Lezione 28 - 15/05/2026

proof of S.G.N. theorem $\quad 1 \le p < d, \quad t > 1$

$$\|u\|^{t}_{L^{\frac{td}{d-1}}(\mathbb{R}^d)} \le t\, \|u\|^{t-1}_{L^{p'(t-1)}(\mathbb{R}^d)} \|\nabla u\|_{L^p(\mathbb{R}^d)} \qquad \forall u \in C^\infty_0(\mathbb{R}^d)$$

$p' : \frac{1}{p} + \frac{1}{p'} = 1$

idea: choose t in such a way that

$$\frac{td}{d-1} = p'(t-1) \qquad \frac{1}{p'} = 1 - \frac{1}{p} = \frac{p-1}{p}$$

$$\frac{td}{d-1} = \frac{p}{p-1}(t-1)$$

$(t-1)($

$$td(p-1) = p(t-1)(d-1)$$

$$tdp - td = ptd - pt - pd + p$$

$$-p + pd = (d-p)t$$

$$t = \frac{pd-p}{d-p} \qquad (t > 1) \text{ since } pd - p > d - p \ (p > 1)$$

$$\frac{td}{d-1} = p'(t-1) = \frac{pd}{d-p}$$

conclusion $\circledast$

$$\|u\|_{L^{\frac{pd}{d-p}}} \le \frac{pd-p}{d-p} \cdot \|\nabla u\|_{L^p} \qquad \forall u \in C^\infty_0(\mathbb{R}^d)$$

$\frac{pd}{d-p} = p^*$, $\quad \frac{pd-p}{d-p} = C(p,d)$

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d} = \frac{d-p}{pd} \qquad p^* = \frac{pd}{d-p}$$

how to obtain $\circledast$ for $u \in W^{1,p}(\mathbb{R}^d)$?

we use the density of $C^\infty_0(\mathbb{R}^d)$ in $W^{1,p}(\mathbb{R}^d)$

given $u \in W^{1,p}(\mathbb{R}^d)$ $\exists (u_n)_n$ in $C^\infty_0(\mathbb{R}^d)$

s.t. $u_n \to u$ in $W^{1,p}(\mathbb{R}^d)$ and a.e.

since $(u_n)_n$ is Cauchy in $W^{1,p}(\mathbb{R}^d)$

then $\circledast$ gives that it Cauchy in L^{p^*}

and $u_n \to v$ in L^{p^*} but $v_n \to v$ a.e.

$\Downarrow$

$v = u \in L^{p^*}$

QED

Rem. given $p \in [1, d[$

Suppose that $\exists q \in [p, +\infty[$, $\exists C_q > 0$

s.t. $\forall u \in W^{1,p}(\mathbb{R}^d)$

$$\|u\|_{L^q(\mathbb{R}^d)} \le C_q \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

now we see that necessarily $q = p^*$

the inequality $\|u\|_{L^q} \le C_q \|\nabla u\|_{L^p}$ holds for all $u \in W^{1,p}(\mathbb{R}^d)$

so that if $u \in C^\infty_0(\mathbb{R}^d)$

the the inequality holds for $v(x) = u(\lambda x)$ for all $\lambda > 0$

$$\|v\|_{L^q} = \left(\int |u(\lambda x)|^q\right)^{\frac{1}{q}} = \left(\int \lambda^{-d} |u(y)|^q dy\right)^{\frac{1}{q}} = \lambda^{-\frac{d}{q}} \|u\|_{L^q}$$

$\lambda x = y \quad dx = \frac{dy}{\lambda^d}$

$\nabla v(x) = \nabla(u(\lambda x)) = \lambda (\nabla u)(\lambda x)$

$$\|\nabla v\|_{L^p} = \left(\int |\lambda \nabla u(\lambda x)|^p\right)^{\frac{1}{p}} = \lambda \left(\int |(\nabla u)(\lambda x)|^p\right)^{\frac{1}{p}} = \lambda \cdot \lambda^{-\frac{d}{p}} \|\nabla u\|_{L^p}$$

