Lesson 31 - 21/5/2026

Examples of boundary value problems

Ex 1 (homogeneous Dirichlet problem $d = 1$)

Let $f \in \mathcal{C}([0,1])$

Problem: find $u \in \mathcal{C}^2(]0,1[) \cap \mathcal{C}([0,1])$ s.t. | classical problem

$$\begin{cases} u - u'' = f & \text{in }]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

Strategy. 1) we introduce a "weak problem" in a suitable setting
2) we solve the weak problem
3) we see that the soln. of weak problem is actually solution for the classical pb.

1) weak hom. Dirichlet pb

let $f \in L^2(0,1)$

Prob. find $u \in H^1_0(0,1)$ $\left(= W^{1,2}_0(]0,1[)\right)$ s.t. | weak Pb

$$\int_0^1 uv + \int_0^1 u'v' = \int_0^1 fv \qquad \forall v \in H^1_0(0,1)$$

2) We use Lax-Milgram th.

$H^1_0(0,1)$ is Hilbert space

$$a : H^1_0 \times H^1_0 \to \mathbb{R}$$
$$(u,v) \longmapsto a(u,v) = \int_0^1 uv + \int_0^1 u'v'$$

bilinear, symmetric, continuous, coercive

$$\left(a(u,u) = \|u\|^2_{H^1} \right)$$

$$\phi : H^1_0 \to \mathbb{R}, \quad v \to \int_0^1 fv \qquad \phi \in (H^1_0)'$$

Lax-Milgram $\exists! \, u \in H^1_0$ s.t. $\underbrace{a(u,v) = \phi(v) \quad \forall v \in H^1_0}_{\text{weak problem}}$

3) let $f \in \mathcal{C}([0,1])$ $(\Rightarrow f \in L^2)$

let u be the unique solution of weak pb.

since $u \in H^1_0(0,1)$ then $u \in \mathcal{C}([0,1])$ and $u(0) = u(1) = 0$

we want to prove that u is $\mathcal{C}^2(]0,1[)$ and $u - u'' = f$ pointwise

classical derivative

we know that

$$\int_I u'v' = \int_I uv + \int_I fv \qquad \forall v \in H^1_0$$

($u \in H^1_0 \Rightarrow u \in L^2$, $u' \in L^2$ and $\int u'\varphi = -\int u\varphi'$ $\forall \varphi \in \mathcal{C}^\infty_0 I$)

$$\Rightarrow \int u'\varphi' = \int (u-f)\varphi \qquad \forall \varphi \in \mathcal{C}^\infty_0(I)$$

$u' \in H^1$ and $(u')' = \underset{L^2}{u} - \underset{L^2}{f} \in L^2$

in the sense of distribution

$$\Rightarrow u \in H^2 \cap H^1_0$$

how to prove that u'' is classical?

1° use Du Bois-Reymond

$u \in H^2 \cap H^1_0 \Rightarrow u \in \mathcal{C}(I)$

$$\begin{cases} u' \in H^1 \Rightarrow u' \in \mathcal{C}(I) \\ u'' = u - f \in \mathcal{C}(I) \end{cases}$$

u'' is the 2nd classical derivative of u

$u \in \mathcal{C}^2$

finally we can integrate by parts

u'' is the 2^{nd} classical derivative of u

finally we can integrate by parts $u \in \mathscr{C}^2$

$$\int u'\varphi' + \int u\varphi - \int f\varphi = 0 \quad \forall \varphi$$

$$= -\int u''\varphi + \int u\varphi - \int f\varphi = 0$$

$$\int_I (-u'' + u - f)\varphi = 0 \quad \forall \varphi \in \mathscr{C}_0^\infty(I)$$

