Discounting methods

- An economic agent has a sum of money:
  - they may decide to spend it immediately or save it:
    immediate need (*certain*) ↔ future need (*uncertain*)
  - they may consider it *more useful to* save it (the utility of spending it today is limited)
  - although they could derive utility from the consumption of the resource today, they could postpone consumption in view of a future *reward*:
    [future value] > [value available today].

- The agent is thus *indifferent* to the *present value* V_0 and the *future value* V_F if

$$V_F = V_0 + \Delta \quad [1]$$

- Δ is a positive quantity *proportional* to V_0:

$$V_F = V_0 + V_0 \cdot r = V_0(1 + r)$$

- The operation of determining the future value of V_0 is called *compounding*.

- In *lending* transactions *r* is called *interest rate* (often denoted by *i*).
- L (*lender*) may grant B (*borrower*) a sum of money V_0 for n periods, if L eventually receives:
  $V_n = V_0 + \Delta$
- At the end of the first period, B would have:

$$V_1 = V_0 + V_0 \cdot r \cdot 1 = V_0(1+r)$$

- *r* is a constant < 1 representing the interest charged by L for 1€ made available to B for a period.

Simple interest

- Assume that L receives at the end of each period t the interest V_0 r :

$$V_1 = V_0 + V_0 \cdot r$$
$$V_2 = V_0 + V_0 \cdot r$$
$$\ldots$$

Lender

- This means that interest is calculated only on the initial capital.
- After *n* periods, L will have obtained from B an interest ($V_0 \cdot r \cdot n$) in addition to the repayment of the initial capital:

$$V_{n,tot} = V_0 + V_0 \cdot r \cdot n = V_0(1 + rn) \quad [2]$$

Compound interest

- Assume that L does not want to collect interest at the end of each period.
- The interest accrued remains *at the disposal of* B.
- Interest is thus accrued on interest:

$$V_1 = V_0 + V_0 \cdot r = V_0(1+r)$$

$$V_2 = V_1 + V_1 \cdot r = V_1(1+r) = V_0(1+r)(1+r) = V_0(1+r)^2$$

$$\ldots$$

$$V_n = V_{n-1} + V_{n-1} \cdot r = V_{n-1}(1+r) = \ldots = V_0(1+r)^n \quad [3]$$

- If an investor wants to obtain a sum V_F in *n* periods and the offered interest rate is *r*, the sum V_0 to be invested today will be

$$V_0 = \frac{V_F}{(1+rn)} \quad [2'] \quad \text{simple interest}$$

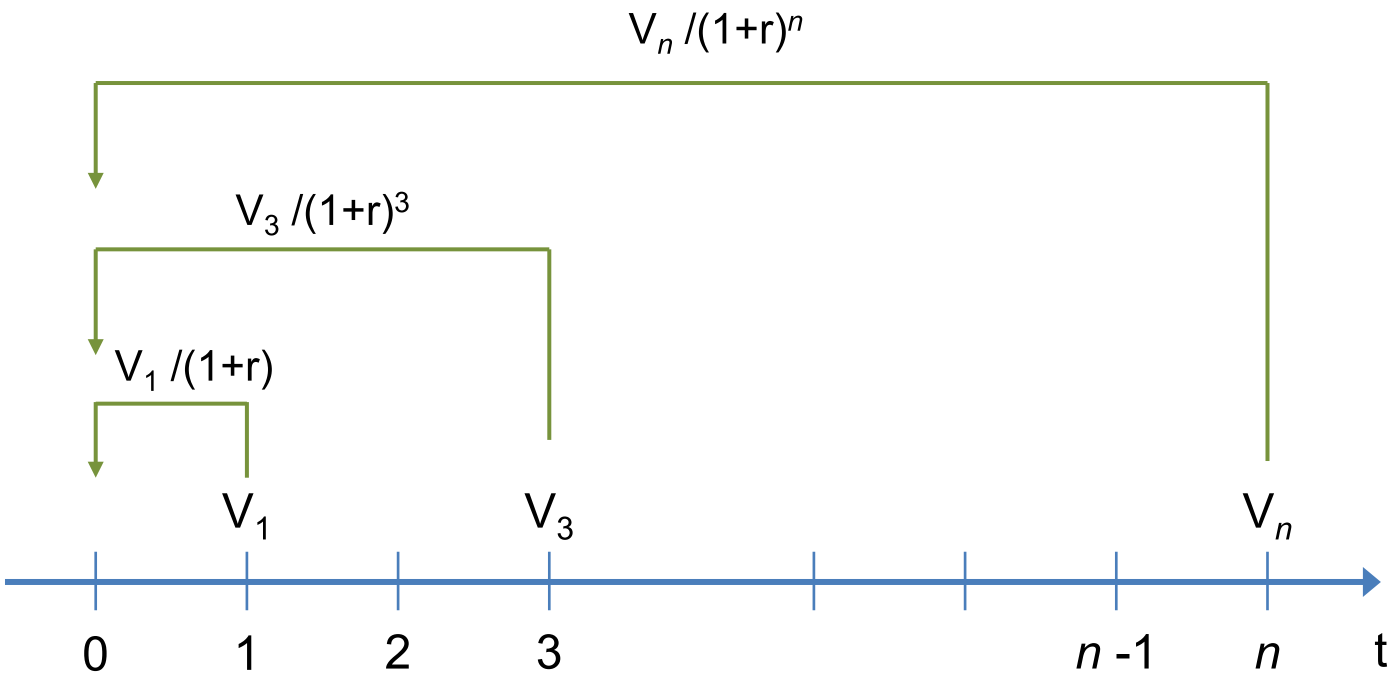
$$V_0 = \frac{V_F}{(1+r)^n} \quad [3'] \quad \text{compound interest}$$