$\|v\|_{L^q} = \lambda^{-\frac{d}{q}} \|u\|_{L^q}$

$\|\nabla v\|_{L^p} = \lambda^{1-\frac{d}{p}} \|\nabla u\|_{L^p}$

so that

$$\lambda^{-\frac{d}{q}} \|u\|_{L^q} \leq C_q \lambda^{1-\frac{d}{p}} \|\nabla u\|_{L^p}$$

finally

$$\frac{\|u\|_{L^q}}{\|\nabla u\|_{L^p}} \cdot \frac{1}{C_q} \leq \lambda^{1-\frac{d}{p}+\frac{d}{q}} \quad \forall \lambda > 0$$

↑ does not dep on λ

if $1-\frac{d}{p}+\frac{d}{q} > 0$ and $\lambda \to 0$ $\frac{\|u\|_{L^q}}{\|\nabla u\|_{L^p}} \frac{1}{C_q} \to 0$

$1-\frac{d}{p}+\frac{d}{q} < 0$ and $\lambda \to +\infty$ ~~infinite~~ the same

so necessarily

$$1-\frac{d}{p}+\frac{d}{q} = 0$$

$$\frac{d}{q} = \frac{d}{p} - 1 = \frac{d-p}{p} \Rightarrow \boxed{q = \frac{pd}{d-p}} = p^*$$

$\lambda^\alpha u(\lambda^\beta x) = v(x)$

↑ "Scaling"

S.G.N $\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p}$

$\Rightarrow u \in W^{1,p}(\mathbb{R}^d) \Rightarrow u \in L^q(\mathbb{R}^d)$ $\forall q \in [p, p^*]$

by interpolation

Corollary let $p = d$

then $W^{1,d}(\mathbb{R}^d) \subseteq L^q(\mathbb{R}^d)$

for all $q \in [d, +\infty[$

and for all $q \in [d, +\infty[$, $\exists C_q > 0$ s.t.

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C_q \|u\|_{W^{1,d}(\mathbb{R}^d)} \quad \forall u \in W^{1,d}(\mathbb{R}^d)$$

proof Go back to the proof of S.G.N.

if $u \in C_0^\infty(\mathbb{R}^d)$

we know

$$\|u\|^t_{L^{t\frac{d}{d-1}}} \leq t \|u\|^{t-1}_{L^{(t-1)p'}} \|\nabla u\|_{L^p}$$

$(t-1)p' = (t-1)\frac{p}{p-1} = (t-1)\frac{d}{d-1}$

$$\|u\|^t_{L^{t\frac{d}{p-1}}} \leq t \|u\|^{t-1}_{L^{(t-1)\frac{d}{d-1}}} \|\nabla u\|_{L^d}$$

true for all $t > 1$

I choose $t = d$

$$\|u\|^d_{L^{\frac{d^2}{d-1}}} \leq d \|u\|^{d-1}_{L^d} \cdot \|\nabla u\|_{L^d}$$

$$\|u\|_{L^{\frac{d^2}{d-1}}} \leq d^{\frac{1}{d}} \|u\|^{1-\frac{1}{d}}_{L^d} \|\nabla u\|^{\frac{1}{q}}_{L^q} \leq d^{\frac{1}{d}} (\|u\|_{L^q} + \|\nabla u\|_{L^q})$$

