepsilon'}+\frac{\varepsilon'}{\varepsilon}-\sin^2\phi_{12}\right) \overset{\varepsilon\approx\varepsilon'}{\approx} \frac{r_e^2}{2}\left(1+\cos^2\phi_{12}\right)$$

$$\underbrace{U(r_e)=\frac{q^2}{4\pi\varepsilon_0 r_e}}_{\text{electron self-energy} = U @ r_e} = \underbrace{m_e c^2}_{\text{rest mass energy}} \Rightarrow r_e = \frac{q^2}{4\pi\varepsilon_0 m_e c^2}$$ classical electron radius

Defining the probability of scattering with angle θ ;

$p(\theta)\,d\theta = p(\mu)\,d\mu$, with $\mu = \cos\theta$ → Note: it also depend on the e^- speed $v_e = \beta c$ ⊛

and the probability ~~of~~ of scattering to an angle θ' ;

$\phi(\mu',\mu)\,d\mu'$

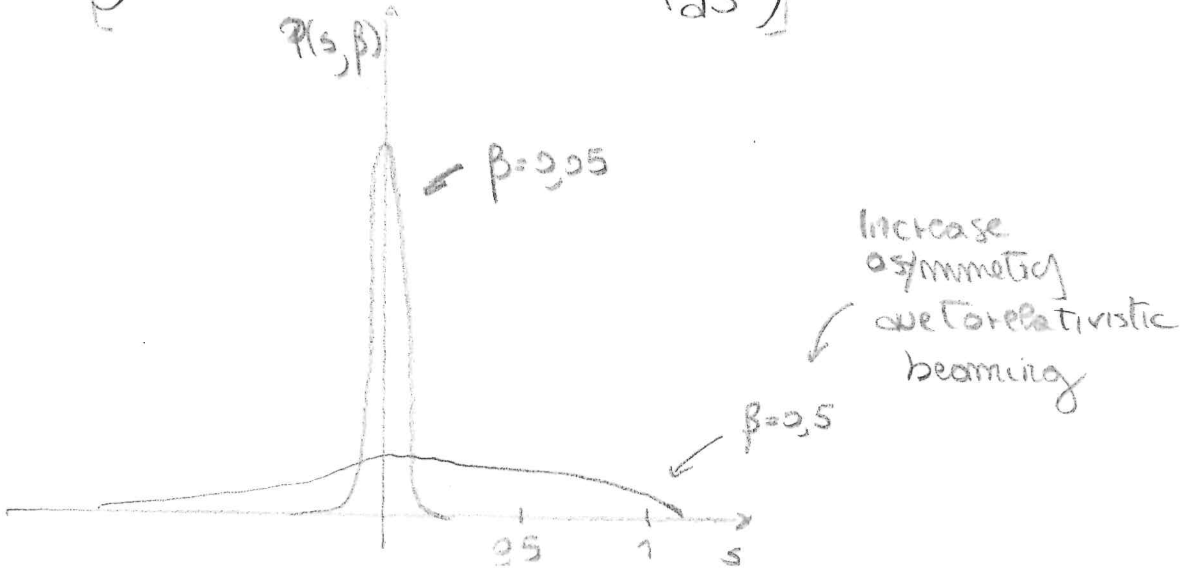
The logarithmic photon frequency shift due to the change of photon direction:

$$s = \log\left(\frac{\nu'}{\nu}\right) = \log\left(\frac{1+\beta\mu'}{1-\beta\mu}\right)^{*}$$

$* \; \nu' = \nu\dfrac{(1+\beta\mu')}{(1-\beta\mu)}$ → observed frequency of the photon after scatter

→ The probability that a single scattering causes a shift s ⊙ from an e^- with speed $v_e = \beta c$ is

$$P(s,\beta)\,ds = \left[\int \overset{⊛}{p}(\mu,\beta)\,\phi(\mu',\mu)\,d\mu\left(\frac{d\mu'}{ds}\right)\right]ds$$



* Scattering of γs by e^- population:

The distribution of γ frequency shifts caused by a populatio of e^- is derived form $P(s,\beta)$ by averaging over the electron β distribution $P_e(\beta)$.