- This operation is called *discounting*.

Comments

- The equations are valid in general, not only in the case of lender-borrower relations.
- The rate *r* has a broader meaning than interest rate: it expresses an *intertemporal preference*.
- Compounding and discounting formulas allow values occurring in different periods to be “moved” to a common (*reference*) period and compared.

For compound interest:



If we denote by $V_{t,\tau}$ the discounting of the cash flow in t to the period τ, then:

$$V_{3,1} = \frac{V_3}{(1+r)^{3-1}} = \frac{V_3}{(1+r)^2}$$

$$V_{1,0}[V_{3,1}] = \frac{V_{3,1}}{(1+r)^1} = \frac{V_3}{(1+r)^2(1+r)} = \frac{V_3}{(1+r)^3}$$

Comments

- It follows from equation [1] that the economic agent considers the sum at the first and second members to be equivalent.
- For example, given [3] and [3'], the agent considers the sum V_0 today and $V_n = V_0 (1+r)^n$ between n periods to be equivalent.
- *Principle* of *equivalence*
  two sums occurring in different periods are *equivalent* if they produce the same (economic) effect.

Example [Loan]

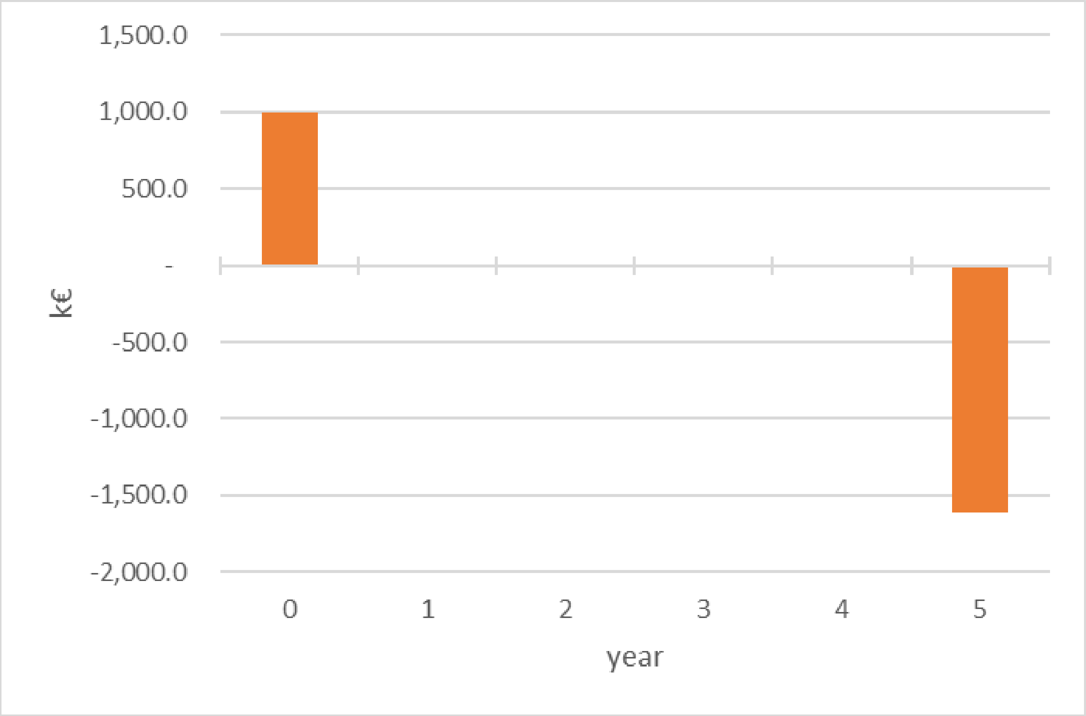
- An entrepreneur applies for a loan of 1,000k€ with a credit institution for five years at a rate of 10%.
- She is presented with 3 possible alternatives for repayment of the *principal* (borrowed – or invested – capital) and payment of interest:
  1. repayment of principal and interest at the end of 5 years in one lump sum
  2. bond with annual payment of accrued interest and repayment of principal at the end of 5 years
  3. repayment of the principal on a straight-line basis over five years with annual payment of interest thereon.