(Young)

$$\|u\|_{L^{\frac{d^2}{d-1}}} \leq d^{\frac{1}{q}} \|u\|_{W^{1,d}}$$

$$\frac{d^2}{d-1} = \frac{d^2-1+1}{d-1} = d+1+\frac{1}{d-1} \geq d + \frac{d}{d-1}$$

$$\boxed{\|u\|_{L^{d+\frac{d}{d-1}}} \leq d^{\frac{1}{d}} \|u\|_{W^{1,d}}}$$

$$\|u\|_{L^{d+\frac{d}{d-1}}} \leq d^{\frac{1}{d}} \|u\|_{W^{1,d}}$$

from this ↗ if $u \in W^{1,d}$

then $u \in L^{d+\frac{d}{d-1}}$

So $u \in L^q \quad \forall q \in [d, d+\frac{d}{d-1}]$

$d \longleftrightarrow d+\frac{d}{d-1}$

we go back to

$$\|u\|^t_{L^{t\frac{d}{d-1}}} \leq d \, \|u\|^{t-1}_{L^{(t-1)\frac{d}{d-1}}} \cdot \|\nabla u\|_{L^d}$$

we choose t in such a way that $(t-1)\left(\frac{d}{d-1}\right) = d + \frac{d}{d-1}$

i.e. $t = d+1$

$\frac{td-d}{d-1} = \frac{d^2}{d-1}$

$t-1 = d$

$t = d+1$

we obtain $t\frac{d}{d-1} = \frac{(d+1)d}{d-1} = \frac{d^2+d}{d-1} = d + 2\frac{d}{d-1}$

now $u \in L^q \quad \forall q \in [d, d+2\frac{d}{d-1}]$

$d \quad d+\frac{d}{d-1} \quad d+2\frac{d}{d-1} \quad d+3\frac{d}{d-1} \dots$

QED

Theorem (Morrey)

let $p > d$

then $W^{1,p}(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d)$ and $\exists C = C(p,d)$

s.t. $\|u\|_{L^\infty} \leq C \|u\|_{W^{1,p}}$

moreover $\exists C' > 0$ s.t.

for a.e. $x, y \in \mathbb{R}^d$, for all $u \in W^{1,p}(\mathbb{R}^d)$

$$|u(x) - u(y)| \leq C' |x-y|^{1-\frac{d}{p}} \|\nabla u\|_{L^p}$$

rem. if $d \geq 2$

$1 \leq p < d \Rightarrow W^{1,p}(\mathbb{R}^d) \subseteq L^q(\mathbb{R}^d) \quad \forall q \in [p, p^*]$

$p = d \Rightarrow W^{1,p}(\mathbb{R}^d) \subseteq L^q(\mathbb{R}^d) \quad \forall q \in [p, +\infty[$

$p > d \Rightarrow W^{1,p}(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d)$

proof. if $p = +\infty$

we have $W^{1,\infty}(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d)$ by def.

and $\|u\|_{L^\infty} \leq \|u\|_{W^{1,\infty}}$

and the second part is consequence of the theorem of characterization of $W^{1,p}(\Omega)$ for $1<p$

(th. let $1 < p \leq +\infty$. let $u \in L^p(\Omega)$ the following are equivalent)

- $u \in W^{1,p}(\Omega)$
- $\exists C > 0 : \forall \varphi \in C_c^\infty(\Omega) \ \forall j \quad \left| \int_\Omega u \frac{\partial \varphi}{\partial x_j} \right| \leq C \|\varphi\|_{L^{p'}(\Omega)}$
- $\exists C' > 0 : \forall \omega \subset\subset \Omega, \forall h : |h| \leq dist(\partial\Omega, \omega)$

  $\|\tau_h u - u\|_{L^p(\omega)} \leq C|h|$

Th. let $1 < p \le +\infty$. let $u \in L^p(\Omega)$ (the following are equivalent)

- $u \in W^{1,p}(\Omega)$
- $\exists C > 0 : \forall \varphi \in C_c^\infty(\Omega)\ \forall j \quad \left| \int_\Omega u \frac{\partial}{\partial x_j} \varphi \right| \le C \|\varphi\|_{L^{p'}(\Omega)}$;
- $\exists C' > 0 : \forall \omega \subset\subset \Omega,\ \forall h : |h| \le dist(\partial\Omega, \omega)$

$$\|\tau_h u - u\|_{L^p(\omega)} \le C |h|$$

in particular $u \in L^\infty(\mathbb{R}^d)$

$$u \in W^{1,\infty}(\mathbb{R}^d) \iff \|\tau_h u - u\|_{L^\infty(\omega)} \le C|h|$$

