Lesson 30 - 19/5/2026

**def.** Let $p \in [1,+\infty]$, let $m \in \mathbb{N} \setminus \{0,1\}$ $(m \geq 2)$

$\Omega$ open (connected) set in $\mathbb{R}^d$

in the sense of distributions

$$W^{m,p}(\Omega) = \{ u \in L^p(\Omega) : \forall \alpha, |\alpha| \leq m \Rightarrow \partial^\alpha u \in L^p(\Omega) \}$$

$$= \{ u \in W^{m-1,p}(\Omega) : \nabla u \in W^{m-1,p}(\Omega) \}$$

ex. $W^{2,p}(\Omega) = \{ u \in W^{1,p}(\Omega) : \nabla u \in W^{1,p}(\Omega) \}$

$u \in W^{m,p}(\Omega)$ $\quad \|u\|_{W^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^p(\Omega)}$ or equivalent norms

in particular $H^m(\Omega) = W^{m,2}(\Omega)$

$$H^m(\Omega) = \{ u \in L^2(\Omega) : \partial^\alpha u \in L^2(\Omega) \text{ for all } |\alpha| \leq m \}$$

$$\|u\|^2_{H^m} = \sum_{|\alpha| \leq m} \|\partial^\alpha u\|^2_{L^2}$$

remind $H^m(\mathbb{R}^d) = \{ u \in L^2(\mathbb{R}^d) : (1+|\xi|^2)^{\frac{m}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^d) \}$

<u>Rem</u> it makes sense to consider embedding of Sobolev type also for these spaces

<u>Ex</u> let $1 \leq p < \frac{d}{2}$

consider $u \in W^{2,p}(\mathbb{R}^d)$

we know $\quad u \in W^{1,p}(\mathbb{R}^d) \Rightarrow u \in L^{p^*}(\mathbb{R}^d)$

$\qquad\qquad \nabla u \in W^{1,p}(\mathbb{R}^d) \Rightarrow \nabla u \in L^{p^*}(\mathbb{R}^d)$

$\Rightarrow u \in W^{1,p^*}(\mathbb{R}^d)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$

then $1 \leq p < p^* < d$

since $p < \frac{d}{2} \Rightarrow \frac{1}{d} < \frac{1}{2p}$

$\frac{1}{p} > \frac{2}{d}$

$d > p^*$ $\Leftarrow$ $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d} > \frac{1}{d}$

$\Rightarrow u \in L^{p^{**}}(\mathbb{R}^d)$

where $\frac{1}{p^{**}} = \frac{1}{p^*} - \frac{1}{d} = \frac{1}{p} - \frac{1}{d} - \frac{1}{d} = \frac{1}{p} - \frac{2}{d}$

conclusion

if $1 \leq p \leq \frac{d}{2}$ then $W^{2,p}(\mathbb{R}^d) \subseteq L^{p^{**}}(\mathbb{R}^d)$ with $\frac{1}{p^{**}} = \frac{1}{p} - \frac{2}{d}$

<u>Th.</u> let $m \in \mathbb{N} \setminus \{0,1\}$, $p \in [1,+\infty]$ then

i) $1 \leq p < \frac{d}{m} \Rightarrow W^{m,p}(\mathbb{R}^d) \subseteq L^q(\mathbb{R}^d)$ where $q \in [p, \tilde{p}]$ $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{m}{d}$

ii) $p = \frac{d}{m} \Rightarrow W^{m,p}(\mathbb{R}^d) \subseteq L^q(\mathbb{R}^d)$ with $q \in [p, +\infty[$

iii) $p > \frac{d}{m} \Rightarrow W^{m,p}(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d)$

equivalent

moreover, in the case iii) i.e. $m - \frac{d}{p} > 0$

denote $k$ the integer part of $m - \frac{d}{p}$

$\theta$ the fractionary part of $m - \frac{d}{p}$ $\quad \left( m - \frac{d}{p} = \underset{\substack{\Downarrow \\ \mathbb{N}}}{k} + \underset{\substack{\Downarrow \\ [0,1[}}{\theta} \right)$

supposing $\theta \in ]0,1[$

then $W^{m,p}(\mathbb{R}^d) \subseteq \mathcal{C}^k(\mathbb{R}^d)$ (bounded with all bounded derivatives)

and $\forall x,y$, $\forall \alpha$ s.t. $|\alpha| = k$

$$|\partial^\alpha u(x) - \partial^\alpha u(y)| \leq C |x-y|^\theta \cdot \|u\|_{W^{m,p}(\mathbb{R}^d)}$$