- The three options are summarised in the following tables.

Example [Loan]

- Option 1.

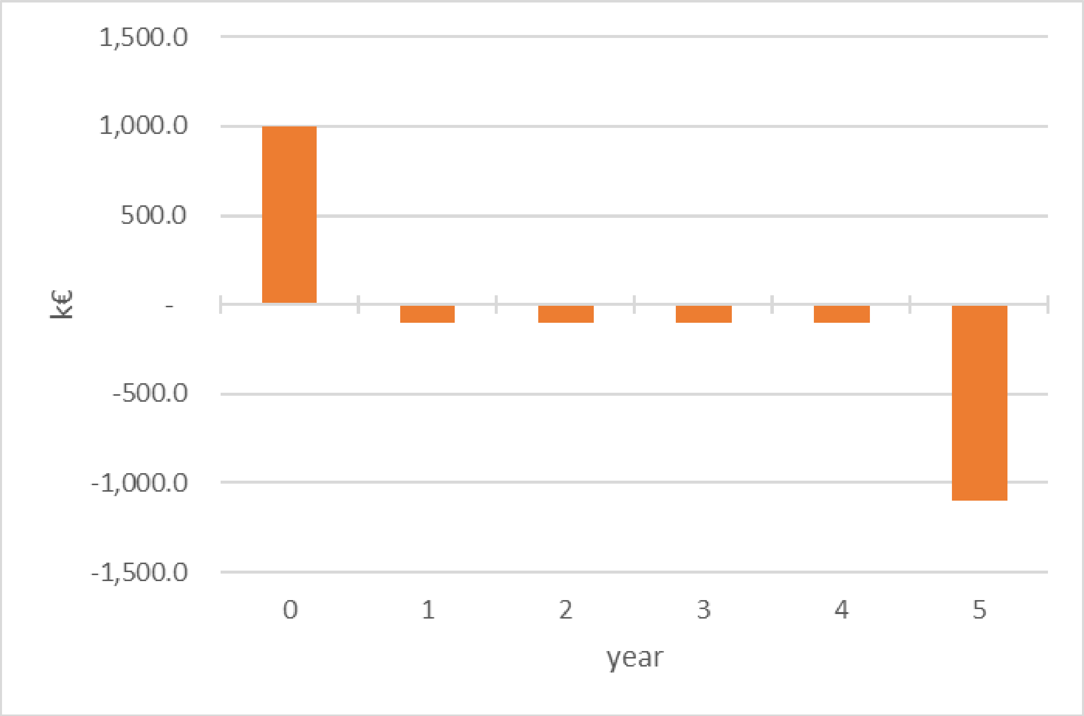
year	cash flow	due	int paid	cap paid
0	1,000.0			
1	-	1,100.0	-	-
2	-	1,210.0	-	-
3	-	1,331.0	-	-
4	-	1,464.1	-	-
5	- 1,610.5	1,610.5	- 610.5	- 1,000.0



Example [Loan]

- Option 2.

