Notes on annuities

- Annuities can be used in different configurations of cash flow profiles.
- Example 1 (MARR=8%)

	0	1	2	3	4	5	6	7	8
Ft		50	50	50	60	60	60	60	60

- It is always possible to obtain the present value of the cash flows by discounting each of them to t=0:

$$PV = \sum_{t=1}^{8} \frac{F_t}{(1+0.08)^t}$$

	0	1	2	3	4	5	6	7	8
Ft		50	50	50	60	60	60	60	60

- However, we see that there are two series of annuities:
  - 50 from 1 to 3
  - 60 from 4 to 8
- We can then calculate the PV adding
  - the contribution of the first series of three values (compounding to t=3 three annuities and discounting the total future value from t=3 to t=0)
  - and the contribution of the second series of five values (compounding to t=8 five annuities and discounting the total future value from t=8 to t=0):

$$PV = 50 \cdot \frac{(1+0.08)^3 - 1}{0.08} \cdot \frac{1}{(1+0.08)^3} + 60 \cdot \frac{(1+0.08)^5 - 1}{0.08} \cdot \frac{1}{(1+0.08)^8}$$

- Example 2 (MARR=8%)

	0	1	2	3	4	5	6	7	8
Ft		50	50	50		60	60	60	60

- In this case, we can calculate the PV adding
  - the contribution of the first series of three values (compounding to t=3 three annuities and discounting from t=3 to t=0)
  - and the contribution of the second series of four values (compounding to t=8 four annuities and discounting from t=8 to t=0):

$$PV = 50 \cdot \frac{(1+0.08)^3 - 1}{0.08} \cdot \frac{1}{(1+0.08)^3} + 60 \cdot \frac{(1+0.08)^4 - 1}{0.08} \cdot \frac{1}{(1+0.08)^8}$$

- In summary, the key aspect is compounding the number of annuities to the last year in which they appear and then discounting such future value to the reference year (t=0).

Notes on the discount rate

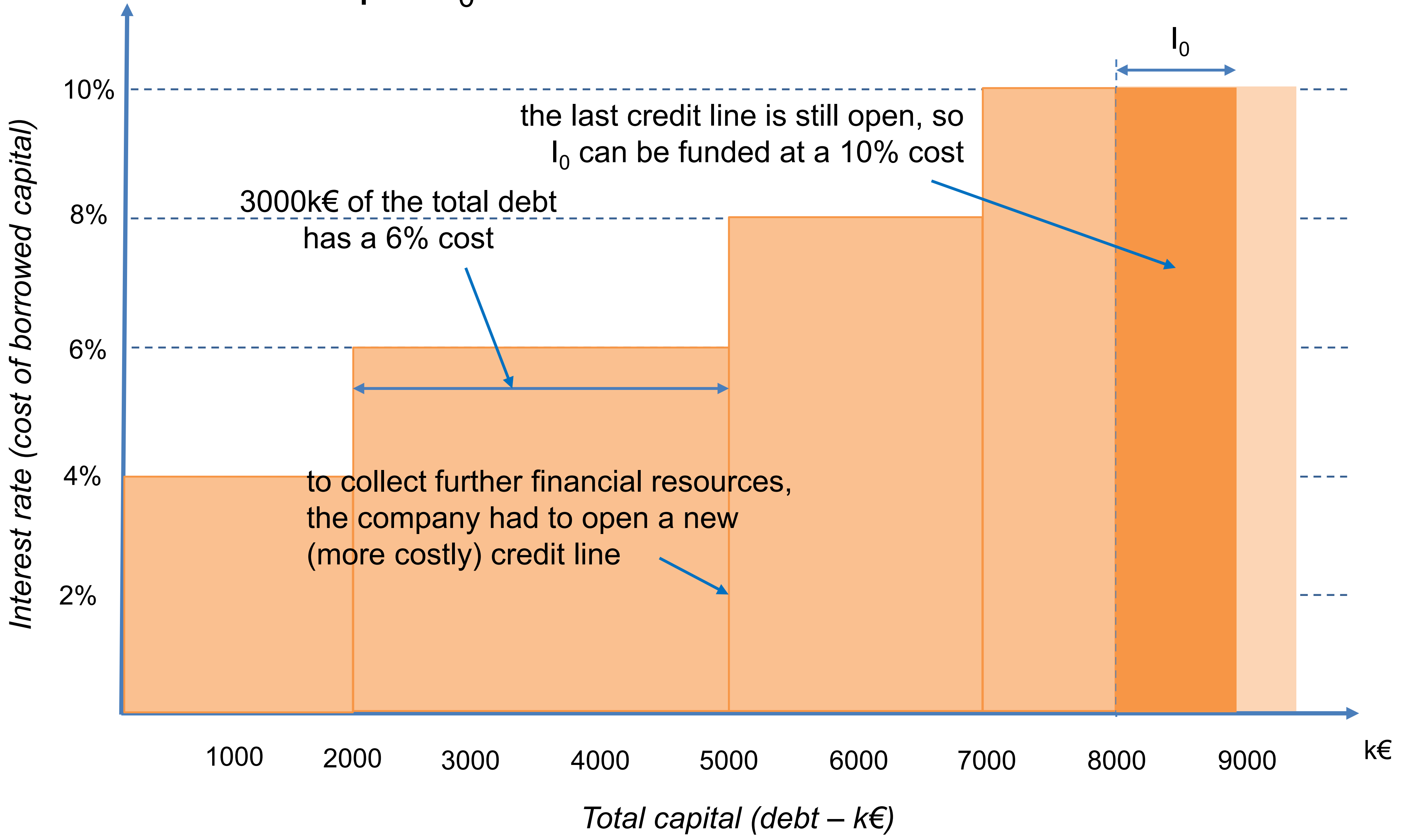
- In financial evaluation the different investment are assessed:
  - through the NPV (or PI) by discounting the cash flows at a rate r
  - through the IRR, which is compared with a hurdle rate.
- In practice, a benchmark representing the minimum acceptable rate of return of the invested capital is often considered.
- This parameter is called *minimum acceptable* (or *attractive*) *rate of return* (MARR) and it is used:
  - in the first case as the discount rate
  - in the second as the hurdle rate.