Thus for γs that have been scattered once we have:

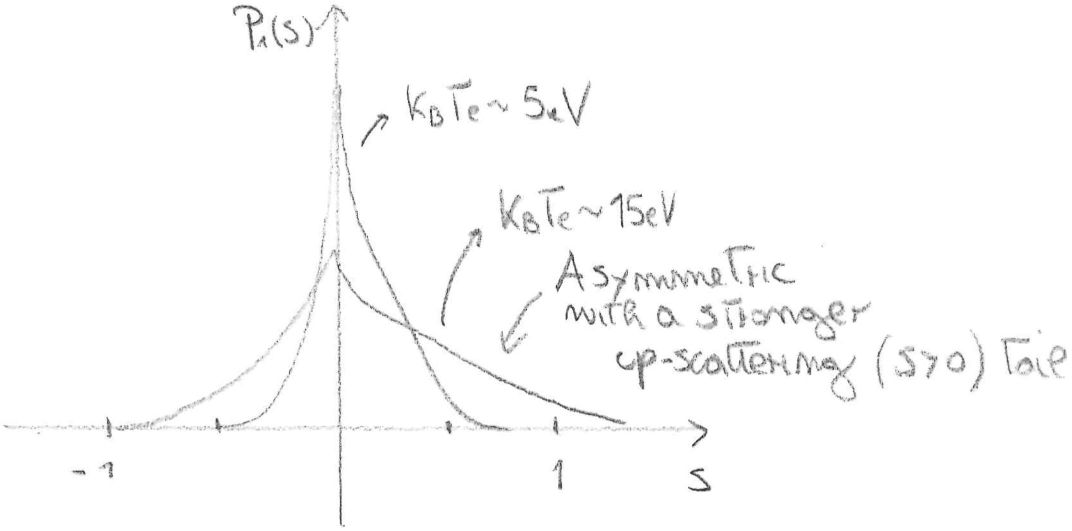
$$P_1(s) = \int_{\beta_{min}(s)}^{1} P_e(\beta)\, P(s,\beta)\, d\beta \quad \Rightarrow$$

Redistribution function: given a γ at ν_0 what is the probability ~~of~~ it ends up at ν after 1 scattering

$\beta_{min}(s)$ ↓ minimum value of β capable of causing the shift s

$$\beta_{min} = \frac{e^{|s|}-1}{e^{|s|}+1}$$

assuming a relativistic Maxwellian distributions** the probability looks like:



**

$$P_e(\beta)d\beta = \frac{\gamma^5 \beta^2 \exp(-\gamma/\Theta)}{\Theta K_2(1/\Theta)} d\beta \quad \text{with } \Theta = \frac{K_BT_e}{m_e c^2}$$

↓ modified Bessel function

∅ Effect on spectrum radiation

→ We can use these results to derive the form of the scattered CMB spectrum:

Incident spectrum: $$I_0(\nu) = \frac{2h\nu^3}{c^2}\left(e^{h\nu/kT_{rad}} - 1\right)^{-1}$$

If every γ is scattered once; the resulting spectrum is

$s = \ln \frac{\nu}{\nu_0}$ ↓ $\nu_0 \to 0$ $s \to$ $\nu_0 \to \infty$ $s \to$

$$\left[\frac{I(\nu)}{h\nu}\right. = \int_0^{\infty} d\nu_0 \, \overset{\text{prob. of scattering from } \nu_0 \text{ to } \nu}{P_1(\nu,\nu_0)} \frac{I_0(\nu_0)}{h\nu_0} = \int_{-\infty}^{+\infty} \overset{d\nu_0 = ds\,\nu_0}{ds} \, \frac{P_1(s)}{\nu} \frac{I_0(\nu_0)}{h}$$

