Lezione 5 maggio 2026

martedì 5 maggio 2026 09:22

7/5 Tutorato

12/5 Ultima lezione relativa al programma

13/5 e 14/5 esercizi

19/5, 20/5 esercizi

21/5, TUTORATO

26/5 Tutor simulazione prova scritta
27/5 Tutor: correzione
28/5 Prova scritta: 9:00 - 11:00

Orali: 3/6 pomeriggio

8/6 : 9:30 scritto
14:30 orali

* * *

GEODETICHE

Def.: S sup. regolare, $f: U \to S$

Cons. $\alpha: I \to f(U)$ curva, regolare
(non nec. semplice: ci possono essere punti di autointersezione sulla traccia)

α si dice **CURVA GEODETICA**

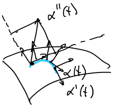
se $\forall t \in I \quad \alpha''(t) \parallel N(\alpha(t))$

N: CVN locale

cioè: $\forall t, \exists \lambda_t$: $\boxed{\alpha''(t) = \lambda_t N(\alpha(t))}$

eventualmente NULLO.

→ l'accelerazione tangenziale è nulla



Se $S = H$ piano affine, sono esatt. le rette.

OSS.: $\alpha''(t) \cdot \alpha'(t) = 0$

$\Longrightarrow \quad 2\, \alpha''(t) \cdot \alpha'(t) = 0$

$$= (\alpha'(t)\cdot\alpha'(t))' = 0$$

$$\Rightarrow \quad \alpha'(t)\cdot\alpha'(t) = \text{cost} = c$$

$$\|\alpha'(t)\|^2 = c$$

Esempi: ⊙ H piano affine

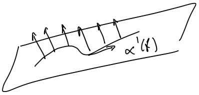
$\alpha(t) = P + w\cdot t$ retta

$\Rightarrow \alpha''(t) \equiv 0 \Rightarrow$ è geodetica

Viceversa: $\alpha : I \to H$ geodetica

$$\alpha''(t) \parallel N(\alpha(t)) \Rightarrow \alpha''(t) = \lambda(t)\cdot N$$

ma su un piano N è costante



Inoltre $\alpha'(t)\cdot N = 0$

$$\Rightarrow (\alpha'(t)\cdot N)' = 0$$

$$\alpha''(t)\cdot N + \alpha'(t)\cdot 0 = 0$$

$$\alpha''(t)\cdot N = 0$$

$$c(t)\cdot\underbrace{N\cdot N}_{=1} \equiv 0$$

$$\Rightarrow c(t)\equiv 0 \quad \Rightarrow \quad \alpha''(t)\equiv 0$$

$$\Rightarrow \alpha(t) = P + w\cdot t$$

CARATT. GEODETICHE IN COORD.

$$\alpha(t) \subseteq f(U)$$

come al solito

$$\alpha(t) = f(u(t), v(t))$$

$$(u(t), v(t)) = (f^{-1}\circ\alpha)(t)$$

$$(f^{-1}\circ\alpha) : I \to U \subseteq \mathbb{R}^2$$

$$\alpha'(t) = d_u f\cdot u'(t) + d_v f\cdot v'(t)$$

$$\alpha''(t) = u''(t)\cdot \partial_u f(u(t),v(t)) + u'\cdot\left(\partial_u f(u(t),v(t))\right)'$$

$$+ v''(t)\cdot \partial_v f(u(t),v(t))$$

$$+ v'(t)\cdot\left(\partial_v f(u(t),v(t))\right)'$$

$$= u''(t)\,\partial_u f + u'\left(\partial_{uu} f\cdot u'(t) + \partial_{uv} f\cdot v'(t)\right)$$

$$+ v''(t)\cdot\partial_v f + v'(t)\left(\partial_{uv} f\cdot u'(t) + \partial_{vv} f\cdot v'(t)\right)$$

$$= u''(t)\,\underbrace{\partial_u f} + v''(t)\cdot\underbrace{\partial_v f} +$$

$$+ \left(\underbrace{\partial_{uu} f}\cdot(u')^2 + 2\,\underbrace{\partial_{uv} f}\cdot(u'v') + \underbrace{\partial_{vv} f}\cdot(v')^2\right)$$

insieme simb. Γ^{i}_{jk}

$$= u''\,\partial_u f + v''\,\partial_v f +$$

$$+ \left(\Gamma^1_{11}\,\partial_u f + \Gamma^2_{11}\,\partial_v f + eN\right)\cdot(u')^2$$

$$+ 2\left(\Gamma^1_{12}\,\partial_u f + \Gamma^2_{12}\,\partial_v f + fN\right)\cdot(u'\cdot v')$$

$$+ \left(\Gamma^1_{22}\,\partial_u f + \Gamma^2_{22}\,\partial_v f + gN\right)(v')^2$$

$$= L_1(t)\,\partial_u f(u(t),v(t)) + L_2(t)\,\partial_v f(\dots)$$

$$+ L_3(t)\cdot N(u(t),v(t))$$

$$\alpha''(t) \parallel N \iff \begin{cases} L_1(t) = 0 \\ L_2(t) = 0 \end{cases}$$

Cioè:

$$\circledast \quad \begin{cases} u'' + (u')^2\,\Gamma^1_{11} + 2u'v'\,\Gamma^1_{12} + (v')^2\,\Gamma^1_{22} = 0 \\ v'' + (u')^2\,\Gamma^2_{11} + 2u'v'\,\Gamma^2_{12} + (v')^2\,\Gamma^2_{22} = 0 \end{cases}$$

EQUAZIONI (DIFFERENZIALI)