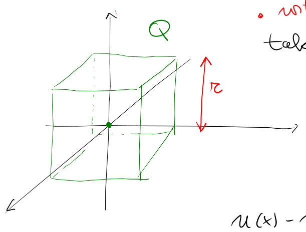
$$\| u(x-h) - u(x) \|_{L^\infty(\omega)} \le C\,|h|$$

i.e. $|u(y) - u(x)| \le C\, |x-y|$ a.e. x, y (C doesn't dep on ω)

let $d < p < +\infty$.

take $u \in C_c^\infty(\mathbb{R}^d)$

take Q a cube
- of side of length r
- containing the origin
- with sides parallel to axis



take $x \in Q$

consider $u(x) - u(0) = v(1) - v(0)$

where $v(t) = u(tx)$

$$v'(t) = \sum_{j=1}^d x_j \cdot \frac{\partial u}{\partial x_j}(tx)$$

$$u(x) - u(0) = \int_0^1 v'(s)\,ds = \int_0^1 \sum_{j=1}^d x_j \cdot \frac{\partial u}{\partial x_j}(sx)\,ds$$

given w I denote $\bar{w} = \frac{1}{|Q|} \int_Q w(x)\,dx$

so that $\bar{u} = \frac{1}{|Q|} \int_Q u(x)\,dx \qquad |Q| = r^d$

$$\bar{u} - u(0) = \frac{1}{|Q|} \int_Q u(x) - u(0)\,dx$$

so that
$$|\bar{u} - u(0)| \le \frac{1}{r^d} \int_Q |u(x) - u(0)|\,dx$$

$$\le \frac{1}{r^d} \int_Q \int_0^1 \sum_{j=1}^d |x_j| \left| \frac{\partial u}{\partial x_j}(sx) \right| ds\,dx \qquad |x_j| < r$$

$$\le \frac{1}{r^{d-1}} \int_0^1 \sum_j \left(\int_Q \left| \frac{\partial u}{\partial x_j}(sx) \right| dx \right) ds$$

$|sQ| = (sr)^d$

$$\int_Q \left| \frac{\partial u}{\partial x_j}(sx) \right| dx \qquad sx = y, \quad dx = \frac{dy}{s^d}$$

$$= s^{-d} \int_{sQ} \left| \frac{\partial u}{\partial x_j}(y) \right| dy \qquad \int_{sQ} 1 \cdot \frac{\partial u}{\partial x_j}, \quad 1 \in L^{p'},\ \frac{\partial u}{\partial x_j} \in L^p$$

$$\le s^{-d} (sr)^{\frac{d}{p'}} \cdot \left\| \frac{\partial u}{\partial x_j} \right\|_{L^p(sQ)} \qquad \left(\int_{sQ} 1 \right)^{\frac{1}{p'}} \cdot \left(\int_{sQ} \left| \frac{\partial u}{\partial x_j} \right|^p \right)^{\frac{1}{p}}$$

$$\le r^{\frac{d}{p'}} \cdot s^{\frac{d}{p'} - d} \cdot \left\| \frac{\partial u}{\partial x_j} \right\|_{L^p(sQ)}$$

$$|\bar{u} - u(0)| \le \frac{1}{r^{d-1}} \cdot r^{\frac{d}{p'}} \cdot \int_0^1 s^{d\left(\frac{1}{p'} - 1\right)} ds \cdot \sum_{j=1}^d \left\| \frac{\partial u}{\partial x_j} \right\|_{L^p(Q)}$$

$$|\bar{u} - u(0)| \le \frac{1}{r^{d-1}} \cdot r^{\frac{d}{p'}} \cdot \int_0^1 s^{d\left(\frac{1}{p'}-1\right)} ds \cdot \underbrace{\sum_{j=1}^{d} \left\| \frac{\partial u}{\partial y_j} \right\|_{L^p(Q)}}_{\|\nabla u\|_{L^p(Q)}}$$

$$|\bar{u} - u(0)| \le r^{1-d+\frac{d}{p'}} \cdot \int_0^1 s^{d\left(\frac{1}{p'}-1\right)} ds \cdot \|\nabla u\|_{L^p(Q)}$$

