Esercitazioni 13 maggio 2026

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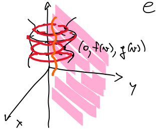
SUP. DI ROTAZIONE

Cons. una curva piana contenuta nel piano yz di eq. $x=0$;

$\alpha(v) = (0, y(v), z(v))$

Per il momento: $\alpha(v) = (0, f(v), g(v))$



$\mathcal{C}^\infty$, regolare, semplice

e supp. $f(v) > 0$

$\forall v \in I$

$\alpha : I \longrightarrow \mathbb{R}^3$

La sup. di rotaz. ottenuta ruotando attorno asse z:

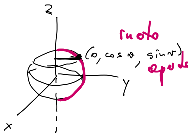
$$f(u,v) = (f(v)\cos u,\ f(v)\sin u,\ g(v))$$

$$f : U \longrightarrow \mathbb{R}^3, \quad U = I \times (0, 2\pi)$$

$f(u, v_0)$ si chiama PARALLELO (curve coordinate)
$\forall v_0 \in I$

$f(u_0, v)$ si chiama MERIDIANO
$\forall u_0 \in (0, 2\pi)$

Es.: Sfera: $\mathbb{S}^2$



ruoto una semicircon.
aperta

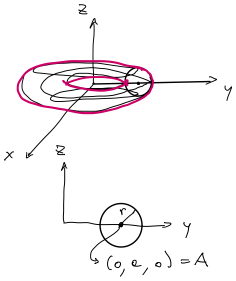
$$\alpha(v) = (0, \cos v, \sin v)$$

$v \in (-\pi/2, \pi/2)$,

$$f(u,v) = (\cos v \cos u,\ \cos v \sin u,\ \sin v)$$

$u \in (0, 2\pi)$

② TORO CIRCOLARE



, con $a > r$

$$\alpha(v) = \underbrace{(0,\ r\cos(v) + a,\ r\sin v)}_{}$$

$(0,\ r\cos v,\ r\sin v)$

$$\|\overrightarrow{A\,\alpha(v)}\|^2 = \left\| \begin{pmatrix} 0-0 \\ r\cos v + a - a \\ r\sin v \end{pmatrix} \right\|^2 = r^2 \ ✓$$

$$f(u,v) = \big((r\cos v + a)\cos u,\ (r\cos v + a)\sin u,\ r\sin v\big)$$

Torniamo a sup. di rotaz. generali

OSS.:

$f(u,v) : (0,2\pi) \times I \to \mathbb{R}^3$

è una param. regolare.

① f $\mathcal{C}^\infty$: $(f(v)\cos u,\ f(v)\sin u,\ g(v))$

②

③ $$Jf = \begin{pmatrix} -f(v)\sin u & f'(v)\cos u \\ f(v)\cos u & f'(v)\sin u \\ 0 & g'(v) \end{pmatrix}$$

$$-ff'\sin^2 u - ff'\cos^2 u = -f \cdot f' \cdot 1$$

$$= -f \cdot f' \quad = 0$$

$$\Longleftrightarrow f' = 0$$

$f(v) > 0$, f'

$\alpha(v) = (0, f(v), g(v))$ α regolare

$\alpha'(v) \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Se $f'(v_0) = 0 \Rightarrow g'(v_0) \neq 0 \Longrightarrow$ $-f(v_0) \cdot g'(v_0)\sin u$, opp.

$-f(v_0)\, g'(v_0)\cos u$

è sicuramente $\neq 0$.

② f è omeo tra U e $f(U)$

$f(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$

$f^{-1}(x,y,z) =$

$\downarrow$

DO CARMO

f iniettiva : facile

$\underline{(f(v)\cos u,\ f(v)\sin u,\ g(v)) = (x,y,z)}$

segue dal fatto che $\alpha(v)$ è semplice (α iniettiva)

$\alpha(v) = (0, f(v), g(v)) = (x,y,z)$

$(u,v) = \left(\sqrt{x^2+y^2},\ z\right)$ (?)

* * *

Coeff. : I^a

$$\partial_u f = \begin{pmatrix} -f(v)\sin u \\ f(v)\cos u \\ 0 \end{pmatrix}, \quad \partial_v f = \begin{pmatrix} f'(v)\cos u \\ f'(v)\sin u \\ g'(v) \end{pmatrix}$$

$$\begin{cases} E = f^2(v) & > 0 \\ F = 0 \\ G = (f'(v))^2 + (g'(v))^2 & = 1 \end{cases}$$

$$\sqrt{EG - F^2} = \sqrt{(f(v))^2} = f(v)$$

$f(v) > 0$

OSS.: $G = \|\alpha'(v)\|^2$

α regolare $\Longrightarrow$ possiamo par. α per LA:

se α è PLA, $\|\alpha'(v)\|^2 = 1 = G$

II^a FF:

$$d_{uu}\varphi = \begin{pmatrix} -f(v)\cos u \\ -f(v)\sin u \\ 0 \end{pmatrix}$$

$$d_{uv}\varphi = \begin{pmatrix} -f'(v)\sin u \\ f'(v)\cos u \\ 0 \end{pmatrix}$$

$$d_{vv}\varphi = \begin{pmatrix} f''(v)\cos u \\ f''(v)\sin u \\ g''(v) \end{pmatrix}$$

$$e = \frac{1}{\sqrt{EG-F^2}} \det\left(d_u\varphi \;\; d_v\varphi \;\; d_{uu}\varphi \right)$$

$$= \frac{1}{f(v)} \begin{pmatrix} -f(v)\sin u & f'(v)\cos u & -f(v)\cos u \\ f(v)\cos u & f'(v)\sin u & -f(v)\sin u \\ 0 & g'(v) & 0 \end{pmatrix}$$

$$= \frac{1}{\cancel{f(v)}} \cdot \left(-g'(v) \cdot (f(v))^{\cancel{2}} \right) = -g'(v)\, f(v)$$