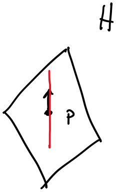
DELLE GEODETICHE

OSS. Siccome Γ^i_{jk} dipendono solo da E, F, G e loro derivate, anche $\circledast$ dipendono solo da E, F, G

Dalla Teorema di Cauchy-Lipschitz

Teorema: S sup. regolare; $\forall\, p \in S$ e $\forall\, v \in T_pS$, $v \neq \begin{pmatrix}0\\0\\0\end{pmatrix}$, $\exists\ \alpha : I \to S$, $I = (-\delta, \delta)$ geodetica t.c. $\alpha(0) = p$

$\alpha'(0) = v$



Inoltre: se $\exists\ \tilde{\alpha} : \tilde{I} \to S$ con le stesse proprietà, allora $\alpha \equiv \tilde{\alpha}$ in $I \cap \tilde{I}$

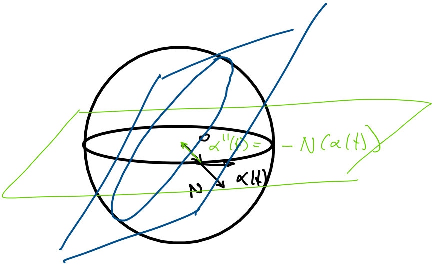
CONSEGUENZA: GEOD. su S^2:

Cons.: l'equatore $z=0$: $(\cos t, \sin t, 0) = \alpha(t)$

$S^2 : x^2+y^2+z^2 = 1$

$$\alpha'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix}$$

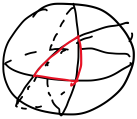
$$\alpha''(t) = \begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix} = (-1) \cdot \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} = \overrightarrow{O\alpha(t)}$$

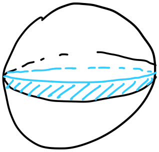


Analog: con un movimento rigido (rotazione) posso mandare ogni circonf. di raggio massimo

(cioè circonf $= S^2 \cap H$, H piano passante per $(0,0,0)$)

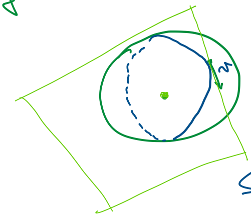
nell'equatore $\Rightarrow$ TUTTI I CERCHI DI RAGGIO MAX sono geodiche





BIANGOLI

AFFERMO: Non ci sono altre geodiche su S^2; infatti:



cons. $H \ni 0, p$

e con giacitura

$\text{Span}(\vec{OP}, v)$

Es. ② CILINDRO: $x^2+y^2=1$

Param. $f(u,v) = (\cos u, \sin u, v)$

$E = 1$ $\quad\partial_u E = \partial_v E = 0$

$F = 0$ $\quad\partial_u F = \partial_v F = 0$

$G = 1$ $\quad\partial_v G = \partial_u G = 0$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Posso calcolare Γ^i_{jk}

$$\begin{pmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2}\partial_u E \\ \partial_u F - \frac{1}{2}\partial_v E \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Analog. si trova $\Gamma^2_{12} = 0 = \Gamma^1_{22} = 0$

$$\Gamma^i_{jk} = 0$$

EQ GEO

$$\Longrightarrow \quad (*) \quad \begin{cases} u'' = 0 \\ v'' = 0 \end{cases}$$

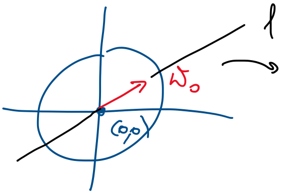
Se fisso $(u(0), v(0))$ e $(u'(0), v'(0))$

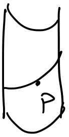
$= (0,0)$ $\quad = w_0 = (w_0^1, w_0^2)$

per semplicità

⇒ soluz. $(u(t), v(t)) = (0 + w_0^1 t, 0 + w_0^2 t)$

soluzione in $U \subseteq \mathbb{R}^2$





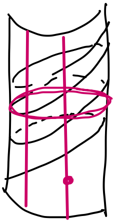
Per trovare la geodetica, compongo con f

$$\alpha(t) = f(w_0^1 t, w_0^2 t)$$

$$\overset{!}{=} (\cos w_0^1 t, \sin w_0^1 t, w_0^2 t)$$

$$\overset{!}{=} (\cos at, \sin at, bt)$$

ELICHE CIRCOLARI



CASI PART.:

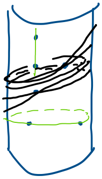
$w_0^1 = a = 0$: $(0, 0, bt)$

GENERATRICI

$w_0^2 = b = 0$, $(\cos at, \sin at, 0)$ circonferenze piene

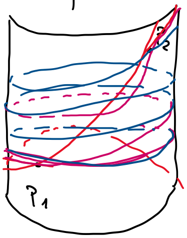
$$w = \begin{pmatrix} w_0^1 \\ 0 \end{pmatrix}$$

OSS.: Se cons. 2 punti sul cilindro



se i 2 punti non giacciono su stesso piano orizz.

ovvero non giacciono su una generatrice, allora per i 2 punti passano infinite geodetiche (eliche):

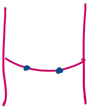
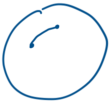


OSS.: Le eq. geod. ⊛ hanno per coeff. Γ^i_{jk}, sono invarianti per ISOM. LOCALI

⇒ hanno LE STESSE SOLUZIONI

⇒ tutte e sole le geod. (localmente) su una sup. sono immagine delle geod. sull'altra sup., tramite l'isometria locale F

FATTO IMPORT. VENERDÌ

le geodetiche sono anche curve che localmente MINIMIZZANO LE LUNGH. DELLE CURVE per 2 punti