This quantity is the # of γ per u. of. time, area, Ω, ν)

We assume that the # of γ is conserved

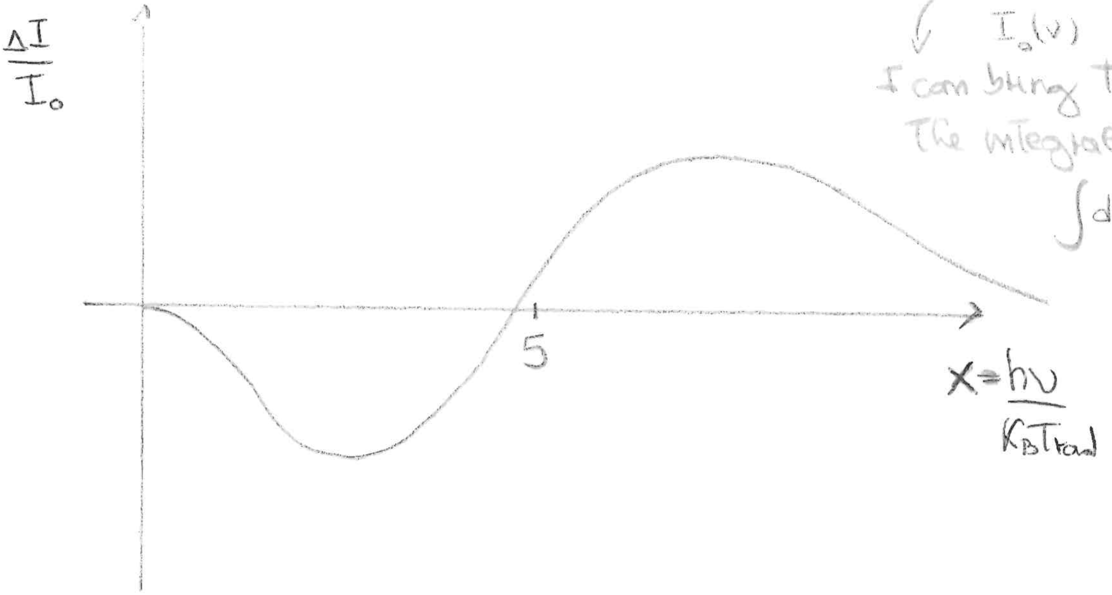
$P_1(\nu,\nu_0)d\nu = P_1(s)$

$P_1(\nu,\nu_0) = P_1(s)\frac{ds}{d\nu}$ $^{1}/\nu$

The change in the radiation spectrum @ ν is:

$$\Delta I(\nu) = I(\nu) - I_0(\nu) = \frac{2h}{c^2}\int_{-\infty}^{+\infty} ds\, P_1(s)\left(\frac{\nu_0^3}{e^{h\nu_0/k_BT_{rad}} - 1} - \underbrace{\frac{\nu^3}{e^{h\nu/k_BT_{rad}} - 1}}_{I_0(\nu)}\right)$$

I can bring this inside the integral since $\int d\nu_0 P_1(\nu,\nu_0) I_0(\nu$ $= I_0(\nu)$



⇝ In the non-relativistic limit the shape of the distortion is independent of T_e, and the amplitude proportional to T_e, for the Kompaneets kernel (see later)

→ To relate ΔI_ν to ΔT, we can interpret the shift of the BB spectrum as a shift in the effective temperature of the BB

$$dI_\nu = \frac{\partial I_\nu}{\partial T} dT \qquad I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1} \qquad x = \frac{h\nu}{k_B T}$$

$$\frac{\partial I_\nu}{\partial T} = \frac{2h\nu^3}{c^2}\left[\frac{-1}{(e^x-1)^2}\right]e^x\left(-\frac{x}{T}\right) = \frac{I_\nu}{T}\frac{xe^x}{e^x-1}$$