- In the case of a company, the MARR takes three aspects into account:
  - the company's *cost of capital* (essentially, the rate of return a company needs to pay to secure funds from its various sources, like debt and equity)
  - the *opportunity cost* of capital (i.e. the return foregone when investing in a particular project instead of choosing the next best alternative with a similar risk profile)
  - the *risk* of the investment project.

- To calculate the MARR, it is first necessary to identify investment alternatives.
- The project analysed will be cost-effective if it guarantees greater profitability than any alternative.
- It should be noted that, in this case, a risk-free investment is an alternative with a low rate of return.
- The actual MARR is defined by the financial department of a company.
- However, we can think of two basic situations:
  - the company cannot cover the project’s investment with internal resources
  - the company can cover the project’s investment with internal resources.

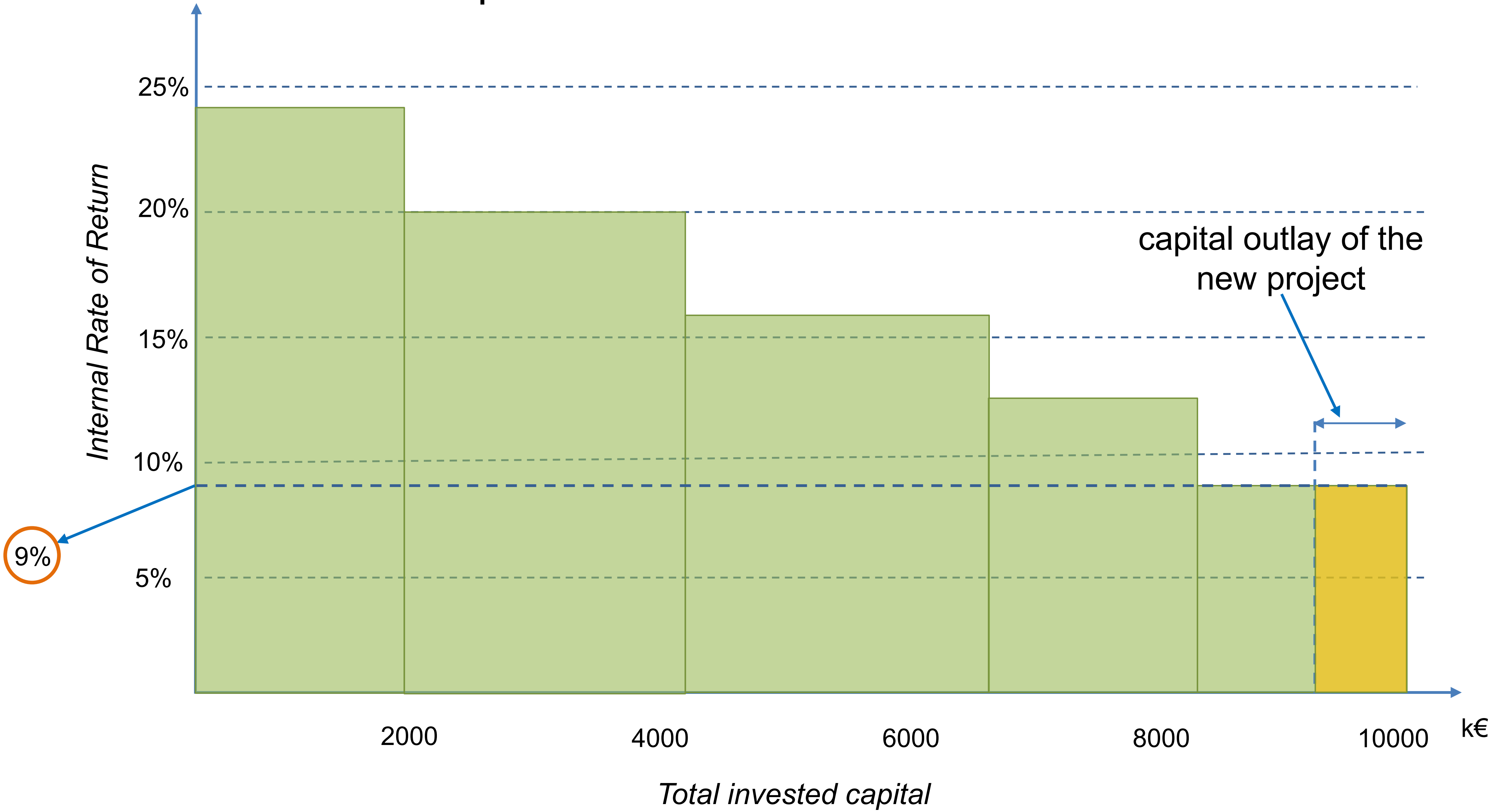
- In the first case, the company will seek financial assistance through loans.
- The return on the project will then have to repay the principal and interest.
- It is possible to represent the *credit lines* of the company in ascending order of the interest rate.
- K_j is the total capital borrowed at the interest rate r_j.
- The cost of capital increases as the debt of the company increases, as more and more expensive sources (credit lines) will have to be accessed.

Example: I_0 = 1000k€



- In the second situation, the company can use a part of its equity to fund the project.