$d\left(\frac{1}{p'}-1\right) = -\frac{d}{p}$

$$\le r^{1-\frac{d}{p}} \cdot \int_0^1 s^{-\frac{d}{p}} ds \cdot \|\nabla u\|_{L^p(Q)}$$

$\frac{1}{p'} = 1 - \frac{1}{p}$

$\frac{1}{p'} - 1 = -\frac{1}{p}$

$$\le r^{1-\frac{d}{p}} \cdot \frac{1}{1-\frac{d}{p}} \cdot \|\nabla u\|_{L^p(Q)}$$

$d < p < +\infty$

$0 < \frac{d}{p} < 1$

$0 > -\frac{d}{p} > -1$

conclusion

$$|\bar{u} - u(0)| \le \frac{r^{1-\frac{d}{p}}}{1-\frac{d}{p}} \cdot \|\nabla u\|_{L^p(Q)}$$

$\forall u \in C_0^\infty(\mathbb{R}^d)$

Q cube ....

by shifting, $\forall x \in \mathbb{R}^d$

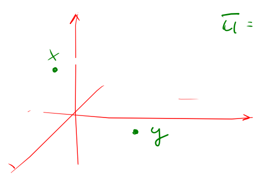
$$|\bar{u} - u(x)| \le \frac{r^{1-\frac{d}{p}}}{1-\frac{d}{p}} \cdot \|\nabla u\|_{L^p(Q)}$$

Q cube of side r containing x

$\bar{u} = \frac{1}{|Q|} \int_Q u(x)\,dx$

now take $x, y \in \mathbb{R}^d$

then $\exists\, Q$ of side $2|x-y|$ containing x, y



then

$$|u(x) - u(y)| \le |\bar{u} - u(x)| + |\bar{u} - u(y)| \le 2\, \frac{r^{1-\frac{d}{p}}}{1-\frac{d}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

$$|u(x) - u(y)| \le 2 \cdot \frac{(2|x-y|)^{1-\frac{d}{p}}}{1-\frac{d}{p}} \cdot \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

$$|u(x) - u(y)| \le C \cdot |x-y|^{1-\frac{d}{p}} \cdot \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

second part of the theorem

with the approximation in $W^{1,p}(\mathbb{R}^d)$ by $C_0^\infty(\mathbb{R}^d)$ functions we have the conclusion

It remains to prove that $u \in L^\infty$ (first part of the theorem.)

take $u \in C^\infty(\mathbb{R}^d)$

consider a cube of length of the sides $= 1$

we have $|u(x)| \le \underbrace{|u(x) - \bar{u}|}_{\le C \|\nabla u\|_{L^p}} + \underbrace{|\bar{u}|}_{\le \|u\|_{L^p}} \le (C+1)\left(\|u\|_{L^p} + \|\nabla u\|_{L^p}\right) \le C' \|u\|_{W^{1,p}}$

and $|\bar{u}| \le \frac{1}{|Q|} \int_Q |u(x)|\,dx$, $|Q| = 1$

Hölder $\longrightarrow$ $\le 1 \cdot \underbrace{\left(\int_Q 1\right)^{\frac{1}{p'}}}_{1} \cdot \underbrace{\left(\int_Q |u(x)|^p\right)^{\frac{1}{p}}}_{\le \|u\|_{L^p}}$