$\Rightarrow \quad -u'' + u - f = 0$ a.e, but $u'', u, f \in \mathscr{C}$

$\Rightarrow \quad -u'' + u = f$ pointwise in $]0,1[$

2nd way we have $u'' = u - f$ in the sense of distribution

then (from the theorem on the continuous rep)

$$u'(x) - u'(y) = \int_y^x (u(t) - f(t))\,dt$$

and $u - f \in \mathscr{C}$ (so we apply the classic fundamental th. of calculus)

u' is $\mathscr{C}^1 \Rightarrow u \in \mathscr{C}^2$ an so on

Ex. 2 non hom. Dirichlet pb.

let $f \in \mathscr{C}([0,1])$ let $a, b \in \mathbb{R}$

prob. find $u \in \mathscr{C}^2(]0,1[) \cap \mathscr{C}([0,1])$ s.t.

$$(*) \begin{cases} -u'' + u = f & \text{in }]0,1[\\ u(0) = a, \quad u(1) = b \end{cases}$$

consider $v(t) = u(t) - a - (b-a)t$

then $v''(t) = u''(t)$

$v(0) = 0, \ v(1) = 0$

u is a solution of $(*)$ $\Leftrightarrow$ v is a solution of $\begin{cases} -v'' + v = \tilde{f} \\ v(0) = v(1) = 0 \end{cases}$

$\tilde{f}(t) = \underline{f(t) + a + (b-a)t}$

problem 1

Ex. 3 (Hom. Neumann problem)

let $f \in \mathscr{C}([0,1])$

find $u \in \mathscr{C}^2(]0,1[) \cap \mathscr{C}^1([0,1])$ s.t.

$$\begin{cases} -u'' + u = f & \text{in }]0,1[\\ u'(0) = u'(1) = 0 \end{cases}$$

weak pb. consider $f \in L^2(0,1)$

find $u \in H^1(0,1)$ s.t

$$\int_0^1 uv + \int_0^1 u'v' = \int_0^1 fv \quad \forall v \in H^1 \quad (*)$$

Lax-Milgram $\Rightarrow \exists!\ u$ sol. of $(*)$

letting now $f \in \mathscr{C}$ in particular $\int_I u\varphi + \int_I u'\varphi' = \int_I f\varphi$

$$\int_I u'\varphi' = \int (u - f)\varphi \quad \Rightarrow \quad u \in H^2 \text{ (as before)}$$

letting now $f \in \mathcal{C}$ in particular $\int_I u\varphi + \int_I u'\varphi' = \int_I f\varphi$

$$\int_I u'\varphi' = \int (u-f)\varphi \implies u \in H^2 \text{ (as before)}$$

as before $u \in \mathcal{C}^2(]0,1[)$ and

$-u'' + u = f$ in $]0,1[$

but $\int_0^1 u'v' = u'(1)v(1) - u'(0)v(0) - \int_0^1 u''v$

so that $\int_0^1 uv + \int_0^1 u'v' = \int_0^1 fv \quad \forall v \in H^1$

$$\Downarrow$$

$$\int_0^1 uv + u'(1)v(1) - u'(0)v(0) - \int_0^1 u''v = \int_0^1 fv$$

finally $0 = \int_0^1 (u - u'' - f)v + u'(1)v(1) - u'(0)v(0) \quad \forall v \in H^1$

but $u - u'' - f = 0$

$\implies u'(1)v(1) - u'(0)v(0) = 0 \quad \forall v \in H^1$

$\implies u'(0) = u'(1) = 0$

what about $\underline{d \geq 2}$ given $f \in \mathcal{C}(\overline{B})$ find $u \in \mathcal{C}^2(B) \cap \mathcal{C}(\overline{B})$

Hom. Dirichlet pb. $\begin{cases} -\Delta u + u = f \\ u|_{\partial B} = 0 \end{cases}$

$B = B(0,1)$

weak $f \in L^2(B)$

$u \in H^1_0(B)$ s.t. $\int_B uv + \int_B \nabla u \cdot \nabla v = \int_B fv \quad \forall v \in H^1_0(B)$

difficult : prove that weak sol. are classical sol.