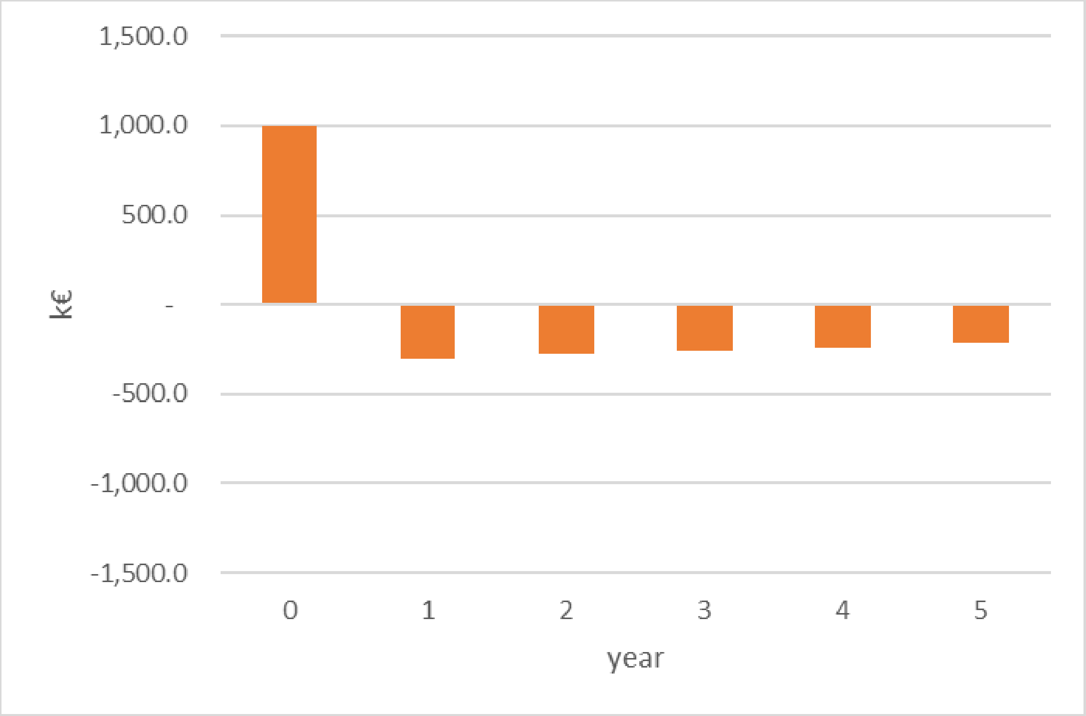
year	cash flow	due	int paid	cap paid
0	1,000.0			
1	- 100.0	1,100.0	- 100.0	-
2	- 100.0	1,100.0	- 100.0	-
3	- 100.0	1,100.0	- 100.0	-
4	- 100.0	1,100.0	- 100.0	-
5	- 1,100.0	1,100.0	- 100.0	- 1,000.0



Example [Loan]

- Option 3.

year	cash flow	due	int paid	cap paid
0	1,000.0			
1	- 300.0	1,100.0	- 100.0	- 200.0
2	- 280.0	880.0	- 80.0	- 200.0
3	- 260.0	660.0	- 60.0	- 200.0
4	- 240.0	440.0	- 40.0	- 200.0
5	- 220.0	220.0	- 20.0	- 200.0



Example [Loan]

- The following table shows the cash flows and discounted cash flows.
- The last row shows that the three options are equivalent.

	Option 1		Option 2		Option 3	
year	Ft	Ft,0	Ft	Ft,0	Ft	Ft,0
0	1,000,0		1,000.0		1,000.0	
1	-	-	- 100.0	- 90.9	- 300.0	- 272.7
2	-	-	- 100.0	- 82.6	- 280.0	- 231.4
3	-	-	- 100.0	- 75.1	- 260.0	- 195.3
4	-	-	- 100.0	- 68.3	- 240.0	- 163.9
5	- 1,610.5	- 1,000.0	- 1,100.0	- 683.0	- 220.0	- 136.6
		- 1,000.0		- 1,000.0		- 1,000.0

discounted value of the profile

Working hypotheses

- In the financial valuation of industrial projects, *compound* interest is most often used.
- We will often employ discounting formulas, taking period t=0 as a reference.
- Cash flow in period *t* occurs entirely *at the end of* period *t*.