Interpolation

Th. $u \in L^{p_1} \cap L^{p_2}$ with $1 \le p_1 < p_2 \le +\infty$

then $u \in L^q \quad \forall q \in [p_1, p_2]$

and $\|u\|_{L^q} \le \|u\|_{L^{p_1}} + \|u\|_{L^{p_2}}$

proof. $q \in [p_1, p_2] \Rightarrow q = \alpha p_1 + (1-\alpha) p_2$
$\alpha \in [0,1]$

then $|u|^q = \underbrace{|u|^{\alpha p_1}}_{\in L^{\frac{1}{\alpha}}} \cdot \underbrace{|u|^{(1-\alpha) p_2}}_{\in L^{\frac{1}{1-\alpha}}}$

$$\left(\int \left(|u|^{\alpha p_1}\right)^{\frac{1}{\alpha}}\right)^{\alpha} = \left(\int |u|^{p_1}\right)^{\alpha} = \|u\|_{L^{p_1}}^{\alpha p_1} \qquad \left(\int \left(|u|^{(1-\alpha)p_2}\right)^{\frac{1}{1-\alpha}}\right)^{\alpha-1} = \left(\int |u|^{p_2}\right)^{\alpha-1} = \|u\|_{L^{p_2}}^{(\alpha-1)p_2}$$

Hölder $\int |u|^q \le \|u\|_{L^{p_1}}^{\alpha p_1} \cdot \|u\|_{L^{2}}^{(1-\alpha)p_2}$

$$\|u\|_{L^q} \le \left(\|u\|_{L^{p_1}}\right)^{\frac{\alpha p_1}{q}} \left(\|u\|_{L^{p_2}}\right)^{\frac{(1-\alpha)p_2}{q}} \le \|u\|_{L^{p_1}} + \|u\|_{L^{p_2}}$$

Young

QED

consequence $1 \le p < d$

S.G.N $\Rightarrow u \in W^{1,p}(\mathbb{R}^d) \Rightarrow u \in L^{p^*}(\mathbb{R}^d)$
$\Downarrow$
$u \in L^p(\mathbb{R}^d)$

$\Rightarrow \forall q \in [p, p^*] \quad \|u\|_{L^q} \le \|u\|_{L^p} + \|u\|_{L^{p^*}}$

SGN $\Rightarrow \|u\|_{L^{p^*}} \le C \|\nabla u\|_{L^p}$

$$\le \|u\|_{L^p} + C\|\nabla u\|_{L^p}$$
$$\le (1+C)\|u\|_{W^{1,p}}$$

$1 \le p < d \Rightarrow$	$W^{1,p}(\mathbb{R}^d) \subseteq L^q(\mathbb{R}^d)$	$q \in [p, p^*]$
$p = d$	$\| \subseteq L^q(\mathbb{R}^d)$	$q \in [p, +\infty[$
$p > d$	$W^{1,p}(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d) \cap \mathcal{C}(\mathbb{R}^d)$	

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take Ω of class $\mathcal{C}^1$ < $\partial\Omega$ bounded / Ω half space

then using the extension operator

$1 \le p < d$	$W^{1,p}(\Omega) \subseteq L^q(\Omega)$	$q \in [p, p^*]$
$p = d$	$W^{1,d}(\Omega) \subseteq L^q(\Omega)$	$q \in [p, +\infty[$
$p > d$	$W^{1,p}(\Omega) \subseteq L^\infty(\Omega) \cap \mathcal{C}(\Omega)$	

with continuous embedding

Th (Rellich)

Suppose Ω open bounded set in $\mathbb{R}^d$, of class $\mathcal{C}^1$

Th (Rellich)

Suppose Ω open bounded set in $\mathbb{R}^d$, of class $\mathcal{C}^1$

if $1 \leq p < d$ then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [p, p^*[$

if $p = d$ $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [p, +\infty[$

if $p > d$ $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}(\bar{\Omega})$ ← the topology of sup norm ($\|\cdot\|_{L^\infty}$)

with compact embeddings

Riesz Fréchet Kolmogoroff (see details on Brezis' book)

From Ascoli-Arzelà

$u_n(x) = n^\alpha u(n^\beta x)$

$u \in \mathcal{C}_0^\infty(B(0,1))$

Ex. prove that $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ continuous but not compact embedding