(does not dep on $u$)

$W^{m,p}(\mathbb{R}^d) \subseteq \mathcal{C}^{k,\theta}(\mathbb{R}^d)$

Hölder space

similar result for $W^{m,p}(\Omega)$ with usual properties of $\Omega$

Properties of $W_0^{1,p}(\Omega)$

def. Let $\Omega$ open (connected) set in $\mathbb{R}^d$ $(d \geq 1)$

Let $1 \leq p < +\infty$

We define $W_0^{1,p}(\Omega)$ = closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$

= closure of $\mathcal{C}_0^1(\Omega)$ in $W^{1,p}(\Omega)$

First case $\boxed{d = 1}$

Th let $I$ be an interval $(I \neq \mathbb{R} \Leftrightarrow \partial I \neq \emptyset)$

let $1 \leq p < +\infty$

let $u \in W^{1,p}(I)$, ($u$ is the continuous representative)

$u \in W_0^{1,p}(I) \iff u_{|\partial I} = 0$

proof. (idea)

"$\Rightarrow$" Let $u \in W_0^{1,p}(I)$

by def. $\exists (\varphi_n)_n$ in $\mathcal{C}_0^\infty(I)$ s.t. $\|\varphi_n - u\|_{W^{1,p}(I)} \xrightarrow{n} 0$

$\varphi_{n|\partial I} = 0$ $\quad (I = ]a,b[,\ \varphi_n \in \mathcal{C}_0^\infty(]a,b[))$

$\varphi_n - u \in L^\infty(I)$ (by Sobolev)

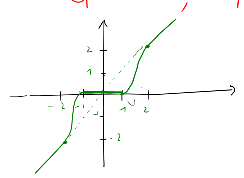
and $\|\varphi_n - u\|_{L^\infty} \leq C \|\varphi_n - u\|_{W^{1,p}}$

then $\|\varphi_n - u\|_{L^\infty} \xrightarrow{n} 0$

$\varphi_n, u$ are continuous on $\bar{I}$ $\Rightarrow$ $\underbrace{\varphi_{n|\partial I}}_{= 0} \xrightarrow{n} u_{|\partial I}$

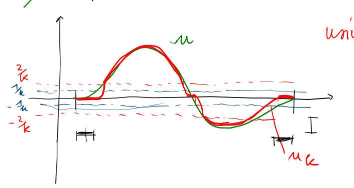
$\Leftarrow$ $\quad u \in W^{1,p}(I)$ and $u_{|\partial I} = 0$

consider $G : \mathbb{R} \to \mathbb{R}$, $G \in \mathcal{C}^1$, $G(s) = \begin{cases} 0 & |s| \leq 1 \\ s & |s| \geq 2 \end{cases}$, $|G(s)| \leq |s|$




define $u_k(x) = \frac{1}{k} G(k u(x))$

$u_k(x) = \begin{cases} 0 & \text{if } |k u(x)| < 1,\ |u(x)| < \frac{1}{k} \\ u(x) & \text{if } |u(x)| > \frac{2}{k} \end{cases}$




using convolution with mollifier $u_k \in W_0^{1,p}(I)$

to conclude

we have to show that

$\|u_k - u\|_{W^{1,p}(I)} \xrightarrow{k} 0$

Th (Poincaré inequality in $d = 1$)

let $p \in [1, +\infty[$, let $I$ be a bounded interval $(I = ]a,b[)$

then there exists $C > 0$ s.t.