Working hypotheses

- The first F_t (= F_1) is at the end of t=1.
- If the investment results in a single exit, this occurs at the end of t=0 (i.e. at the beginning of t=1).
- Discounting to the period t=0 shows future flows (or future capital) at the end of t=0.

- To sum up, the discounted value at time 0 of a flow occurring at t is:

$$F_{t,0} = \frac{F_t}{(1+r)^t} = F_t \cdot v(t)$$

where v(t) (*discount factor*) is a function

- defined and positive in [0,n].
- monotonic decreasing
- $v(0) = 1$
- $v(t) > 0 \ \forall t$

Annuities

- A special case is when the cash flows have the same value in each period of the analysed horizon.
- Assume that an investment produces a constant annual income A (*annuity*) over the next *n* years.
- Each annuity is produced at the end of the year starting in year 1.

- Assuming that each flow A is reinvested at rate r, we can apply discounting to determine the present value (PV) of the profile:

$$PV = \frac{A}{(1+r)^1} + \frac{A}{(1+r)^2} + \ldots + \frac{A}{(1+r)^n} = \sum_{t=1}^{n} \frac{A}{(1+r)^t}$$

- It can be proved that this sum is equal to:

$$PV = A \boxed{\frac{(1+r)^n - 1}{r(1+r)^n}} \quad [4]$$

discount factor of a series of n annuities

- It is also possible to calculate the future value (FV) of a series of constant annuities:

$$FV = A\frac{(1+r)^n - 1}{r} \quad [4']$$

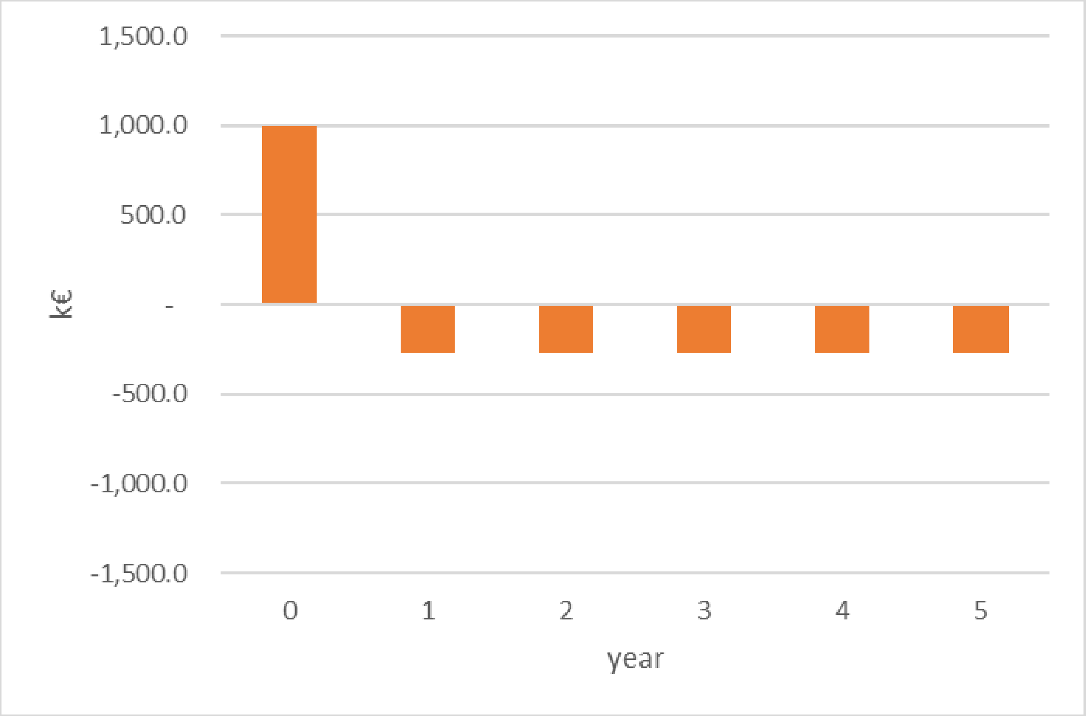
- An entrepreneur takes out a loan of V_0 today at an annual interest rate r for n years.
- The loan may be repaid in constant annual instalments:

$$A = V_0 \frac{r(1+r)^n}{(1+r)^n - 1} \quad [5]$$

Example [Loan]

- With this option, the following results are obtained in the case of [Loan]:

year	cash flow	due	int paid	cap paid
0	1,000.0			
1	- 263.8	1,100.0	- 100.0	- 163.8
2	- 263.8	919.8	- 83.6	- 180.2
3	- 263.8	721.6	- 65.6	- 198.2
4	- 263.8	503.6	- 45.8	- 218.0
5	- 263.8	263.8	- 24.0	- 239.8



- Constant annuities repay decreasing instalments of interest and increasing instalments of principal.