$$\frac{dI_\nu}{I_\nu} = \left(\frac{xe^x}{e^x-1}\right)\frac{dT}{T}$$

→ More generally a γ entering an e^- distribution may be scattered 0, 1, 2, … N times, ~~depen~~ with a probability depending on the optical depth τ_e; e.g. the prob.ty to penetrate the cloud unscattered is $e^{-\tau_e}$, the prob.ty that is scattered once is $\tau_e e^{-\tau_e}$, and in general the prob.ty of N scattering is:

$$P_N = \frac{\tau_e^N e^{-\tau_e}}{N!} \qquad \tau_e = \int dl\, n_e \sigma_T$$

and the full frequency redistribution function from scatterin is:

$$P(s) = e^{-\tau_e}\Big(\underset{\substack{\uparrow \\ \text{NO SCATTERING}}}{\delta(s)} + \tau_e P_1(s) + \frac{1}{2}\tau_e^2 P_2(s) + \dots\Big)$$

where $P_m(s)$ is given by a repeated convolution

$$P_2(s) = \int dt_1\, P_1(t_1) P_1(s-t_1) \,, \quad P_3(s) = \int dt_1 dt_2\, P_1(t_1) P_1(t_2) P_1(t-t_1-t$$

→ In the case of an optical thin medium: $(\tau_e \ll 1)$

$$\underbrace{P(s) \cong (1-\tau_e)\delta(s) + \tau_e P_1(s)}_{\text{first order in } \tau_e} \qquad e^{-\tau_e} \approx 1-\tau_e$$

so that the resulting intensity change is given by:

$$\Delta I(\nu) = \frac{2h}{c^2} \tau_e \int_{-\infty}^{+\infty} ds\, P_1(s) \left(\frac{\nu_0^3}{e^{h\nu_0/k_B T_{rad}} - 1} - \frac{\nu^3}{e^{h\nu/k_B T_{rad}} - 1} \right)$$

Independent from redshift, it depends only on the intrinsict property of medium through τ_e and $P_1(s)$

ø Non-Relativistic Limit: Kompaneets Approximation

→ In this limit the scattering kernel has a Gaussian form:

$$P_K(s)=\frac{1}{\sqrt{4\pi y}}\exp\left(-\frac{(s+3y)^2}{4y}\right) \quad \text{where } \overset{\text{Compton y-parameter}}{y}=\int n_e\sigma_T\,dl\,\frac{k_BT_e}{m_ec^2}$$

$$=\tau_e\frac{k_BT_e}{m_ec^2}$$

→ The difference with P(s) is relevant for mildly relativistic e^-, and entails spectral differences in the predicted $\Delta I(\nu)$: changing positions of minimum, zero, maximum with T_e

in this approximation (assuming $y \ll 1$) the spectral distortion is:

$$\Delta I(x)=\frac{2h}{c^2}\left(\frac{k_BT_{rad}}{h}\right)^3 x^3\Delta n(x)$$

$$\Delta n(x)=xy\frac{e^x}{(e^x-1)^2}\left(x\coth\left(\frac{x}{2}\right)-4\right)$$

change in photon occupation number due to diffusion process

$$x=\frac{h\nu}{k_BT_{rad}}$$

→ It can be seen that in this approximation the location of min/max/zero are independent of T_e, while the amplitude depends only on $y \propto T_e\tau_e$

→ Instead in the relativistic equation: (first order in θ)

$$x_{min}=2.26$$

$$x_{zero}=3.83(1+1.3\theta)$$

$$x_{max}=6{,}51(1+2.15\theta)$$

$$\theta=\left(\frac{k_BT_e}{m_ec^2}\right)$$

dimensionless e^- temperature

and the amplitude depends on complicated function of T_e

¢ The Kinematic SZ effect

→ KSZ arises if the scattering medium causing the tSZ is moving relative to the Hubble flow.

→ In the reference frame of the scattering gas the CMB radiation appears anisotropic, and the effect of the inverse Compton scattering is to re-isotropize the radiation slightly.