- However, all “internal” capital resources are invested in some investment lines.
- To fund the project, the company must disinvest some of the invested capital.
- The investment lines can be sorted by decreasing profitability (e.g. their IRR).
- The company would disinvest part of the less profitable investment line.

Example:



25%
20%
15%
10%
5%
9%
Internal Rate of Return
capital outlay of the new project
2000
4000
6000
8000
10000
k€
Total invested capital

- This is an opportunity cost: the company will accept the project if its return is not lower than that provided by the disinvested line.

- If the project is partly funded with debt ($I_{0,D}$, r_D) and partly with equity ($I_{0,E}$, r_E), the MARR is the weighted average of the two components:

$$MARR = \frac{I_{0,D} \cdot r_D + I_{0,E} \cdot r_E}{I_{0,D} + I_{0,E}}$$

MARR as expected return on investment

- The concept of *expected return* suggests using as a rate of return that the investor considers reasonable based on the nature and risk of the investment.
- The rate can be considered to be the sum of three components:
  - the rate of return on a *risk-free investment* (rf)
  - the *risk premium for the industry* in which the project takes place (pi)
  - *the project-specific risk premium* (pp).

$$MARR = rf + pi + pp$$

- For *rf* the average yield of medium-term government bonds (in Italy, BTP_7 ≅ 3.25% in 2025*) can be used.
- *pi* depends on the country (the average value for Italy is estimated at about 3.5%**) and the sector (e.g. metalworking, 5-6%).
- *pp* can have a value between 0 and 5% depending on the risk.