Example [SLOG]

- The logistics service company SLOG wants to offer a service for the storage and management of spare parts for some customers.
- For this purpose, an investment I_0 = 200k€ is planned for purchasing and installation of automated storage equipment.
- Such a system would allow for an estimated annual net operating profit of 40k€ over the 6-year period.
- We want to assess whether, at a rate r=8%, the estimated annuity allows the initial investment to be recovered.

Example [SLOG]

- We calculate the present value of the annuities:

$$PV = A\frac{(1+r)^n - 1}{r(1+r)^n} = 40\frac{(1+0,08)^6 - 1}{0,08(1+0,08)^6} \cong 185\,k€$$

- PV of cash inflow is lower than the initial outlay, taking into account an expected return on capital of 8%.
- We note that the *payback* of this investment is 5 years (< 6 years).

Example [SLOG]

- Suppose SLOG plans to renew its current storage equipment in 6 years.
- The estimated investment for this project is 260k€.
- How much should the company invest annually, at an annual return rate of 4%, to accumulate the necessary capital?

$$A = I_F \frac{r}{(1+r)^n - 1} = 260.000 \frac{0,04}{(1+0,04)^6 - 1} \cong 39.200\,€$$

- Annuities are largely used for the evaluation of projects with operating cash flows which are assumed to stay constant for some periods.
- Some assumptions are often adopted in the economic analysis of an energy conversion system:
  - if E_S is the energy to be supplied to users (useful energy output) and E_P is the total energy input in the system, $\eta=E_S/E_P$ is the efficiency of the conversion process
  - so, the energy input (and paid) is $E_P=E_S/\eta$
  - if c_E is the unit cost of the energy input, then the annual cost of energy is

$$\overline{CE_y} = c_E \cdot E_P \quad (\text{e.g. } \frac{€}{kWh} \cdot \frac{kWh}{yr} = \frac{€}{yr})$$

Geometric gradient

- A particular case of annuities that is used in industrial projects is represented by a cash flow starting in t=1 at a value A and then changing at a constant rate g.
- g can be positive or negative, thus the series of values can be increasing or decreasing.
- Examples:
  - constant increase in energy demand due to production increase
  - constant decrease of output capacity due to degradation of equipment.

- The present value of a profile with value A in t=1 and escalating with (positive) g until t=n is:

$$PV = \frac{A}{(1+r)^1} + \frac{A(1+g)}{(1+r)^2} + \ldots + \frac{A(1+g)^{t-1}}{(1+r)^t} + \ldots + \frac{A(1+g)^{n-1}}{(1+r)^n}$$

- It can be noted that: $A_t = A_{t-1} \cdot (1+g)$

- The following equation is valid (g can be positive or negative):

$$PV = A \cdot \left[1 - \frac{(1+g)^n}{(1+r)^n} \right] \cdot \frac{1}{(r-g)} \quad [5]$$

Example [PMMC]

- PMMC is a medium-sized manufacturer that produces precision mechanical components for CNC machine tools.
- PMMC plans to install a grid connected photovoltaic plant with an output in the first year of 300,000 kWh. The study period (n) is 10 years.
- Other technical and economic data are:

Self-consumption share	75%	
Avoided grid price	0.165	(€/kWh)
Feed-in tariff	0.070	(€/kWh)
Annual degradation	-0.5%	
Discount rate	6.0%	

Example [PMMC]

- The first-year gross revenue can be calculated as

$$R_1 = 0.75 \times 300000 \times 0.165 + (1 - 0.75) \times 300000 \times 0.07 = 42{,}375€$$