→ In the observer rest frame the radiation field is not isotropic anymore, but shows a structure toward the scattering medium with amplitude proportional to $\tau_e v_z/c$, where v_z is component of peculiar velocity of the scattering medium along the l.o.s.

↦ Derivation of the KSZ signal:

Assume that both tSZ and KSZ are small, and that only single scatterings are important ⇒ in this approx. the TSZ e KSZ decouples, and the KSZ can be derived by taking the e⁻ to be at rest in the frame of the scattering medium.

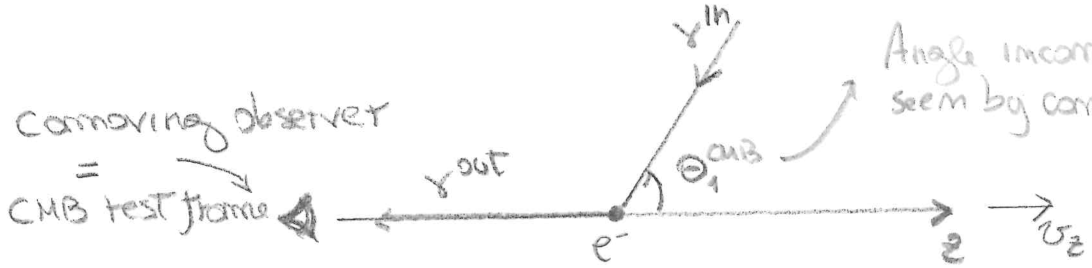
The occupation number n_α, in the scattering rest frame moving at v_z along the z-axis away from the observer is:

$$n_\alpha = \left(e^{x_1 \gamma_z (1 - \underbrace{\beta_z}_{v_z/c} \mu_1)} - 1 \right)^{-1} \qquad x_1 = \frac{h\nu_1}{K_B T_1}$$

ν_1 → frequency of the γ in the scattering medium rest frame

$\mu_1 = \cos\Theta_1$, where Θ_1 is angle of the γ w.r.t the z-axis in the scattering medium rest frame

T_1 → The CMB temperature for a comoving observer (i.e. at rest w.r.t. Hubble flow) at the redshift of the medium $T_1 = T_{CMB}(1+$



Angle incoming γ seen by comoving observer

$$\cos\theta_1 = \frac{\cos\theta_1^{CMB} - \beta_z}{1-\beta_z\cos\theta_1^{c}}$$

The γ frequency in the medium rest frame $\overset{\nu_1}{\nu}$, is related to that of the CMB rest frame by:

$$\nu_1 = \gamma_z(1+\beta_z)\nu$$

$$\mu_{CMB} = \cos\overset{180°}{\theta}_{CMB} = -1$$

→ The transport equation in the presence of scattering processes reads: (w/o absorption or emission, and $\frac{\partial I_\nu}{\partial t} = 0$

$$\frac{1}{\;}\frac{dI_\nu}{dz} = \alpha_{\nu,scat}\left[\left(\int \Phi_\nu(\hat{k},\hat{k}')\,I_\nu(\hat{k}')\,d\Omega'\right) - I_\nu(\hat{k})\right] *$$

$\hat{k}\cdot\nabla I_\nu$

scattering coefficient: fractional loss of intensity of the radiation per unit length of propagation because of scattering

Scattering redistribution function: probability of scattering from direction $\hat{k}'$ to k

$$\tau_e = \int \overset{n_e\sigma_T}{\overset{\shortparallel}{\alpha}}_{\nu,scat}\,dz \Rightarrow d\tau_e = \alpha_{\nu,scat}\,dz \Rightarrow \frac{dI_\nu}{dz} = \frac{dI_\nu}{d\tau_e}\alpha_{\nu,scat}$$