(*Source: www.dt.tesoro.it/it/debito_pubblico/dati_statistici/principali_tassi_di_interesse
**Source: pages.stern.nyu.edu/~adamodar/New_Home_Page/datafile/ctryprem.html)

Valorisation of costs and revenues

- Costs and revenues are expressed in terms of market values (generally, adjusted for indirect taxes like VAT*).
- It is possible to use:
  - constant values
  - current values.

(* Value Added Tax)

- The criterion of *constant* (*real*) *values* adopts a homogeneous monetary unit throughout the time horizon of analysis.
- This represents the value of goods or services at a specific point in time, often the base year (t_0=0), to remove the impact of inflation.
- With this criterion, financial items are estimated as if the inflation rate were zero.
- Price differences in different periods for the same input or output item will be justified by *real* changes in value.

- The criterion of *current* (*nominal*) *values* adopts a specific monetary unit in each year.
- It depends on the generalised increase in prices.
- If *f* is the expected inflation rate (future projection), the future value of F_0 estimated at t_0 will be

$$F_{0,t} = F_0 \cdot (1+f)^t$$

- The MARR is usually a *gross* rate r_c (i.e. it includes *f*), so the present value of $F_{0,t}$ is:

$$PV(F_{0,t}) = F_0 \cdot \frac{(1+f)^t}{(1+r_c)^t}$$

- Example: we have the following investment and an estimated average annual price change $f = 2\%$.

		constant			current		
Year	index number	costs	revenues	net flow	costs	revenues	net flow
0	100.00	4000		-4000	4000		-4000.0
1	102.00	110	350	240	112.2	357.0	244.8
2	104.04	110	350	240	114.4	364.1	249.7
3	106.12	110	350	240	116.7	371.4	254.7
4	108.24	110	350	240	119.1	378.9	259.8
5	110.41	110	350	240	121.4	386.4	265.0
6	112.62	110	350	240	123.9	394.2	270.3
7	114.87	110	350	240	126.4	402.0	275.7
8	117.17	110	350	240	128.9	410.1	281.2
9	119.51	110	350	240	131.5	418.3	286.8
10	121.90	110	350	240	134.1	426.6	292.6

- The choice of constant or current values implies a consistent choice of discount rate.
- If cash flows are expressed in current values, the gross rate (r_c) is used.
- If the flows are expressed in constant values, the rate should be adjusted, removing the effect of inflation (*real rate* r_r):

$$(1+r_c)=(1+r_r)(1+f)$$

$$r_r=\frac{1+r_c}{1+f}-1=\frac{r_c-f}{1+f}$$

- If, in the example reported in the previous table, MARR is 8%, we can calculate the NPV using it as r_c.
- The calculation of NPV using the net cash flow as *current* values and r_c gives NPV=2,223.69€.
- To calculate NPV using the net cash flow as *constant* values we first obtain r_r:

$$r_r = \frac{r_c - f}{1+f} = \frac{0.08-0.02}{1+0.02} \cong 0.059$$

- The calculation of NPV then gives NPV=2,223.69€.
- Note that $r_c - f \cong 0.06$; if *f* is small, this approximated calculation can be accepted.

- Inflation is typically measured using a basket of goods and services, representing a general average for the entire economy.
- If the cash flow depends only on a specific good or service (e.g. cost of natural gas or electricity), *escalation* is employed.
- If 1€ in period 0 is subject to escalation (e), it is subject to the effect of the inflation (f) *and* the real price change of the specific good or service (f’) with respect to the inflation:

$$(1+e) = (1+f)(1+f') = 1+f+f'+f\cdot f'$$

$$e = f+f'+f\cdot f'$$

- The equations to calculate r_r can be applied replacing *f* with *e*.

Total life-cycle cost (TLCC)

- TLCC is used to evaluate differences in costs and the timing of (cost) cash flows for different projects.
- It considers all relevant costs over the life of a project.
- In sum, it corresponds to the present worth of the cost flows (CF):

$$TLCC = \sum_{t=0}^{n} \frac{Ct}{(1+r)^t}$$

- TLCC can be used to select among alternatives that provide the same economic returns.