- After the first year the material degradation affects the annual output of the PV system.
- The revenue decreases at a constant rate; the present value (PV) of the flow of gross revenues is:

$$PV = A \cdot \left[1 - \frac{(1+g)^n}{(1+r)^n}\right] \cdot \frac{1}{(r-g)} = 42375 \cdot \left[1 - \frac{(1-0.005)^{10}}{(1+0.06)^{10}}\right] \cdot \frac{1}{(0.06+0.005)} \cong 305690.04$$

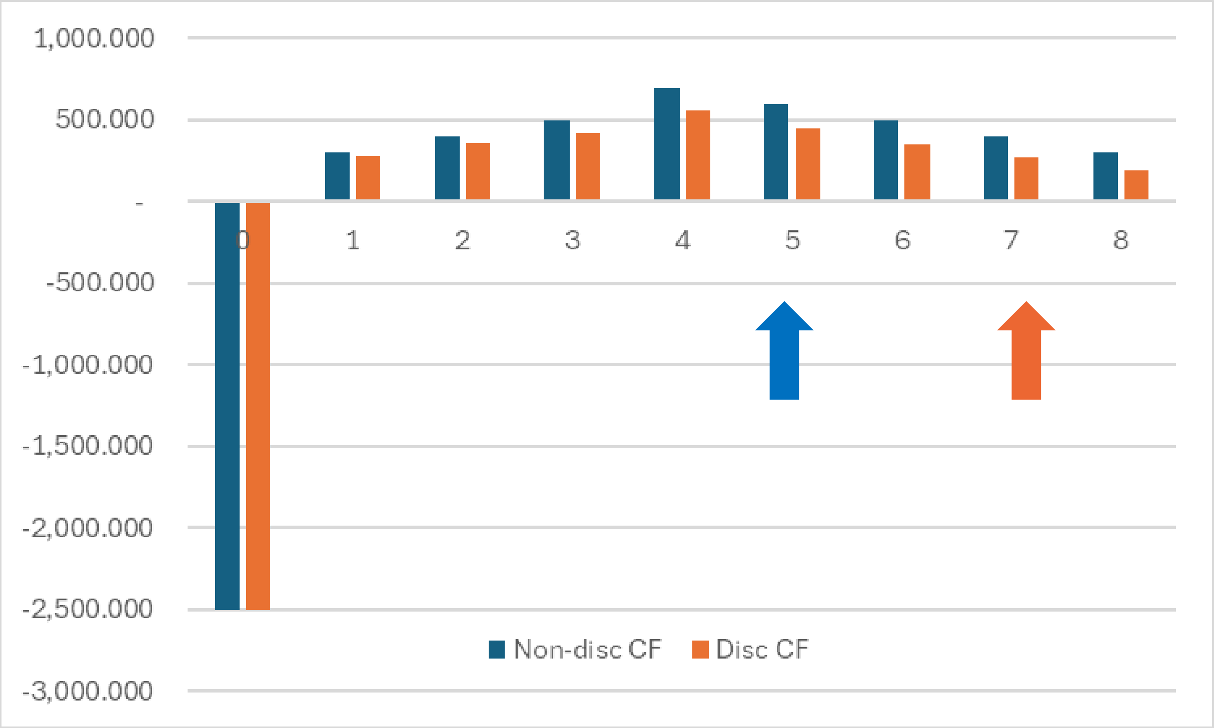
Discounted payback period

- The *discounted payback period* (DPB) is the year in which the progressive sum of the discounted flows equals the investment.
- In the case of investment I_0 at year 0 and positive flows from year 1, we have:

$$DPB = t^*: \quad \sum_{t=1}^{t^*} \frac{F_t}{(1+r)^t} = I_0$$

We can compare the simple PB with the DPB for the project below, assuming r=6%

Year	Non-disc CF (k€)	Disc CF (k€)
0	- 2,500.000	- 2,500.000
1	300.000	283.019
2	400.000	355.999
3	500.000	419.810
4	700.000	554.466
5	600.000	448.355
6	500.000	352.480
7	400.000	266.023
8	300.000	188.224
	5 years	7 years



- If the cash flows are constant over the years, after obtaining the simple PB, the following equation can be used:

$$DPB = \frac{-\ln(1 - PB \cdot r)}{\ln(1 + r)}$$

- In the EMot example we had PB=2.02 years; if r=5% we get

$$DPB = \frac{-\ln(1 - 2.02 \times 0.05)}{\ln(1 + 0.05)} \simeq 2.19 \text{ years}$$

- The difference between PB and DPB, in this case, is not significant.