$\forall u \in W_0^{1,p}(I)$, $\quad \|u\|_{W^{1,p}(I)} \leq C \|u'\|_{L^p(I)}$

(corollary: in $W_0^{1,p}(I)$ $\quad \|u'\|_{L^p}$ is an equivalent norm w.r.t. $\|u\|_{W^{1,p}}$)

proof $\quad$ let $u \in W_0^{1,p}(]a,b[)$

$u(x) = \underbrace{u(a)}_{=0} + \int_a^x u'(t)\,dt = \int_a^x u'(t)\,dt$

$\forall x \in I$, $\quad |u(x)| \leq \int_a^b \underset{L^{p'}}{1}\, \underset{L^p}{|u'(t)|}\,dt \underset{\text{Hölder}}{\leq} \underbrace{\left(\int_a^b 1\,dt\right)^{\frac{1}{p'}}}_{=(b-a)^{\frac{1}{p'}}} \left(\int_a^b |u'(t)|^p\right)^{\frac{1}{p}}$

$$|u(x)| \leq (b-a)^{\frac{1}{p'}} \|u'\|_{L^p(]a,b[)}$$

$$|u(x)|^p \leq (b-a)^{\frac{p}{p'}} \|u'\|^p_{L^p}$$

$$\int_a^b |u(x)|^p \leq (b-a)^{\frac{p}{p'}+1} \|u'\|^p_{L^p}$$

$$\|u\|_{L^p(I)} \leq (b-a) \|u'\|_{L^p}$$

$$\boxed{\|u\|_{W^{1,p}(I)} \leq (|I|+1) \|u'\|_{L^p(I)}}$$

QED

case $\boxed{d \geq 2}$

rem. if $d=1$, and $I \neq \mathbb{R}$ then $W^{1,p}_0(I) \subsetneq W^{1,p}(I)$

if $d \geq 2$ and $\Omega \neq \mathbb{R}^d$ then it is not $W^{1,p}_0(\Omega) \subsetneq W^{1,p}(\Omega)$ for all $\Omega$.

Exercise

take $\Omega \subseteq \mathbb{R}^2$, $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$

consider the closure of $\mathcal{C}^\infty_0(\Omega)$ in $W^{1,p}(\Omega)$ with $1 \leq p < 2$

show that this set is $W^{1,p}(\Omega) \overset{?}{=} W^{1,p}(\mathbb{R}^2)$

?

Theorem let $p \in [1,+\infty[$ and let $\Omega$ be a bounded open (connected) set in $\mathbb{R}^d$, of class $\mathcal{C}^1$

let $u \in W^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$

then $u \in W^{1,p}_0(\Omega) \iff u_{|\partial\Omega} = 0$

rem. if $p>d$ this is not required (thanks to morrey)

(proof in the book of Brezis)

Th (Poincaré inequality)

let $p \in [1,+\infty[$

let $\Omega$ be a bounded (connected) open set in $\mathbb{R}^d$ (no hyp on $\partial\Omega$)

Then $\exists C>0$ s.t.

$\forall u \in W^{1,p}_0(\Omega)$, $\|u\|_{W^{1,p}(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}$

Lemma 1. let $p \in [1,+\infty[$ let $\Omega$ be bounded open set in $\mathbb{R}^d$

if $u \in W^{1,p}_0(\Omega)$ then

$\exists C>0 : \forall \varphi \in \mathcal{C}^\infty_0(\mathbb{R}^d)$, $j=1,\dots,d$

$$\left|\int_\Omega u \, \partial_j \varphi\right| \leq C \|\varphi\|_{L^{p'}(\mathbb{R}^d)} \qquad \frac{1}{p}+\frac{1}{p'}=1$$

proof. $u \in W^{1,p}_0(\Omega)$

$\Downarrow$

$\exists (\varphi_k)_k$ in $\mathcal{C}^\infty_0(\Omega)$ s.t. $\|\varphi_k - u\|_{L^p(\Omega)} \xrightarrow{k} 0$, $\|\partial_j\varphi_k - \partial_j u\|_{L^p(\Omega)} \xrightarrow{k} 0$

$$\int_\Omega u \, \partial_j \varphi = \lim_k \int_\Omega \varphi_k \partial_j \varphi = \lim_k \int_{\mathbb{R}^d} \varphi_k \partial_j \varphi$$

$$= \lim_k - \int_{\mathbb{R}^d} \partial_j \varphi_k \, \varphi$$

$$= - \int_\Omega \partial_j u \, \varphi$$

$$\left| -\int_\Omega \partial_j u \, \varphi \right| \le \underbrace{\|\nabla u\|_{L^p(\Omega)}}\|\varphi\|_{L^{p'}(\Omega)}$$
$$\le C \|\varphi\|_{L^{p'}(\mathbb{R}^d)}$$