$* \int \Phi_\nu(\hat{k},\hat{k}')\,d\Omega' = 1$ quindi $I_\nu(\hat{k}) = \int \phi(\hat{k},\hat{k}')\,I_\nu(\hat{k})\,d\Omega'$

ovvero lo posso portare dentro l'integrale

→ For small optical depth, $\tau_e \ll 1$, we can assume the single scattering limit, and the derivative approximate to a difference over the entire path

$$\frac{dI_{\nu_1}}{d\tau_e} \approx \frac{I_{\nu_1}(\tau_e,\mu) - I_{\nu_1}(0,\mu)}{\Delta\tau_e}$$

$\Delta\tau_e$ = $\tau_e - 0 = \tau_e$

Intensity after passing through the medium

outgoing direction along the l.o.s. of the observer

Initial intensity befor enteri
the medium ($\tau_e = 0$)

→ so we get:

original direction

$$\frac{I_{\nu_1}(\tau_e,\mu) - I_{\nu_1}(0,\mu)}{\tau_e} \simeq \int_{-1}^{1} d\mu_1\, \phi(\mu,\mu_1)\big(I_{\nu_1}(0,\mu_1) - I_{\nu_1}(0,\mu$$

For $\mu = -1$ (i.e. $\cos\theta = -1 \Rightarrow \theta = 180°$, see figure)

γ-out point toward the observer

$\phi(\mu,\mu_1) = \frac{3}{8}(1+\mu_1^2)$ and we can write:

$$\frac{\Delta I_{\nu_1}}{I_{\nu_1}} = \tau_e \int_{-1}^{1} d\mu_1 \frac{3}{8}(1+\mu_1^2)\left(\frac{I_{\nu_1}(0,\mu_1)}{I_{\nu_1}(0,1)} - 1\right)$$

This quantity is relativistic invariant: the same fractional shift would be seen by an observer in the CMB rest frame at frequency ν. The same fractional intensity change is seen by a distant observer, for whom the scattering medium lies at z_m, after redshifting the frequency and CMB temperature

Rembering That $I_{\nu_1} = \frac{2h\nu_1^3}{c^2} n(x_1)$, using The expression for $n(x)$ and working in Terms of The frequency seen at $z=0$, ν ; The eq. becomes:

$$\frac{\Delta I_\nu}{I_\nu} = \tau_e \int_{-1}^{+1} d\mu_1 \, \frac{3}{8}\left(1+\mu_1^2\right)\left(\frac{e^x-1}{e^{x_2}-1} - 1\right)^*$$

see eq. To Transform frequencies

where: $x_2 = x_1\gamma_z(1+\beta_z) = x\gamma_z^2(1+\beta_z)(1-\beta_z\mu_1)$ and $x = \frac{h\nu}{K_B T_{rad}}$

$x_1 \rightarrow \underbrace{x\gamma_z(1-\beta_z\mu_1)}$

The medium sees an incoming photon at frequency x_1

frequency seen at $z=0$

Frequency of The γ before being scattered, as measured in The scattering medium, but expressed in Term of The observed frequency at $z=0$

→ For small β_z, we can Taylor expand The integrand and keep only The Terms linear in β_z:

$$\gamma_z \simeq 1 \qquad x_2 \simeq x(1+\beta_z)(1-\beta_z\mu_1) \simeq x + x\beta_z(1-\mu_1)$$

$$e^{x_2} \simeq e^x + \frac{de^x}{dx}(x_2-x) \simeq e^x + e^x\left[x\beta_z(1-\mu_1)\right]$$

$$^{**}\, e^{x_2} - 1 = (e^x-1) + e^x x\beta_z(1-\mu_1) = (e^x-1)\Big[1 + \underbrace{\frac{x\beta_z(1-\mu_1)}{e^x-1}}\Big]$$

This will be II order on β_z

$$^*\, \frac{e^x - e^{x_2}}{e^{x_2} - 1^{**}} \simeq -\beta_z(1-\mu_1)\frac{xe^x}{e^x-1}$$