Example: TLCC

- A small manufacturing company wants to estimate the TLCC of a 40kW industrial compressor that should be used in the finishing department.
- The purchase and installation cost is 32,000€.
- Estimated electric energy cost for year 1 is 22,000€ and preventive maintenance cost for year 1 is 2,560€.
- Financial data are:

MARR =	0.12	
f =	0.02	inflation (both energy and maintenance)
f' =	0.015	real price change (energy)
e =	0.0353	escalation (energy)

Example: TLCC

- The table reports the cost cash flows (current values) over a 10-year period. (Disc are the discounted values, using the MARR):

Year	Investment	El energy cost	Ord Man cost	Disc EE cost	Disc Man cost
0	32,000.00	-			
1		22,000.00	2,560.00	19,642.86	2,285.71
2		22,776.60	2,611.20	18,157.37	2,081.63
3		23,580.61	2,663.42	16,784.22	1,895.77
4		24,413.01	2,716.69	15,514.91	1,726.51
5		25,274.79	2,771.03	14,341.59	1,572.35
6		26,166.99	2,826.45	13,257.01	1,431.97
7		27,090.68	2,882.98	12,254.45	1,304.11
8		28,046.98	2,940.64	11,327.71	1,187.67
9		29,037.04	2,999.45	10,471.05	1,081.63
10		30,062.05	3,059.44	9,679.18	985.06
				141,430.33	15,552.42

Example: TLCC

- The TLCC is 188,892.75.
- It is worth noting that the total discounted values of energy and maintenance costs (last row) could have been obtained using the geometric gradient equation.
- For example, if CE_1 is the cost of energy in the first year, the present value of energy cost (PVCE) is:

$$PVCE = CE_1 \cdot \left[1 - \frac{(1+e)^n}{(1+r)^n} \right] \cdot \frac{1}{(r-e)} = 22000 \cdot \left[1 - \frac{(1+0.0353)^{10}}{(1+0.12)^{10}} \right] \cdot \frac{1}{(0.12-0.0353)}$$

Example [PNT]: use of incremental investment

- A manufacturing facility produces stainless steel cabinets to protect electrical panels.
- The project involves the installation of a coating line, with the investment outlay (I) and operating costs (Ct) estimated for each alternative configuration.
- The identified solutions have the same performance characteristics and revenues will not be affected by the choice.

- The costs (investment I and annual operation Ct) of the proposed solutions are:

Altern	I	Ct
A	- 650,000	- 91,800
B	- 780,000	- 52,600
C	- 600,000	- 105,000
D	- 750,000	- 68,400
E	- 720,000	- 74,900

- The operational lifetime of each system was estimated at 5 years and the chosen (real) MARR is 15%.
- The residual value of the solutions can be neglected.
- In this case, it is therefore a question of evaluating solution flows in terms of costs: the objective will be to minimise the present value (PV) of the costs.

- Since the Ct's are constant (for each solution) over the 5 years, the present value of the constant annuities can be calculated:

$$PV = Ct \cdot \frac{(1+r)^n - 1}{r(1+r)^n}$$

- Adding up the investment, we get:

Altern	I	Ct	VA(t=1,n)	VA
A	- 650,000	- 91,800	- 307,728	- 957,728
B	- 780,000	- 52,600	- 176,323	- 956,323
C	- 600,000	- 105,000	- 351,976	- 951,976
D	- 750,000	- 68,400	- 229,287	- 979,287
E	- 720,000	- 74,900	- 251,076	- 971,076

- Alternative C gives the lowest present value of total costs.

- We can also apply the *incremental investment* valuation method, which adopts a differential logic.
- Starting with a good solution, we assess whether the increased investment required to adopt a different one pays off and adds value.
- Note that the IRR of the individual solutions cannot be calculated, as the cash flows are always negative.

- The procedure involves a series of steps.

1. the alternatives are ordered in ascending order of the investment required (I)
2. the “best” starting alternative (A_0) is the one with the lowest I
3. A_0 is compared with the alternative which, of the remaining alternatives, is the one requiring the lowest investment I (A_1)
4. the IRR of the incremental investment, thus defined, is assessed:

$$F(A_1 - A_0) = F(A_1) - F(A_0)$$

if $IRR[F(A_1-A_0)] > HR$ (e.g. MARR)

A_0 is discarded and the alternative A_1 is adopted; in fact, this means that A_1 adds to the (economic) result of A_0 further economic benefits;

if $IRR[F(A_1-A_0)] \leq HR$

A_1 is discarded and A_0 is retained as "best" alternative.