Remarks on payback

$$PB \leq DPB$$

in fact, the present value of the cash flow, when discounted, is lower than its non-discounted counterpart.

- This difference increases as time *t* increases, as the flows are multiplied by $(1+r)^{-t}$
- This method produces a time indicator using only part of the cash flows.
- It would be appropriate to use indicators that consider the overall cash flow profile of a project.

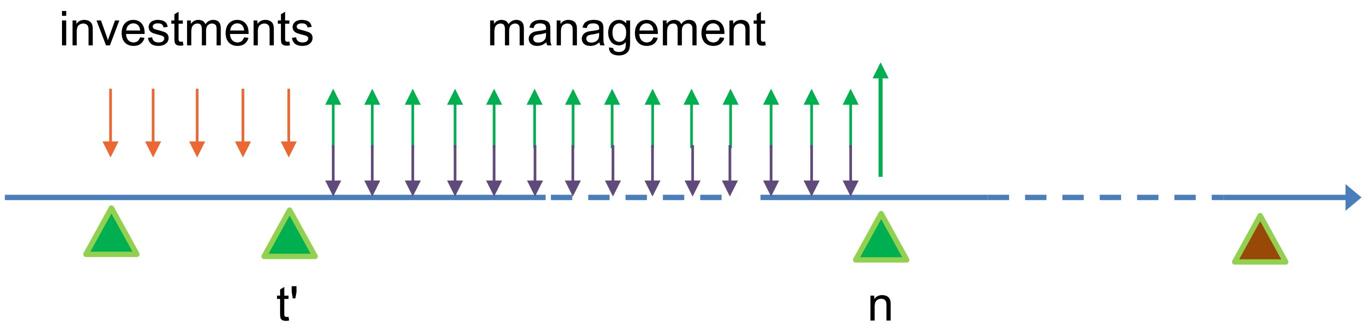
Profitability Index

- A first method that takes into consideration the cash flows during the *entire* project useful life is the *profitability index* (PI).
- If the investment outlays (I_t) are clearly defined and distinguished from the operating cash flows (F_t), we have
  - total investments: $$\sum_{t=0}^{n} \frac{I_t}{(1+r)^t}$$
  - operating cash flows: $$\sum_{t=0}^{n} \frac{F_t}{(1+r)^t}$$

- The profitability index is then:

$$PI = \frac{\sum_{t=0}^{n} \frac{F_t}{(1+r)^t}}{\sum_{t=0}^{n} \frac{I_t}{(1+r)^t}}$$

- Conceptually, PI is a measure of investment *efficiency*: it relates the (discounted) output produced by the investment to the (discounted) value of the resources absorbed in the investment.
- The investment is profitable if PI>1: in this case, for every €1 invested, the operation allows capital recovery and additional value generation.



- In the simplest cases, investment outlays only occur in the first few years (up to a year t') and, during operation, revenues exceed costs (F_t is positive for t>t')

$$PI = \frac{\sum_{t=0}^{n} \frac{F_t}{(1+r)^t}}{\sum_{t=0}^{t'} \frac{I_t}{(1+r)^t}}$$

- If PI>1 the investment is profitable: in this case, for every €1 invested, the operation allows recovery and value generation.
- In the case of several investments, the one with the highest PI will be preferred*.

- It is worth noting that the discounting operator is a linear operator: it therefore enjoys the property of value additivity.
- This means, for example, that years in which I_t or F_t are absent can be excluded from the summation.

(* We will see, however, that it is not entirely correct to base a comparison of solutions on the largest PI).

Example [SLOG]

- Let us take the case of the logistics service company again:
  - investment, I_0 = 200k€.
  - estimated annual net profit, 40k€
  - 6 years of operation
  - rate r=8%.

Example [SLOG]

- So:

$$PI = \frac{\sum_{t=0}^{n} \frac{F_t}{(1+r)^t}}{\sum_{t=0}^{t'} \frac{I_t}{(1+r)^t}} = \frac{A \frac{(1+r)^n - 1}{r(1+r)^n}}{I_0} = \frac{40 \frac{(1+0.08)^6 - 1}{0.08(1+0.08)^6}}{200,000}$$

$$= \frac{185}{200} = 0.925$$

- Each € invested yields €0.925.