Plugging This into The integral we get:

$$\frac{\Delta I_\nu}{I_\nu} = -\tau_e\beta_z\left(\frac{xe^x}{e^x-1}\right)\int_{-1}^{+1}\frac{3}{8}\overset{= 8/3}{\underbrace{\left(1+\mu_1^2\right)\left(1-\mu_1\right)}}d\mu_1$$

only Term with even power of μ_1 survive since The integral is symmetric

$\Delta I / T$ Vedi prima

$$\rightarrow \frac{\Delta I_\nu}{I_\nu} = -\tau_e \beta_z \frac{x e^x}{e^x - 1} \Rightarrow$$ Doppler shift of the CMB spectrum

if the cluster is moving away ($\beta_z > 0$) the signal decreases.

That is:

$$\Delta I_\nu = -\beta \tau_e I_0 \frac{x^4 e^x}{(e^x-1)^2} \leftarrow I_\nu = I_0 \frac{x^3}{e^x-1}$$

$$\downarrow$$

$$\frac{2h}{c^2}\left(\frac{k_B T_{rad}}{h}\right)^3$$

$$\Delta T_{RJ} = -\beta \tau_e T_{rad} \frac{x^2 e^x}{(e^x-1)^2} \Rightarrow \frac{\Delta T_{rad}}{T_{rad}} \approx -\tau_e \beta_z$$ decrease of the radiation temperature

$$\downarrow$$

$T_{RJ}(\nu) = \frac{c^2 I_\nu}{2 k_B \nu^2}$ = Brightness Temperature, i.e. the temperature of a thermal radiation field which in the RJ limit of low frequencies would have the same brightness $I_\nu = 2 k_B T_{rad} \nu^2 / c$

→ For a cluster of $k_B T_e \simeq 7.5$ keV, central optical density $\tau_{e0} \simeq 0{,}01$

$$\Delta T_{kSZ} \simeq -\tau_{e0} \beta_z T_{rad} \simeq -0{,}1 \left(\frac{v_z}{1000\,km/s}\right) mK << \Delta T_{tSZ} \simeq -0.8$$

$\downarrow$ for v_z of hundreds km/s

In general

$$\frac{\Delta T_{kSZ}}{\Delta T_{tSZ}} = \frac{1}{2} \beta_z \left(\frac{k_B T_e}{m_e c^2}\right)^{-1} = 0{,}085 \left(\frac{v_z}{1000\,km/s}\right) \left(\frac{k_B T_e}{10\,keV}\right)^{-1}$$

→ kSZ can be separated from tSZ using their different spectra:

The kSZ effect produces its maximum intensity change at the frequency at which tSZ effect is zero

(show this)

↓

→ ∴ in the Kompaneets approximation

- $$\Delta I_{tSZ} \propto \frac{x e^x}{e^x - 1}\left(x \coth\left(\frac{x}{2}\right) - 4\right) = g(x)$$

$$g(x) = 0 \Rightarrow x \coth\left(\frac{x}{2}\right) = 4 \Rightarrow x_0 \approx 3{,}83 \quad \nu_0 \approx 217\,\text{GHz}$$

- $$\Delta I_{kSZ} \propto \frac{x^2 e^x}{(e^x - 1)^2} = h(x) \Rightarrow \frac{d}{dx} h(x) = 0$$

$$(4 + x)(e^x - 1) - 2 x e^x = 0$$

$$x \underbrace{\frac{e^x + 1}{e^x - 1}}_{\coth\left(\frac{x}{2}\right)} = 4 \Rightarrow x \coth\left(\frac{x}{2}\right) = 4$$

C.V.C

— Given the weakness of the kSZ signal one can increase the S/N by computing the mean ΔT_{kSZ} of tracers (e.g. galaxies, ctc) separated by a distance r:

- Pairwise kSZ estimator: (Hand et al 2012)
$$\hat{T}_{pkSZ}(r) = - \frac{\sum_{i<j} (\Delta T_i - \Delta T_j)\, c_{ij}(r)}{\sum_{i<j} c_{ij}^2(r)}$$