4. step 3 is repeated until all alternatives have been compared
5. the overall "best" alternative is the one that remains at the end of the process.

- In the Example we have the following initial sorting:

Altern	I	C1	C2	C3	C4	C5
C	-600000	-105000	-105000	-105000	-105000	-105000
A	-650000	-91800	-91800	-91800	-91800	-91800
E	-720000	-74900	-74900	-74900	-74900	-74900
D	-750000	-68400	-68400	-68400	-68400	-68400
B	-780000	-52600	-52600	-52600	-52600	-52600

- Thus (if MARR=15%):

	0	1	2	3	4	5	IRR	
A-C	- 50,000	13,200	13,200	13,200	13,200	13,200	10.0%	<MARR
E-C	- 120,000	30,100	30,100	30,100	30,100	30,100	8.1%	<MARR
D-C	- 150,000	36,600	36,600	36,600	36,600	36,600	7.0%	<MARR
B-C	- 180,000	52,400	52,400	52,400	52,400	52,400	14.0%	<MARR

- Note that the *differential* cash flows from period 1 are positive.
- Alternative C is the preferred alternative (as for NPV).

Example [PNT]: different useful lives

- PNT is considering the purchase of three different plant solutions for the coating process.
- The three solutions have comparable performance characteristics, and the choice will not affect revenues.
- The investment costs I, annual operating costs Ct and the different useful lives (years) are:

Altern	I (€)	Ct (€)	Duration
A	- 680,000	- 40,000	3
B	- 560,000	- 60,000	7
C	- 600,000	- 50,000	5

- Each alternative has a residual value (at the end of its useful life) of zero.
- It would, however, be possible to obtain salvage values (SV) if plants B and C were decommissioned before the end of their useful life.
- The technical lives of A and C are longer than their useful lives, but new maintenance and operating costs (Cupd) will have to be incurred if they were used beyond their useful lives.
- These data are shown in the table below:

Altern	I	Ct	Duration	SV(3)	SV(5)	Cupd(t)
A	- 680,000	- 40,000	3	-	-	- 60,000
B	- 560,000	- 60,000	7	240,000	70,000	-
C	- 600,000	- 50,000	5	180,000	-	- 60,000

- The company uses r=MARR=15%.
- As the durations are different, the alternatives must be referred to a same time frame (or *study period*).
- Three scenarios can therefore be elaborated:
  - study period of 3 years
  - study period of 5 years
  - study period of 7 years.
- To do this, the cash flow profile must be revised taking into account salvage values or changes in operating and maintenance costs.

- Study period n=3 years
- B and C will benefit in the third year from the salvage value, which will be added to the operating costs of the third year:

Altern	0	1	2	3	PV
A	- 680,000	- 40,000	- 40,000	- 40,000	- 771,329
B	- 560,000	- 60,000	- 60,000	180,000	- 539,190
C	- 600,000	- 50,000	- 50,000	130,000	- 595,808

- B has the lowest present value of total costs.

- Study period n=5 years
- B will benefit in the fifth year from the salvage value, which will be added to the operating costs of the fifth year.
- A will see increased values of operating costs in the fourth and fifth years:

Altern	0	1	2	3	4	5	PV
A	- 680,000	- 40,000	- 40,000	- 40,000	- 60,000	- 60,000	- 835,465
B	- 560,000	- 60,000	- 60,000	- 60,000	- 60,000	10,000	- 726,327
C	- 600,000	- 50,000	- 50,000	- 50,000	- 50,000	- 50,000	- 767,608

- B has the lowest present value of total costs.

- Study period n=7 years
- A and C will have higher operating costs from the fourth and sixth year, respectively:

Altern	0	1	2	3	4	5	6	7	PV
A	- 680,000	- 40,000	- 40,000	- 40,000	- 60,000	- 60,000	- 60,000	- 60,000	- 883,961
B	- 560,000	- 60,000	- 60,000	- 60,000	- 60,000	- 60,000	- 60,000	- 60,000	- 809,625
C	- 600,000	- 50,000	- 50,000	- 50,000	- 50,000	- 50,000	- 60,000	- 60,000	- 816,104

- B still has the lowest present value of total costs.
- Different situations do not always favour the same alternative: this depends on the cash flow profile and the value of r.
- If each solution can be used for the shorter time frame (3 years), the PV limited to this time frame will be used, provided that a decision is made about the residual life of the assets with longer life.