Lemma 2 Let $p$ and $\Omega$ as before

Suppose $u \in L^p(\Omega)$

Suppose $\exists C > 0$ st. $\forall \varphi \in C_0^\infty(\mathbb{R}^d)$

$$\left| \int_\Omega u \, \partial_j \varphi \right| \le C \|\varphi\|_{L^{p'}(\mathbb{R}^d)}$$

then $\bar{u}(x) = \begin{cases} u(x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$ defining

$\bar{u} \in W^{1,p}(\mathbb{R}^d)$

proof. $u \in L^p(\Omega) \Rightarrow \bar{u} \in L^p(\mathbb{R}^d)$ by hyp

$\varphi \in C_0^\infty(\mathbb{R}^d)$ $\left| \int_{\mathbb{R}^d} \bar{u} \, \partial_j \varphi \right| = \left| \int_\Omega u \, \partial_j \varphi \right| \le C \|\varphi\|_{L^{p'}(\mathbb{R}^d)}$

then $\bar{u} \in W^{1,p}(\mathbb{R}^d)$ by the proof of the theorem of Characterization of $W^{1,p}(\Omega)$ $1 \le p < +\infty$

proof (Poincaré) Step 1 $1 \le p < d$

let $u \in W_0^{1,p}(\Omega)$

apply Lemma 1 and 2 we have that

$\bar{u} \in W^{1,p}(\mathbb{R}^d)$ and $1 \le p < d$

Apply Sob.-Gagl.-Nir.

$\Rightarrow \bar{u} \in L^{p^*}(\mathbb{R}^d)$ and $\underset{\|u\|_{L^{p^*}(\Omega)}}{\overset{\|}{\|\bar{u}\|_{L^{p^*}(\mathbb{R}^d)}}} \le C_{p,d} \underset{\|\nabla u\|_{L^p(\Omega)}}{\overset{\|}{\|\nabla \bar{u}\|_{L^p(\mathbb{R}^d)}}}$ ? to think about

$p < p^*$ and $\Omega$ is bounded

$\Rightarrow \|u\|_{L^p(\Omega)} \le C' \|u\|_{L^{p^*}(\Omega)}$

$\|u\|_{L^p(\Omega)} \le C'' \|\nabla u\|_{L^p(\Omega)}$ $\longrightarrow$ $C'' = |\Omega|^{\frac{1}{p} - \frac{1}{p^*}}$

Second step : $d \le p < +\infty$

$u \in W_0^{1,p}(\Omega)$ in particular $u \in L^p(\Omega)$
and since $|\Omega| < +\infty$
we have $u \in L^q(\Omega)$ for all $q \in [1, p]$

similarly $\nabla u \in L^p(\Omega)$

then $\nabla u \in L^q(\Omega)$ for all $q \in [1,p]$

$\frac{1}{q} = \frac{d+p}{dp} = \frac{1}{p} + \frac{1}{d}$

I choose $q = \dfrac{dp}{d+p}$ $\left( \text{remark that } \frac{dp}{d+p} = p \cdot \frac{d}{d+p} < p \quad \frac{dp}{d+p} = d \frac{p}{d+p} < d \right)$

we have $u \in W_0^{1,q}(\Omega)$ $q < d$

I apply the previous step

$\bar{u} \in L^{q^*}(\mathbb{R}^d)$ but $q^* = ?$

$\bar{u} \in L^p$ $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{d} = \frac{1}{p} + \cancel{\frac{1}{d}} - \cancel{\frac{1}{d}} = \frac{1}{p}$

with the estimate

$$\|u\|_{L^{q^*}} \leq C \, \|\nabla u\|_{L^q}$$

$$\|u\|_{L^p} \leq C \, \|\nabla u\|_{L^q} \leq C' \, \|\nabla u\|_{L^p}$$

QED