- where ΔT_i is the redshift corrected Temperature decrement measured at location i

- $c_{ij}(r) \equiv \cos \theta_{ij}$ is the cosine between the average l.o.s. $\frac{\vec{r}_i + \vec{r}_j}{2}$ and the pair separation $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$, i.e. $\vec{r}_{ij} \cdot \frac{\vec{r}_i + \vec{r}_j}{2}$

→ Due To The Tendency of two halos (e.g. Traced by clusters) To move Toward each other under The influence of gravity, we expect:

$$T_{pkSZ}(\vec{r}) \equiv -\langle \Delta T_{i,kSZ}(\vec{x}+\vec{r}) - \Delta T_{j,kSZ}(\vec{x}) \rangle \neq 0$$

and ~~by~~ by simmetry $T_{pkSZ}(\vec{r}) = T_{pkSZ}(r)\cos\theta_{ij}$

The signal can be associated to The mean pairwise velocity of The Tracers as: (Soergel et al 2018):

$$T_{pkSZ}(r) \sim -\langle T_i - T_j \rangle \simeq \frac{T_{CMB}}{c} \langle \tau_i v_{iz} - \tau_j v_{jz} \rangle$$

$$\simeq \frac{T_{CMC}}{c} \bar{\tau} \langle v_{iz} - v_{jz} \rangle = T_{CMB} \bar{\tau} \frac{\overset{\substack{\text{mean pairwise velocity} \\ \uparrow}}{v_{12}}}{c}$$

ø Pairwise kSZ: Cosmological dependence

- From the continuity equation: (comoving coordinates)

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla\cdot\left[(1+\delta)\vec{u}\right] = 0$$

- For the pairwise velocity, we consider the statistical average for a pair of particles separated by r:

remember divergence in spherical coordinate $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f(r))$

$$\frac{\partial \xi(r,a)}{\partial t} + \frac{1}{ar^2}\frac{\partial}{\partial r}\left[r^2(1+\xi(r,a))\,\underset{\text{mean pairwise velocity}}{\underbrace{v(r,a)}}\right] = 0$$

Integro

$$\int_0^r dr'\frac{\partial}{\partial r'}\left[r'^2(1+\xi(r',a))v(r',a)\right] = \int_0^r a r'^2 \frac{\partial \xi}{\partial t}\, dr'$$

$$(1+\xi)v = -\frac{ar}{3}\frac{\partial \bar{\xi}}{\partial t} \rightarrow$$ Volume-averaged correlation function $\bar{\xi} = \frac{3}{r^3}\int_0^r dr' r'^2 \xi(r',a$

- In linear theory

$\delta(a) \propto D(a)$ Growth Factor $\rightarrow \xi(r,a) \propto D^2(a)\,\xi_0(r)$

$$\partial_t \xi = 2D(a)\dot{D}(a)\xi_0 = \frac{d\ln D}{dt} = \frac{1}{D}\dot{D} \Big/ \frac{d\ln a}{dt} = \frac{1}{a}\dot{a} \qquad = 2D\dot{D}\frac{\xi}{D^2} = 2\frac{\dot{D}}{D}\xi$$

Growth Rate: $f(a) = \frac{d\ln D}{d\ln a} = \frac{a}{\dot{a}}\frac{\dot{D}}{D} = \frac{1}{H}\frac{\dot{D}}{D}$

$$\frac{\partial \bar{\xi}}{\partial t} = 2Hf\bar{\xi}$$

Substituting $\partial_t \bar{\xi}$:

$$\Rightarrow (1+\xi)v = -\frac{ar}{3}\left(2Hf\bar{\xi}\right)$$

Solving for v:

$$v(r,a) = -\frac{2}{3}Harf\frac{\bar{\xi}(r,a)}{1+\xi(r,a)}$$

The − sign indicate that on average particles are streaming toward each other due to their mutual gravitational attraction