Con le formule $f = 0$

$$g = g'(v) f''(v) - g''(v) f'(v)$$

$$\Longrightarrow K = \frac{eg - f^2}{EG - F^2} = \frac{-g'(v)\,\cancel{f(v)}\,\left(g'(v)f''(v) - g''(v)f'(v)\right)}{(f(v))^{\cancel{2}}}$$

$$= \frac{-(g'(v))^2 f''(v) + \overbrace{g'(v)\, g''(v)}\, f'(v)}{f(v)}$$

ORA: $f'(v)^2 + g'(v)^2 = 1$

$= \|\alpha'(v)\|^2$

$\Longrightarrow \left(f'(v)^2 + g'(v)^2\right)' = 0$

$2f'(v)f''(v) + 2g'(v)g''(v) = 0 \quad /:2$

$\Longrightarrow g'(v)\, g''(v) = -f'(v)\, f''(v)$

$$= \frac{-(g'(v))^2 f''(v) - f'(v) f''(v) f'(v)}{f(v)}$$

$$= \frac{-f''(v)\left((g'(v))^2 + (f'(v))^2\right)}{f(v)} = \boxed{-\frac{f''(v)}{f(v)} = K}$$

INOLTRE $H = \frac{1}{2}\frac{eG + gE - 2fF}{EG - F^2}$

$$= \frac{1}{2}\frac{eG + gE}{EG} = \frac{1}{2}\frac{-g'(v)f(v)\cdot 1 + (f(v))^2\left(g'(v)f''(v) - g''(v)f'(v)\right)}{(f(v))^2}$$

In generale, quando $F = 0 = f$

$$K = \frac{eg}{EG} = \frac{e}{E}\cdot\frac{g}{G} = k_1 \cdot k_2$$

$$H = \frac{1}{2}\frac{eG + gE}{EG} = \frac{1}{2}\left(\frac{e}{E} + \frac{g}{G}\right) = \frac{1}{2}(k_1 + k_2)$$

In generale k_1 e k_2 sono soluzioni di

$$x^2 - 2Hx + K = 0$$

$\frac{e}{E}$ e $\frac{g}{G}$ sono soluzioni $\Rightarrow$ sono LE SOLUZIONI

$\Rightarrow$ $k_1 = \frac{e}{E}$, $k_2 = \frac{g}{G}$ o viceversa

Nel nostro caso $\frac{e}{E} = -\frac{g'(v)}{f(v)}$

$$\frac{g}{G} = g'(v)f''(v) - g''(v)f'(v)$$

Caso del toro:

$$K = -\frac{f''(v)}{f(v)} \quad \text{se} \quad \alpha(v) \text{ è PLA}$$

Per r arbitrario NO:

$$f(u,v) = ((r\cos v + a)\cos u, (r\cos v + a)\sin u, r\sin v)$$

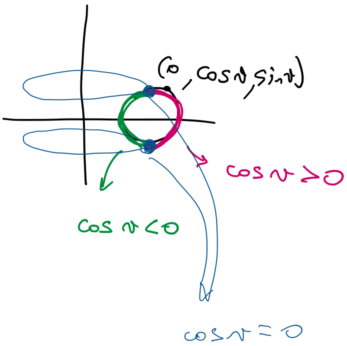
$$E = (a + r\cos v)^2 \qquad\qquad \partial_v f = ($$

$$F = 0$$

$$G = ((r\cos v + a)')^2 + (r\sin v)'$$

$$= (-r\sin v)^2 + (r\cos v)^2 = r^2$$

$$K = \frac{\cos v}{\underbrace{r(\underbrace{a + r\cos v}_{a>r})}_{>0}}$$



$$K \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

$$\underbrace{a - r}_{>0} < a + r\cos v < a + r$$

SIMBOLI DI CHRISTOFFEL:

$\alpha(v)$ PLA ; $E = (f(v))^2$

$F = 0$

$G = 1$

$$\partial_u E = 0$$

$$\partial_v E = 2f(v)f'(v)$$

$$\partial_u F = \partial_v F = 0$$

$$\partial_u G = \partial_v G = 0$$

$$\Rightarrow \Gamma^1_{11} = 0 = \Gamma^2_{12} = \Gamma^1_{22} = \Gamma^2_{22}$$

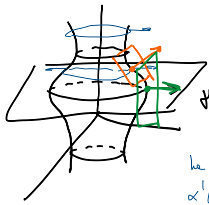
$$\Gamma^2_{11} = -f(v)f'(v), \quad \Gamma^1_{12} = \frac{f(v)f'(v)}{(f(v))^2}$$

Scrivere eq. delle geodetiche

si può verificare che tutti i meridiani

sono geodetiche: $f(u_0, v)$

alcuni paralleli sono geodetiche:



$\gamma''(v_0) \parallel N(p)$

se $\alpha(v_0)$

ha tg verticale

$\alpha'(v_0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$